

# Automated Theorem Proving: Satisfiability Modulo Theories



# One-Slide Summary

- An **automated theorem prover** is an algorithm that determines whether a mathematical or logical proposition is **valid** (**satisfiable**).
- A **theory** is a set of sentences with a deductive system that can determine satisfiability.
- A **satisfiability modulo theories** (SMT) instance is a proposition that can include logical connectives, equality, and terms from various theories.
- The theory of **equality** can be decided via **congruence closure** using **union-find**.
- **DPLL(T)** is an SMT algorithm that uses a modified DPLL SAT solver, a well-defined interface for Theories, and a **mapping** between propositional variables and Theory literals.
- We can use logical rules of inference to encode **proofs**. Proof checking is like type checking.

# Combined Motivation

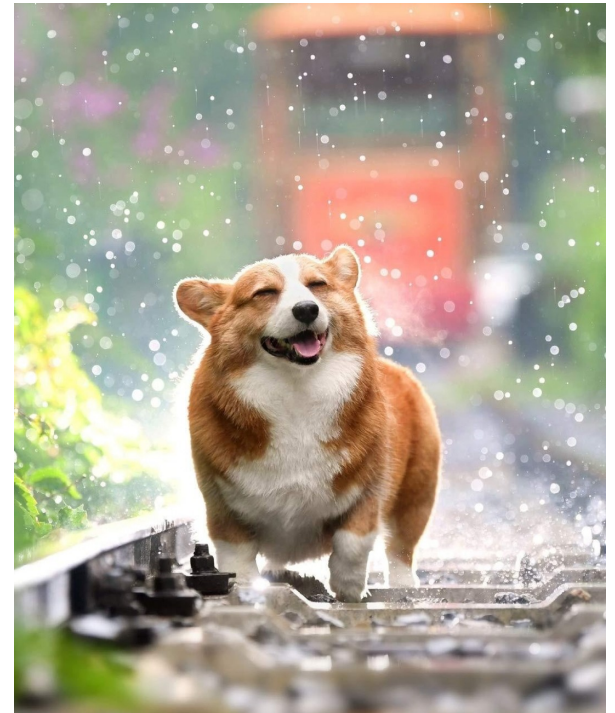
- We have seen:
  - How to handle  $(A \mid \mid !B) \Rightarrow (!A \mid \mid C)$ 
    - Satisfied by  $\{A, C\}$ , for example
    - Arbitrary boolean expressions of boolean variables
  - How to handle  $(x + y \leq 5) \&\& (2x \geq 10)$ 
    - Satisfied by  $\{x = 5, y = 0\}$ , for example
    - Conjunctions of linear inequalities of real variables
- But what about:
  - $(\text{strlen}(x) + y \leq 5) \Rightarrow (\text{strcat}(x,x) \neq \text{“abba”})$ 
    - Satisfied by  $\{x = \text{“abc”}, y = 3\}$ , for example

# High-Level Approach

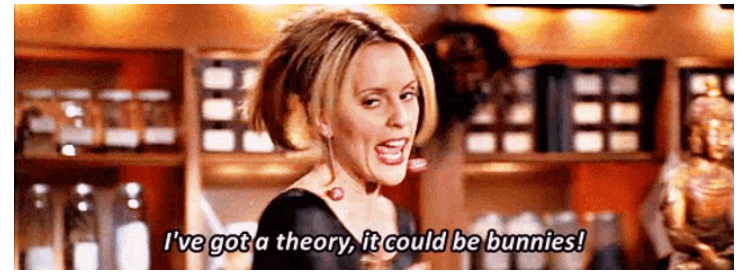
- Beyond basic logic, we want to reason about
  - **Strings**: `strlen(x)`, `regexp_match(x, “[0-9]+”)`, ...
  - **Equality**:  $a = b \Rightarrow f(a) = f(b)$ , ...
  - **Linear Arithmetic**:  $2x + 3y \leq 10$ , ...
  - **Bitvectors**:  $(x \gg 2) \mid y == 0\text{xff}$ , ...
  - **Lists**: `head(cons(p,q)) = p`
- **All at the same time!**
- We will handle each domain separately (as a theory) and then combine them all together using DPLL and SAT as the “glue”.

# Overall Plan

- Theory Introduction
- Theory of Equality
  - Congruence Closure
- The DPLL(T) solver for SMT
  - Formal Theory Interface
  - Changes to DPLL
- Proof and Proof Checking



# We've Got a Theory



- In general, a **theory** is a set of sentences (*syntax*) with a deductive system that can determine satisfiability (*semantics*).
- Usually, the set of sentences is formally defined by a grammar of **terms** over **atoms**. The **satisfying assignment** (or **model**, or **interpretation**) maps **literals** (terms or negated terms) to booleans.
- We will consider theories that reason about **conjunctions** of literals.

# Theory of Linear Inequalities

- Given a finite set of variables  $V$  and a finite set of real-valued constants  $C$
- Term  $::= C_1 V_1 + \dots + C_n V_n \leq C_{n+1}$
- Conjunctions of terms can be decided via the Simplex method

# Theory of Equality and Uninterpreted Functions

- Given a finite set of predicate symbols  $P$  (each of which has an associated *arity*)
- Predicate  $::= P_i(\text{Predicate}_1, \dots \text{Predicate}_n)$
- Term  $::=$       Predicate  
                  |      Predicate<sub>1</sub> = Predicate<sub>2</sub>
- Term Examples:
  - $f(g(a,b),a)$                       //  $a, b$  have arity 0 (constants)
  - $f(a) = f(b)$                         //  $f$  has arity 1
  - $f(f(x)) = f(x)$                     //  $g$  has arity 2



# Theory of Equality Definition

- Theory of equality with *uninterpreted functions*
- Symbols:  $=$ ,  $\neq$ ,  $f$ ,  $g$ , ...
- Axiomatically defined ( $A, B, C$  are Predicates):

$$\begin{array}{cccc} \frac{}{A=A} & \frac{B=A}{A=B} & \frac{A=B \quad B=C}{A=C} & \frac{A=B}{f(A) = f(B)} \end{array}$$

- Reflexive, Symmetric, Transitive, Definition of A Function (Extensionality)

# Solving Equality

- Consider this conjunction of literals in the theory of equality:

$$g(g(g(x)))=x \ \&\& \ g(g(g(g(g(x)))))=x \ \&\& \ g(x)\neq x$$

- Is it satisfiable?



The screenshot shows a web browser window with the address bar displaying `www.surveywriter.net/in/DM_ProcessQ5_2.asp`. The page content includes a red error message: "For the question, 'What is your zip code?', the answer(s) must be between 5 and 5." Below this, the question "What is your zip code?" is displayed. A text input field contains the value "20008". A "Next" button is located below the input field. In the bottom right corner, there is a "Progress Meter" showing a blue bar and the text "7%".


# Solving Equality

- Consider this conjunction of literals in the theory of equality:

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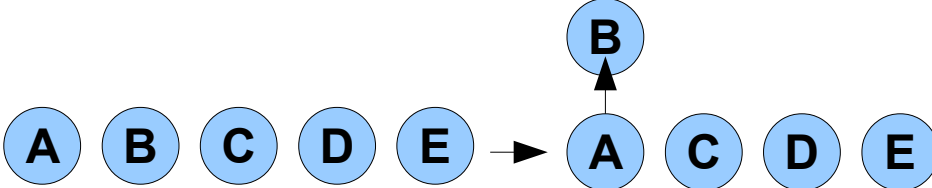
- Is it satisfiable? No.
  - $gggg(x) = g(x)$  by extensionality (1)
  - $ggggg(x) = gg(x)$  by extensionality (4)
  - $x = gg(x)$  by transitivity (2, 5)
  - $g(x) = ggg(x)$  by extensionality (6)
  - $g(x) = x$  by transitivity (1, 7)
  - false by contradiction (8, 3)

# Tracking Equality

- To decide the theory of equality mechanically, we track equivalence classes of predicates
- Two operations
  - Set Equal (**Union**)
  - Check Equal (**Find** Parents)
- Basic approach: build a tree
- $A = B$
- $D = E$  
- $C = B$

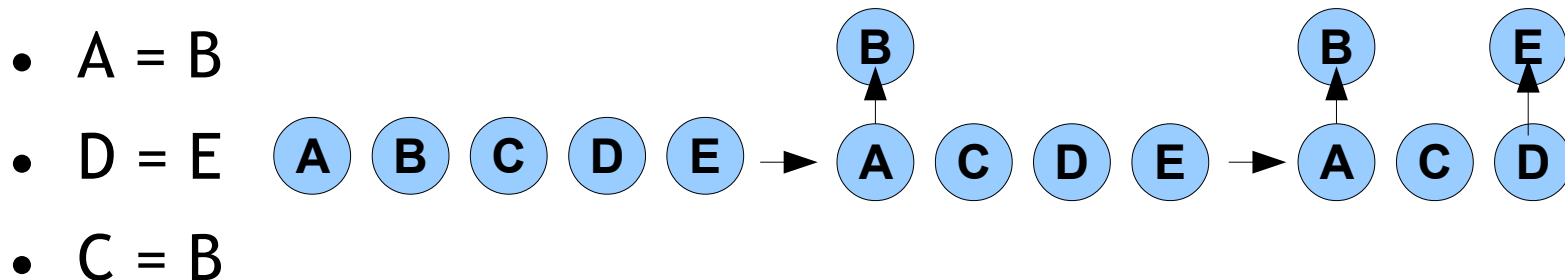
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- 
- The diagram illustrates the 'Set Equal (Union)' operation. It shows two initial sets: Set A containing elements {A, B, C, D, E} and Set B containing element {B}. An arrow points from Set A to a new set A, which now contains {A, C, D, E}. Above this new set A is a node B, with an arrow pointing from A to B, indicating that B is now the parent of A in the equivalence class tree.

# Tracking Equality

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# Tracking Equality

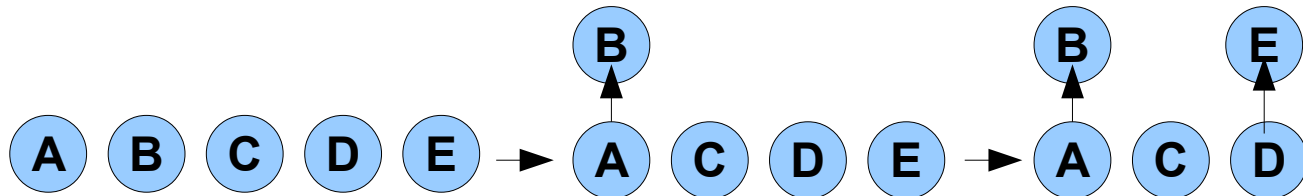
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- Basic approach: build a tree

- $A = B$

- $D = E$

- $C = B$



# Analogy: Raindrops Merging





# Two Optimizations

let **union** x y =

let x\_root, y\_root = find x, find y in

if x\_root = y\_root then return ()

if x\_root.depth < y\_root.depth

    x\_root.parent := y\_root

else if y\_root.depth < x\_root.depth

    y\_root.parent := x\_root

else

    y\_root.parent := x\_root

    x\_root.depth := x\_root.depth + 1

Keep trees short!  
Add smaller tree  
to bigger tree.  
Only increases total  
Depth if depths  
Were equal!

# Analogy: Temporary Forwarding vs. Change of Address

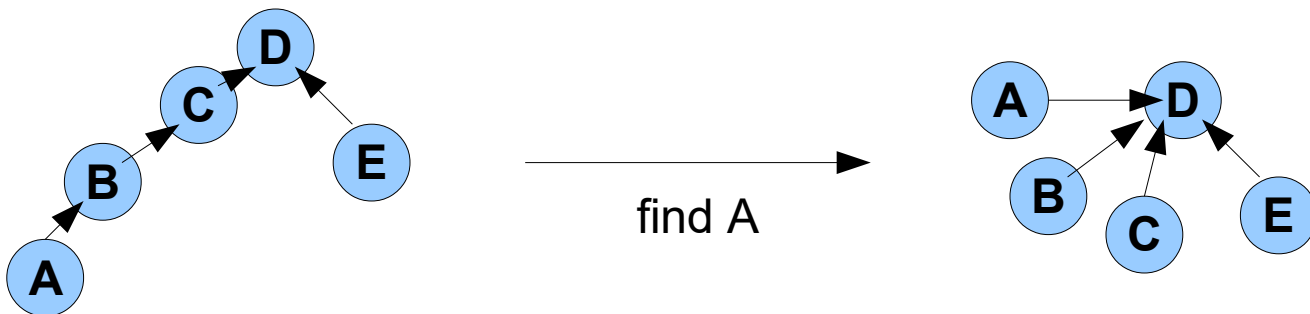


# Two Optimizations

```
let rec find x =  
  if x.parent != x then  
    x.parent := find(x.parent)  
  return x.parent
```



- This is called **path compression**.



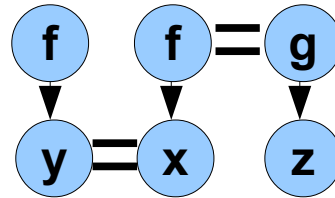
# Union-Find Analysis

- This is known as a “union-find” or “disjoint union” or “congruence closure” data structure.
- With the two optimizations, the amortized running time is  $O(\text{inverse\_ackermann}(N))$ 
  - $\text{inverse\_ackermann}(N) \leq 5$  for  $N \leq 2^{2^{65536}}$
- So the amortized time is effectively constant.
- “Fast Decision Procedures based on Congruence Closure”, Nelson & Oppen, 1980

# Equality Decision Procedure Idea

- Idea: start with the graph based on the conjunction of literals given

-  $f(x) = g(z) \ \&\& \ x = y$   
 $\&\& f(y) \neq g(z)$



- There is a unique minimal graph corresponding to the congruence closure of the equality relation (i.e., if you know  $a=b$  and  $b=c$ , add the  $a=c$  edge).
- Compute that graph via union-find, but when you learn  $a=b$ , also add  $f(a)=f(b)$ .

# Equality Decision Procedure

## Intermediate Steps

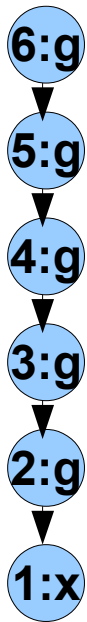
```
let rec merge u v = (* add “u == v” *)  
  if find(u) = find(v) then return ()  
  let p_u = preds_of (eq_class_of u) in  
  let p_v = preds_of (eq_class_of v) in  
  union u v ;  
  for each (x,y) in p_u, p_v do (*  $u=v \Rightarrow f(u) = f(v)$  *)  
    if find x <> find y && congruent x y then  
      merge x y  
let congruent x y =  
  return (out_degree x = out_degree y)  
  && forall i. find x.child[i] = find y.child[i]
```

# Equality Decision Procedure

- Input:
  - $t_1 = t'_1 \ \&\& \ t_2 = t'_2 \ \&\& \ \dots \ t_p = t'_p$
  - $r_1 \neq r'_1 \ \&\& \ \dots \ r_2 \neq r'_2 \ \&\& \ \dots \ r_q \neq r'_q$
- Construct the graph where the vertices correspond to the terms and the edges correspond to function application
- for  $i = 1 \dots p$ : merge  $t_i \ t'_i$
- for  $i = 1 \dots q$ : if find  $r_i =$  find  $r'_i$  then FALSE
- TRUE

# Equality Example

$g(g(g(x)))=x \ \&\& \ g(g(g(g(g(x))))=x \ \&\& \ g(x) \neq x$



Input Term Graph

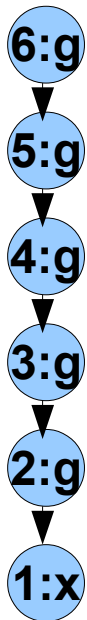


Union-Find Forest



# Equality Example

$$g(g(g(x)))=x \ \&\& \ g(g(g(g(g(x))))=x \ \&\& \ g(x)\neq x$$



Input Term Graph

Steps:

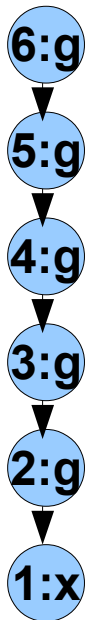
1. merge 4 1
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3. check find 2 1



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Input Term Graph

Steps:

1. merge 4 1
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3. check find 2 1

$u = 4, v = 1$

$p_u = \{5,6\}$

$p_v = \{2,3,4,5,6\}$

merge  $u \ v$

if congruent  $p_u \ p_v$

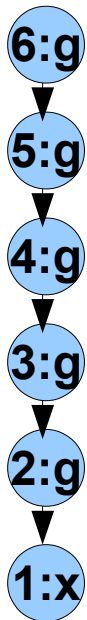
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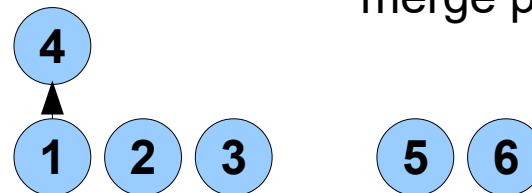
$p_u = \{5,6\}$

$p_v = \{2,3,4,5,6\}$

**merge u v**

if congruent  $p_u p_v$

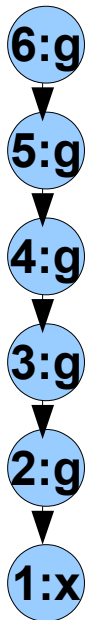
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# Equality Example

$$g(g(g(x)))=x \ \&\& \ g(g(g(g(g(x))))=x \ \&\& \ g(x) \neq x$$



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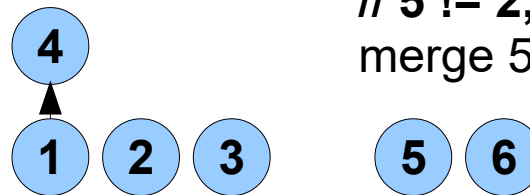
$p_v = \{2, 3, 4, 5, 6\}$

merge  $u \ v$

**if congruent 5 2**

**// 5 != 2, 5.child = 2.child**

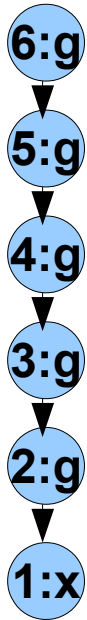
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Union-Find Forest

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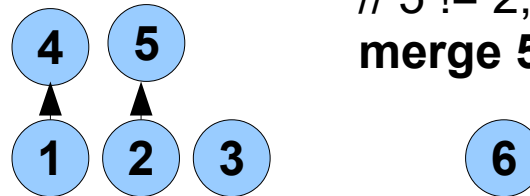
$p_v = \{2,3,4,5,6\}$

merge  $u \ v$

if congruent 5 2

//  $5 \neq 2, 5.child = 2.child$

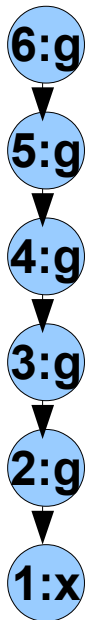
**merge 5 2**



Union-Find Forest

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$$g(g(g(x)))=x \ \&\& \ g(g(g(g(g(x))))=x \ \&\& \ g(x)\neq x$$



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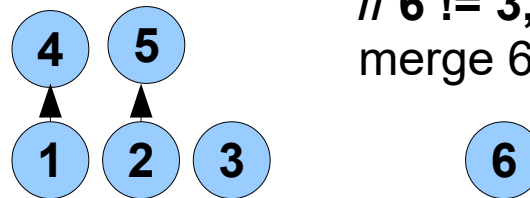
$p_v = \{2, 3, 4, 5, 6\}$

merge  $u \ v$

**if congruent 6 3**

**// 6 != 3, 6.child = 3.child**

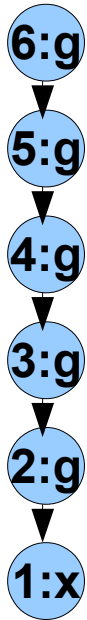
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Union-Find Forest

# Equality Example

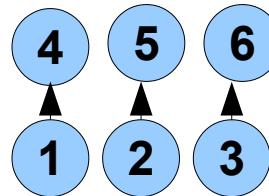
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Input Term Graph

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Union-Find Forest

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merge  $u \ v$

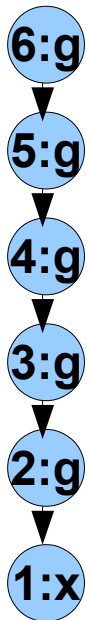
if congruent 6 3

//  $6 \neq 3, 6.child = 3.child$

**merge 6 3**

# Equality Example

$$g(g(g(x)))=x \ \&\& \ g(g(g(g(g(x)))))=x \ \&\& \ g(x) \neq x$$



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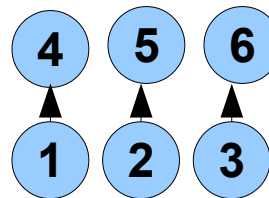
$p_u = \{4,5,6\}$

$p_v = \{1,2,3,4,5,6\}$

merge  $u \ v$

if congruent  $p_u \ p_v$

merge  $p_u \ p_v$

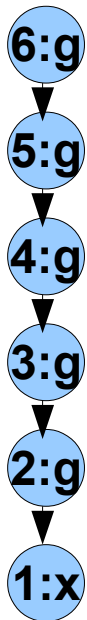


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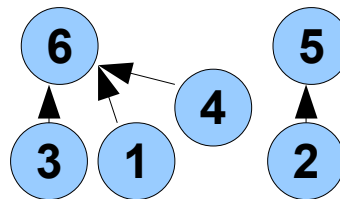
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**merge u v**

if congruent  $p\_u \ p\_v$

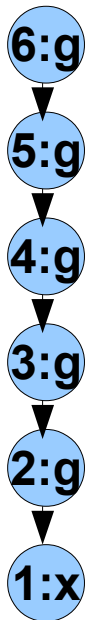
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Union-Find Forest

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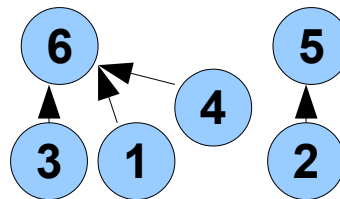
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**if congruent 4 2**

**// 4 != 2, 4.child = 2.child**

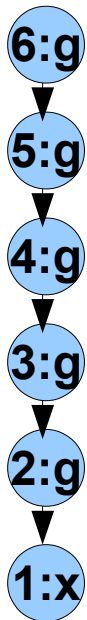
**merge 4 2**



Union-Find Forest

# Equality Example

$$g(g(g(x)))=x \ \&\& \ g(g(g(g(g(x))))=x \ \&\& \ g(x) \neq x$$



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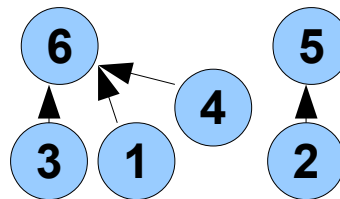
$p_v = \{1, 2, 3, 4, 5, 6\}$

merge  $u \ v$

if congruent 4 2

//  $4 \neq 2, 4.child = 2.child$

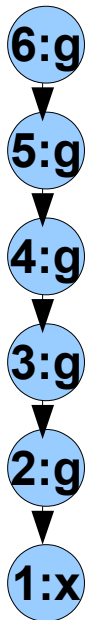
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Union-Find Forest

# Equality Example

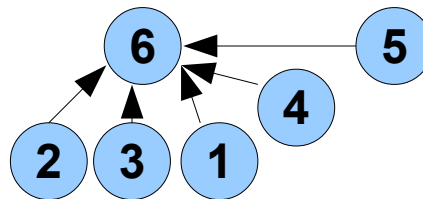
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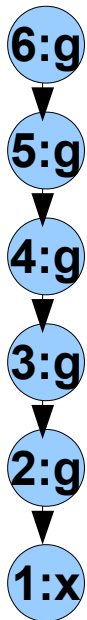


Union-Find Forest

$u = 6, v = 1$   
 $p\_u = \{4,5,6\}$   
 $p\_v = \{1,2,3,4,5,6\}$   
 merge  $u \ v$   
 if congruent 4 2  
 //  $4 \neq 2, 4.child = 2.child$   
**merge 4 2**

# Equality Example

$$g(g(g(x)))=x \ \&\& \ g(g(g(g(g(x))))=x \ \&\& \ g(x)\neq x$$



Input Term Graph

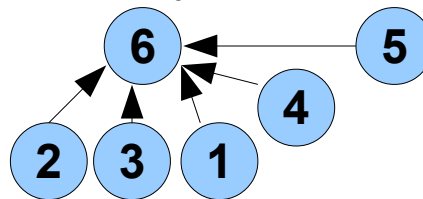
Steps:

1. merge 4 1

2. merge 6 1

3. **check find 2 1**

they are equivalent: return false!



Union-Find Forest

# Q. Computer Science

- This algorithmic strategy is applicable to decomposable problems that exhibit the *optimal substructure property* (in which the optimal solution to a problem P can be constructed from the optimal solutions to its overlapping subproblems). The term was coined in the 1940's by Richard Bellman. Problems as diverse as “shortest path”, “sequence alignment” and “CFG parsing” use this approach.

# Classic YA Vampire Stories

- Consider *Twilight*, *Buffy the Vampire Slayer*, and *The Vampire Diaries*. All three feature a steamy love triangle, all three feature werewolves, all three feature mind or memory control, all three feature some blood being more powerful or more tasty than other blood, but only two of them feature magical jewelry to protect vampires from damaging sunlight. Name the franchise that does not.

# Geography: University Towns in WV

- Situated on the Monongahela River, just south of Pittsburgh Pennsylvania, *this town* is home to West Virginia University. It features a “Personal Rapid Transit” system connecting the campus to downtown and a historic Metropolitan Theater from the 1920s.



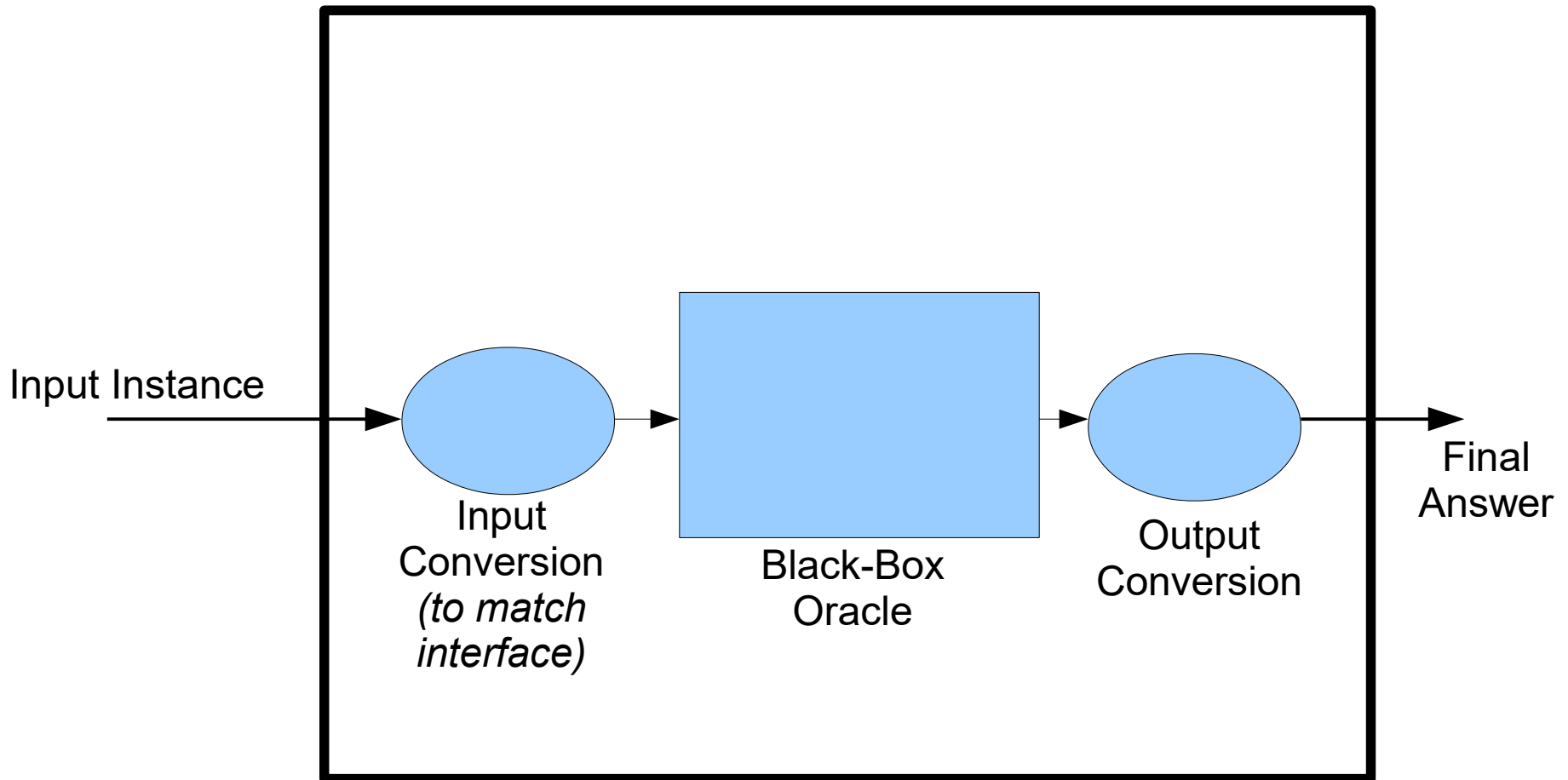


# Q. Computer Science

(olsonchs memorial)

- *This* family of protocols solves distributed consensus among unreliable processors. The participants must agree on one result in the face of failures. *This* protocol was first published in 1989 by Lamport, named after a fictional legislative system on a Greek island. It was so difficult to understand that a subsequent paper, “*This Made Simple*”, was published: “At the PODC 2001 conference, I got tired of everyone saying how difficult it was to understand the *This* algorithm ...” The abstract of the new paper is merely: “The *This* algorithm, when presented in plain English, is very simple.”

# Analogy: (NP-Hard) Reduction



# Theory Interface

- **Initialize**(universe : Literal Set)
- **SetTrue**(l : Literal) : Literal Set
  - Raise exception if l is inconsistent. Otherwise, add l to set of known facts. Return newly implied set of true facts (e.g., “a=c” after “a=b” and “b=c”)
- **Backtrack**(n : Nat)
  - Forget last n facts from “SetTrue”.
- **IsDefinitelyTrue**(l : Literal) : Bool
- **Explanation**(l : Literal) : Literal Set
  - If l is true, return a model (proof) of it.

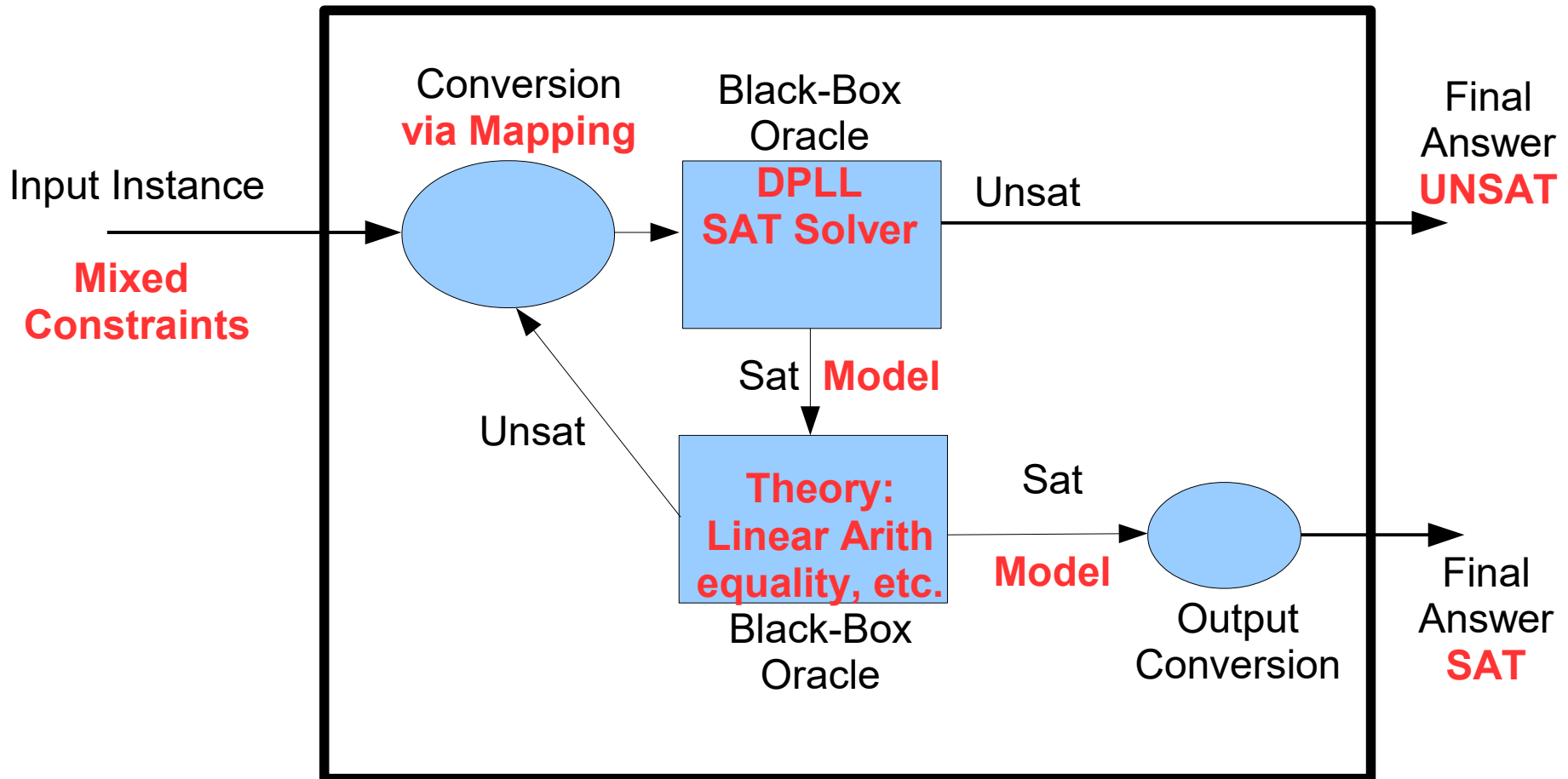
# Satisfiability Modulo Theories

- A **satisfiability modulo theories** (SMT) solver operates on propositions involving both logical terms and terms from theories.
- Modern SMT solvers can use any theory that satisfies the **Theory Interface** shown before.
- Replace Theory clauses with special propositional variables.
- Use a pure SAT solver. If the solution involves some theory clauses, ask the Theory if they can all be true. If not, **add constraints and restart**.

# SMT Basic Idea

- Given a query like
  - $(x > 5) \ \&\& \ (p \ || \ (x < 4)) \ \&\& \ !p$
- Note that almost everything can be handled by SAT:
  - $(\underline{x > 5}) \ \&\& \ (p \ || \ (\underline{x < 4})) \ \&\& \ !p$
  - Only the highlighted parts require a Theory.
- So ask SAT to consider:
  - $T1 \ \&\& \ (p \ || \ T2) \ \&\& \ !p$
- And then whenever SAT gives a model, ask the theories if that model makes sense.

# SMT Architecture (Reduction Redux)



# SMT Example

- Input:  $(x > 5) \ \&\& \ (p \ || \ (x < 4)) \ \&\& \ !p$
- Rewrite:  $T1 \ \&\& \ (p \ || \ T2) \ \&\& \ !p$ 
  - $T1 = "x > 5"$       *// mapping*
  - $T2 = "x < 4"$
- SAT solver returns  $\{T1, T2, !p\}$
- Ask Theory about  $T1 \ \&\& \ T2$ 
  - Theory Query:  $(x > 5) \ \&\& \ (x < 4)$
  - Theory Result: Unsatisfiable!
- $T1 \ \&\& \ (p \ || \ T2) \ \&\& \ !p \ \&\& \ !(T1 \ \&\& \ T2)$

# Analogy: SMT vs. SLAM (super cool!)

- SLAM converted the **C program** to a **Boolean Program** in a sound manner:
  - If **Label** is reachable in the **Boolean Program** via a **Path P** and the ground truth (**Newton**) agrees with that **Path P**, the **Label** is truly reachable.
  - If **Newton** disagrees with that **Path P** because XYZ cannot be true in the **C program**, add XYZ to the abstraction and restart.
- SMT converts the **Mixed Constraints** into **Boolean Constraints** in a sound manner:
  - If the **Boolean Constraints** are **satisfiable** via a **Model M** and the ground truth (**Theories**) agree with that **Model M**, the constraints are truly **satisfiable**.
  - If the **Theories** disagree with **Model M** because XYZ cannot be true in the **Mixed Constraints**, add XYZ to the constraints and restart.



# DPLL(T)

- **DPLL(T)** is an SMT algorithm based on the DPLL SAT solver but parameterized with respect to a set of Theories T.
- DPLL(T) Algorithm
  - Convert mixed constraints to boolean constraints
  - Run DPLL, but with two changes:
    - No Pure Variable Elimination optimization
    - Unit Propagation uses `T.setTrue`, may add clauses
  - Whenever `T.setTrue` raises an exception, add the negation of the explanation to the constraints

(homework #3 coding hint)

# No Pure Variable Elimination

- In pure propositional logic, variables are necessarily **independent**.
  - So if P only appears positively, you can set P=true without loss and save time.
- With Theories, variables may be dependent.
- Consider:
  - $(x > 10 \mid\mid x < 3) \&\& (x > 10 \mid\mid x < 9) \&\& (x < 7)$
- “ $x > 10$ ” is always used positively
  - But just skipping to “ $x > 10$ ” = true as part of the model leads you astray (makes the others false)!

# Unit Propagation Additions

- Consider:
  - $(A = B) \ \&\& \ (B = C) \ \&\& \ (A \neq C \ || \ X)$
- Convert:
  - $P1 \ \&\& \ P2 \ \&\& \ (!P3 \ || \ X)$
- Unit Propagation: add P1 ( $A = B$ ) to model
- Now:  $P2 \ \&\& \ (!P3 \ || \ X)$
- Unit Propagation: add P2 ( $B = C$ ) to model
  - **But Wait! Theory reports: P3 ( $A = C$ ) now true**
- Now:  $(!P3 \ || \ X) \ \&\& \ P3 \dots$

# DPLL(T) Example

- Input Clauses:
  - $(f(a) \neq f(b) \vee d \neq e) \wedge \wedge$  // T1 =  $f(a) = f(b)$
  - $(a = b \vee a \neq c) \wedge \wedge$  // T2 =  $d = e$
  - $(a = b \vee c = b) \wedge \wedge$  // T3 =  $a = b$
  - $(a = c \vee X) \wedge \wedge$  // T4 =  $a = c$
  - $(c \neq b \vee !X)$  // T5 =  $c = b$
- Converted:
  - $(!T1 \vee !T2) (T3 \vee !T4) (T3 \vee T5) (T4 \vee X) (!T5 \vee !X)$
- No Unit Clauses
- Heuristically choose T1 = False, add to model

# DPLL(T) Example

- Input Clauses:
  - $(f(a) \neq f(b) \vee d \neq e) \wedge \wedge$  // T1 =  $f(a) = f(b)$
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- Converted:
  - $(!T1 \vee !T2) (T3 \vee !T4) (T3 \vee T5) (T4 \vee X) (!T5 \vee !X)$
- No Unit Clauses
- Heuristically choose T1 = False, add to model
  - Theory returns: !T3

# DPLL(T) Example

- Input Clauses:
  - $(f(a) \neq f(b) \mid\mid d \neq e) \ \&\&$        $// \ T1 = f(a) = f(b)$
  - $(a = b \mid\mid a \neq c) \ \&\&$        $// \ T2 = d = e$
  - $(a = b \mid\mid c = b) \ \&\&$        $// \ T3 = a = b$
  - $(a = c \mid\mid X) \ \&\&$        $// \ T4 = a = c$
  - $(c \neq b \mid\mid !X)$        $// \ T5 = c = b$
- Converted:
  - $(!T4) \ (T5) \ (T4 \mid\mid X) \ (!T5 \mid\mid !X)$
- Model:  $!T1, !T3$

# DPLL(T) Example

- Input Clauses:
  - $(f(a) \neq f(b) \vee d \neq e) \wedge \wedge$  // T1 =  $f(a) = f(b)$
  - $(a = b \vee a \neq c) \wedge \wedge$  // T2 =  $d = e$
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  - $(c \neq b \vee !X)$  // T5 =  $c = b$
- Converted:
  - $(!T4) (T5) (T4 \vee X) (!T5 \vee !X)$
- Model:  $!T1, !T3$
- Unit Clauses:  $!T4, !T5$ , add to model

# DPLL(T) Example

- Input Clauses:
  - $(f(a) \neq f(b) \mid\mid d \neq e) \ \&\&$        $// \ T1 = f(a) = f(b)$
  - $(a = b \mid\mid a \neq c) \ \&\&$        $// \ T2 = d = e$
  - $(a = b \mid\mid c = b) \ \&\&$        $// \ T3 = a = b$
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  - $(c \neq b \mid\mid !X)$        $// \ T5 = c = b$
- Converted:
  - $(X) \ (!X)$
- Model:  $!T1, !T3, !T4, !T5$
- Unit Clause:  $(X)$ , add to model



# DPLL(T) Example

- Input Clauses:
  - $(f(a) \neq f(b) \vee d \neq e) \wedge \wedge$  // T1 =  $f(a) = f(b)$
  - $(a = b \vee a \neq c) \wedge \wedge$  // T2 =  $d = e$
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  - $(c \neq b \vee !X)$  // T5 =  $c = b$
- Converted:
  - $(X) (!X)$
- Model:  $!T1, !T3, !T4, !T5$
- Unit Clause:  $(X)$ , add to model, contradiction!
  - Original input is not satisfiable!

# DPLL(T) Conclusion

- DPLL(T) is widely used as a basis for modern SMT solving.
  - It is typically much **faster** than eagerly encoding all of the variables into bits (e.g., 32-bit integer: 32 boolean variables). It is **general** in that it allows many types of theories.
- For example, Microsoft's popular and powerful **Z3** automated theorem prover handles many theories, but uses DPLL(T) + Simplex for linear inequalities.
  - “A Fast Linear-Arithmetic Solver for DPLL(T)”, 2006

# Time Permitting: Proofs

“Checking proofs ain’t like dustin’ crops, boy!”



# Proof Generation

- We want our theorem prover to **emit proofs**
  - **No need to trust the prover**
  - Can find bugs in the prover
  - Can be used for proof-carrying code
  - Can be used to extract invariants
  - Can be used to extract models (e.g., in SLAM)
- Implements the soundness argument
  - On every run, **a soundness proof is constructed**

# Proof Representation

- Proofs are trees

- Leaves are hypotheses/axioms
- Internal nodes are inference rules

- Axiom: “true introduction”

- Constant:  $\text{truei} : \text{pf}$
- $\text{pf}$  is the type of proofs

$$\frac{}{|- \text{true}} \text{truei}$$

- Inference: “conjunction introduction”

- Constant:  $\text{andi} : \text{pf} \rightarrow \text{pf} \rightarrow \text{pf}$

$$\frac{|- A \quad |- B}{|- A \ \&\& \ B} \text{andi}$$

- Inference: “conjunction elimination”

- Constant:  $\text{andel} : \text{pf} \rightarrow \text{pf}$

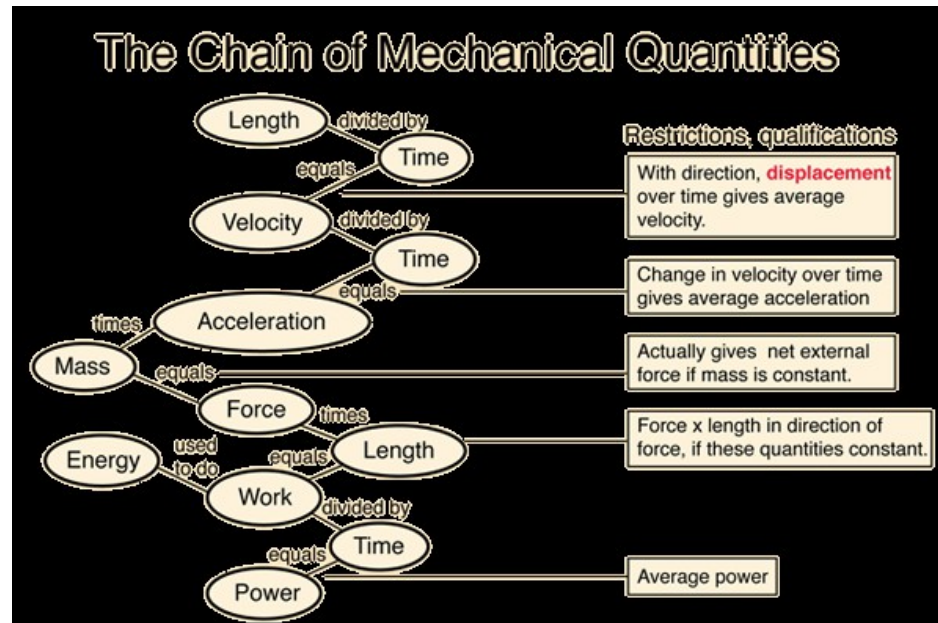
$$\frac{|- A \ \&\& \ B}{|- A} \text{andel}$$

- Problem:

- “ $\text{andel truei} : \text{pf}$ ” but does not represent a valid proof
- Need a powerful *system that checks content*

# Analogy: Integer Subtypes (cf. Hoare paper)

- We don't just need any number, we need a “number *representing distance over time*”
- WE don't just need any proof, we need a “proof *representing A || B*”



# Dependent Types, Preview

- Make **pf** a family of types indexed by formulas
  - $f : \text{Type}$  (type of encodings of formulas)
  - $e : \text{Type}$  (type of encodings of expressions)
  - $\text{pf} : f \rightarrow \text{Type}$  (the type of proofs indexed by formulas: it is a proof *that  $f$  is true*)
- Examples:
  - $\text{true} : f$
  - $\text{and} : f \rightarrow f \rightarrow f$
  - $\text{truei} : \text{pf } \text{true}$
  - $\text{andi} : \text{pf } A \rightarrow \text{pf } B \rightarrow \text{pf } (\text{and } A \ B)$
  - $\text{andi} : \Pi A:f. \Pi B:f. \text{pf } A \rightarrow \text{pf } B \rightarrow \text{pf } (\text{and } A \ B)$
  - ( $\Pi A:f.X$  means “forall  $A$  of type  $f$ , dependent type  $X$ ”, see future lecture)

# Proof Checking

- Validate proof trees by **recursively checking** them
- Given a proof tree  $X$  claiming to prove  $A \ \&\& \ B$
- Must check  $X : \text{pf} \ (\text{and } A \ B)$
- We use “**expression tree equality**”, so
  - andel (andi “ $1+2=3$ ” “ $x=y$ ”) does ***not*** have type  $\text{pf} \ (3=3)$
  - This is already a proof system! If the proof-supplier wants to use the fact that  $1+2=3 \Leftrightarrow 3=3$ , she can **include a proof of it** somewhere!
- Thus **Type Checking = Proof Checking**
  - And it’s quite easily ***decidable***!



# Proof Inference Rules

- What are some rules of inference and function types for:
  - Or introduction
    - Hint: `or_introduction_left : pf A → pf (or A B)`
  - Or elimination
  - Not introduction
  - Not elimination
  - Implies introduction
  - Implies elimination
  - False elimination

# Bonus Question

- If we want to use Simplex to handle our Theory of Linear Inequalities, how do we handle ...
  - Equality  $(x = 10)$
  - Negative Literals  $!(x \leq 10)$
  - Disequality  $(x \neq 10)$
- A **slack variable** converts an inequality to an equality.
  - Given:  $Ax \leq b$ , create fresh  $y \geq 0$
  - Obtain:  $Ax + y = b$

# Homework

- HW2 due Friday
- (Future) HW3 Coding Hint:
  - You are only being asked to code up the “check this candidate model against the Theories and add new clauses if it doesn't work” part of SMT.