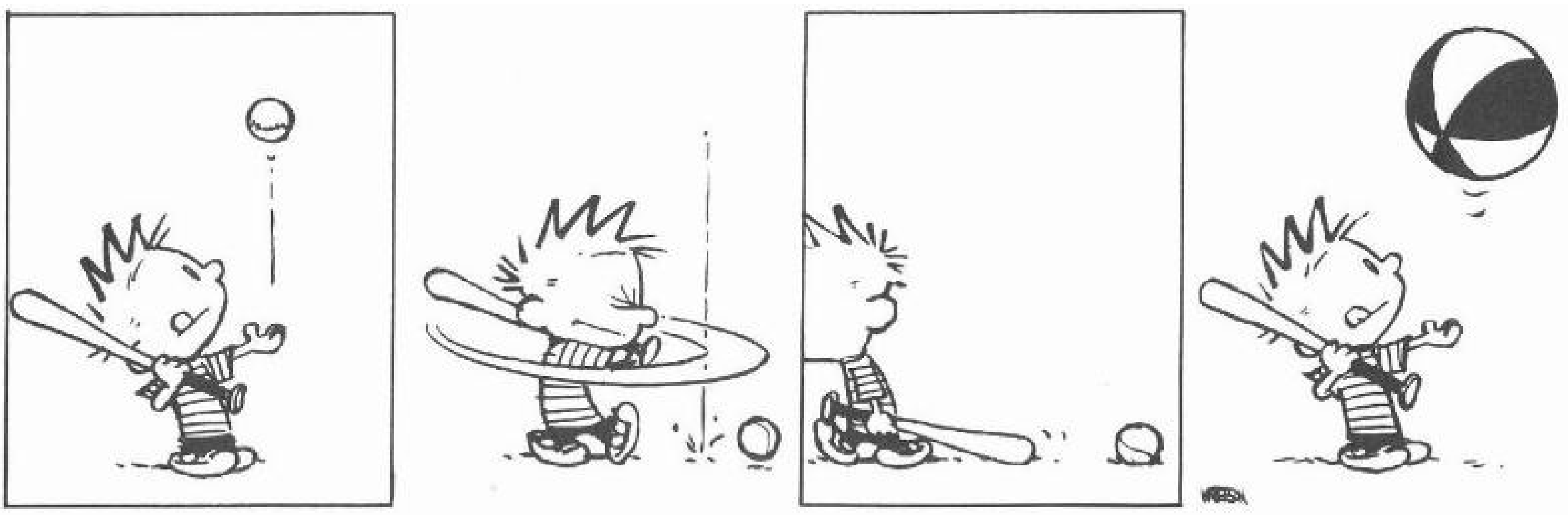
# Recursive Types and Subtyping



# One-Slide Summary

- **Recursive types** (e.g., $\tau$ list) make the typed lambda calculus as powerful as the untyped lambda calculus.
- If $\tau$ is a **subtype** of $\sigma$ then any expression of type $\tau$ can be used in a context that expects a $\sigma$; this is called **subsumption**.
- A **conversion** is a function that converts between types.
- A subtyping system should be **coherent**.

## Recursive Types: Lists

- We want to define recursive data structures
- Example: lists
  - A list of elements of type $\tau$ (a $\tau$ list) is *either* empty *or* it is a pair of a $\tau$ and a $\tau$ list

$$\boldsymbol{\tau \text{ list} = \text{unit} + (\tau \times \tau \text{ list})}$$

  - This is a recursive equation. We take its solution to be the smallest set of values L that satisfies the equation

$$L = \{ * \} \cup (T \times L)$$

    where T is the set of values of type $\tau$

  - Another interpretation is that the recursive equation is taken up-to (modulo) set isomorphism

# Recursive Types

- We introduce a recursive type constructor $\mu$ (mu):

$$\mu t.\ \tau$$

  - The type variable t is bound in $\tau$
  - This stands for the solution to the equation

    $t \simeq \tau$ (t is isomorphic with $\tau$)
  - Example: **$\tau$ list = $\mu t.$ (unit + $\tau \times$ t)**
  - This also allows "unnamed" recursive types
- We introduce syntactic (sugary) operations for the conversion between $\mu t.\tau$ and $[\mu t.\tau/t]\tau$
- e.g. between "$\tau$ list" and "unit + ($\tau \times \tau$ list)"

$$e ::= \ldots \qquad | \ \text{fold}_{\mu t.\tau}\ e \ | \ \text{unfold}_{\mu t.\tau}\ e$$

$$\tau ::= \ldots \qquad | \ t \ | \ \mu t.\tau$$

# Example with Recursive Types

- Lists

  $\tau$ list = $\mu$t. (unit + $\tau \times$ t)

  nil$_\tau$ = fold$_{\tau \text{ list}}$ (injl *)

  cons$_\tau$ = $\lambda$x:$\tau$.$\lambda$L:$\tau$ list. fold$_{\tau \text{ list}}$ injr (x, L)

- A list length function

  length$_\tau$ = $\lambda$L:$\tau$ list.

  case (unfold$_{\tau \text{ list}}$ L) of injl x $\Rightarrow$ 0

  | injr y $\Rightarrow$ 1 + length$_\tau$ (snd y)

- (At home …) Verify that
  - nil$_\tau$ : $\tau$ list
  - cons$_\tau$ : $\tau \to \tau$ list $\to \tau$ list
  - length$_\tau$ : $\tau$ list $\to$ int

# Type Rules for Recursive Types

$$\frac{\Gamma \vdash e : \mu t.\tau}{\Gamma \vdash \texttt{unfold}_{\mu t.\tau}\ e : [\mu t.\tau / t]\tau}$$

$$\frac{\Gamma \vdash e : [\mu t.\tau / t]\tau}{\Gamma \vdash \texttt{fold}_{\mu t.\tau}\ e : \mu t.\tau}$$

- The typing rules are syntax directed
- Often, for syntactic simplicity, the fold and unfold operators are omitted
  - This makes type checking somewhat harder

# Dynamics of Recursive Types

- We add a new form of values

  $v ::= \ldots \mid \mathbf{fold}_{\mu t.\tau}\ v$

  – The purpose of fold is to ensure that the value has the recursive type and not its unfolding

- The evaluation rules:

$$\frac{e \Downarrow v}{\mathtt{fold}_{\mu t.\tau}\ e \Downarrow \mathtt{fold}_{\mu t.\tau}\ v} \qquad \frac{e \Downarrow \mathtt{fold}_{\mu t.\tau}\ v}{\mathtt{unfold}_{\mu t.\tau}\ e \Downarrow v}$$

  - The folding annotations are for type checking only
  - They can be dropped after type checking

# Recursive Types in ML

- The language ML uses a simple syntactic trick to avoid having to write the explicit fold and unfold
- In ML recursive types are *bundled with union types*

  **type t = $C_1$ of $\tau_1$ | $C_2$ of $\tau_2$ | … | $C_n$ of $\tau_n$**
  **(* t can appear in $\tau_i$ *)**

  - e.g., "type intlist = Nil of unit | Cons of int * intlist"
- When the programmer writes Cons (5, l)
  - the compiler treats it as $\text{fold}_{\text{intlist}}$ (injr (5, l))
- When the programmer writes
  - case e of Nil ⇒ … | Cons (h, t) ⇒ …

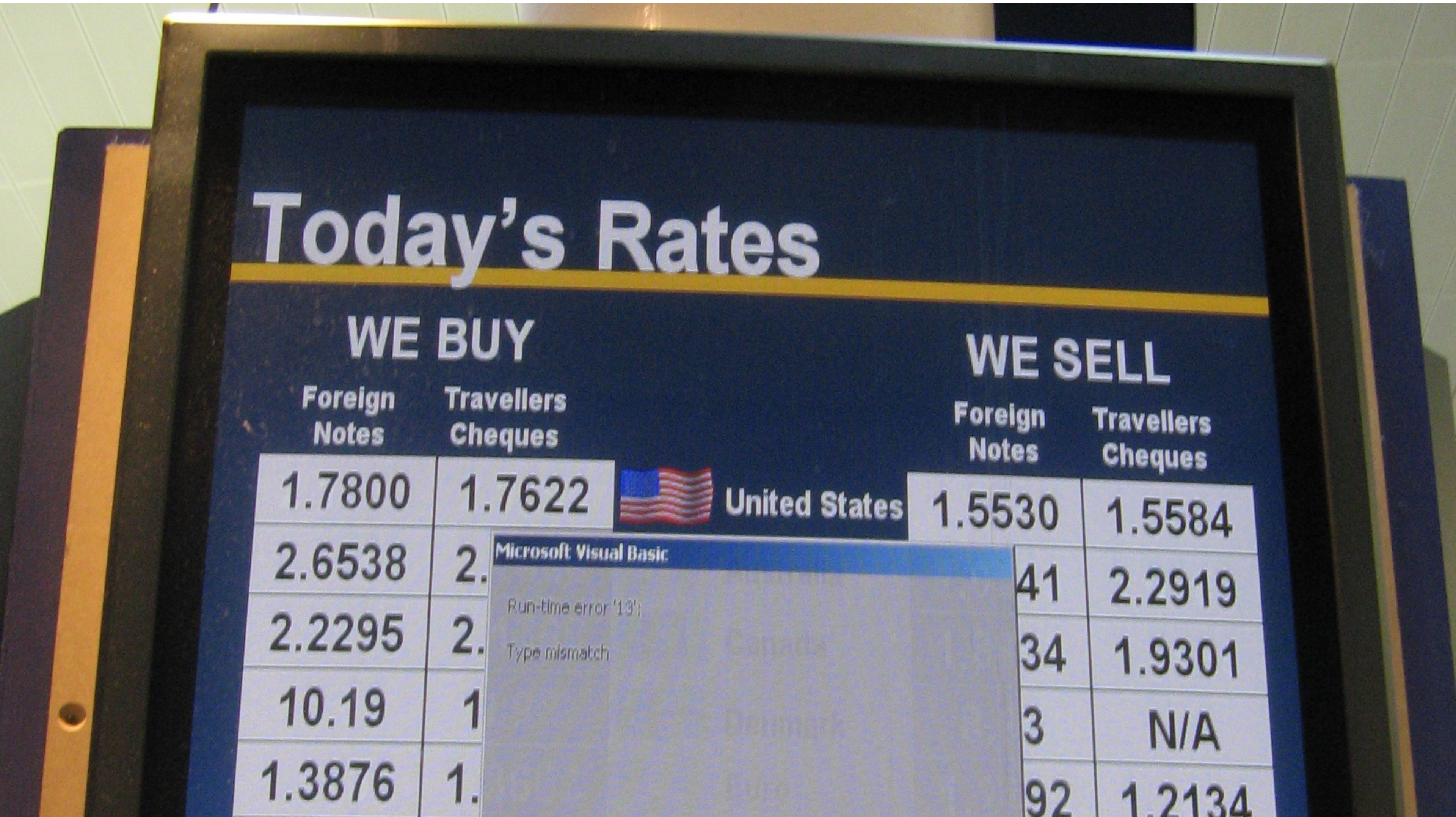
  the compiler treats it as
  - case $\text{unfold}_{\text{intlist}}$ e of Nil ⇒ … | Cons (h,t) ⇒ …

## Encoding Call-by-Value $\lambda$-calculus in $F_1^\mu$

- So far, $F_1$ was so weak that we could not encode non-terminating computations
  - Cannot encode recursion
  - Cannot write the $\lambda x.x\ x$ (self-application)
- The addition of recursive types makes typed $\lambda$-calculus *as expressive as untyped $\lambda$-calculus*!
- We could show a conversion algorithm from call-by-value untyped $\lambda$-calculus to call-by-value $F_1^\mu$

# Smooth Transition

- And now, on to subtyping …



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| 1.7800 | 1.7622 | United States | 1.5530 | 1.5584 |
| 2.6538 | 2. | | 41 | 2.2919 |
| 2.2295 | 2. | | 34 | 1.9301 |
| 10.19 | 1 | | 3 | N/A |
| 1.3876 | 1. | | 92 | 1.2134 |

```
Microsoft Visual Basic

Run-time error '13':

Type mismatch
```

# Introduction to Subtyping

- We can view types as denoting *sets of values*
- Subtyping is a relation between types induced by the *subset relation between value sets*
- Informal intuition:
  - If $\tau$ is a subtype of $\sigma$ then any expression with type $\tau$ also has type $\sigma$ (e.g., $\mathbb{Z} \subseteq \mathbb{R}$, $1 \in \mathbb{Z} \Rightarrow 1 \in \mathbb{R}$)
  - If $\tau$ is a subtype of $\sigma$ then any expression of type $\tau$ can be used in a context that expects a $\sigma$
  - We write $\tau < \sigma$ to say that $\tau$ is a subtype of $\sigma$
  - Subtyping is reflexive and transitive

# Cunning Plan For Subtyping

- Formalize Subtyping Requirements
  - Subsumption
- Create Safe Subtyping Rules
  - Pairs, functions, references, etc.
  - Most easy thing we try will be wrong
- Subtyping Coercions
  - When is a subtyping system correct?

# Subtyping Examples

- FORTRAN introduced int < real
  - 5 + 1.5 is well-typed in many languages
- PASCAL had [1..10] < [0..15] < int
- Subtyping is a fundamental property of object-oriented languages
  - If S is a subclass of C then an instance of S can be used where an instance of C is expected
  - "subclassing ⇒ subtyping" philosophy

# Subsumption

- Formalize the requirements on subtyping
- Rule of subsumption
  - If $\tau < \sigma$ then an expression of type $\tau$ has type $\sigma$

$$\frac{\Gamma \vdash e : \tau \qquad \tau < \sigma}{\Gamma \vdash e : \sigma}$$

- But now type safety may be in danger:
  - If we say that int < (int → int)
  - Then we can prove that "11  8" is well typed!
- There is a way to construct the subtyping relation to preserve type safety

## Subtyping in POPL 20

- *Decidable Subtyping for Path Dependent Types*
- *Graduality and Parametricity: Together Again for the First Time*
  - By UM's Max New!
- *Partial Type Constructors: Or,Making Ad Hoc Datatypes Less Ad Hoc*
- *What Is Decidable about Gradual Types?*
- *... (out of space)*

## Subtyping in POPL/PLDI 14

- *Backpack: Retrofitting Haskell with Interfaces*
- *Getting F-Bounded Polymorphism into Shape*
- *Optimal Inference of Fields in Row-Polymorphic Records*
- *Polymorphic Functions with Set-Theoretic Types (Part 1: Syntax, Semantics, and Evaluation)*
- *... (out of space)*

# POPL 2025

- Many programming languages need to check whether two recursive types are in a subtyping relation. Traditionally recursive types are modelled in two different ways: equi- or iso- recursive types. While efficient algorithms for subtyping …
  - **QuickSub: Efficient Iso-Recursive Subtyping**
- We first propose a subtyping system based on type graphs, offering more efficient (quadratic) subtype-checking than the existing (exponential) inductive algorithm …
  - **Top-Down or Bottom-Up? Complexity Analyses of Synchronous Multiparty Session Types**
- However, existing type inference implementations lack solid theoretical foundations when dealing with non-structural subtyping and intersection and union types, which were not studied before.
  - **Bidirectional Higher-Rank Polymorphism with Intersection and Union Types**

# Defining Subtyping

- The formal definition of subtyping is by derivation rules for the judgment $\tau < \sigma$
- We start with subtyping on the base types
  - e.g. int < real or nat < int
  - These rules are language dependent and are typically based directly on types-as-sets arguments
- We then make subtyping a preorder (reflexive and transitive)

$$\frac{}{\tau < \tau} \qquad \frac{\tau_1 < \tau_2 \quad \tau_2 < \tau_3}{\tau_1 < \tau_3}$$

- Then we build-up subtyping for “larger” types

# Subtyping for Pairs

- Try

$$\frac{\tau < \sigma \quad \tau' < \sigma'}{\tau \times \tau' < \sigma \times \sigma'}$$

  - Show (informally) that whenever a s × s' can be used, a t × t' can also be used:
  - Consider the context H = H'[fst •] expecting a s × s'
    - Then H' expects a s
    - Because t < s then H' accepts a t
    - Take e : t × t'. Then fst e : t so it works in H'
    - Thus e works in H
  - The case of "snd •" is similar

# Subtyping for Records

- Several subtyping relations for records
- Depth subtyping

$$\frac{\tau_i < \tau_i'}{\{\, l_1 : \tau_1, \ldots, l_n : \tau_n \,\} < \left\{\, l_1 : \tau_1', \ldots, l_n : \tau_n' \,\right\}}$$

  - e.g., {f1 = int, f2 = int} < {f1 = real, f2 = int}
- Width subtyping

$$\frac{n \geq m}{\{\, l_1 : \tau_1, \ldots, l_n : \tau_n \,\} < \{\, l_1 : \tau_1, \ldots, l_m : \tau_m \,\}}$$

  - E.g., {f1 = int, f2 = int} < {f2 = int}
  - Models subclassing in OO languages
- Or, a combination of the two

## Subtyping for Functions

$$\frac{\tau < \sigma \qquad \tau' < \sigma'}{\tau \to \tau' < \sigma \to \sigma'}$$

Example Use:

rounded_sqrt : $\mathbb{R} \to \mathbb{Z}$

actual_sqrt : $\mathbb{R} \to \mathbb{R}$

Since $\mathbb{Z}$ < $\mathbb{R}$, rounded_sqrt < actual_sqrt

So if I have code like this:

```
float result = rounded_sqrt(5); // 2
```

… I can replace it like this:

```
float result = actual_sqrt(5); // 2.23
```

… and everything will be fine.

# Chinese Literature ( 紅樓夢 )

- This semi-autobiographical novel is one of China's Four Great Classic Novels. It mirrors the rise and fall of the author's family and is presented as a memorial to the women he knew in his youth. It describes 18th-century Chinese society using many characters, including the compassionate Jia Baoyu ( 賈寶玉 ) and the sickly and spiritual Lin Daiyu ( 林黛玉 ). It also features a sentient stone and romantic rivalry.

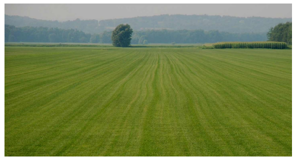
# Q: General (455 / 842)

- This numerical technique for finding solutions to boundary-value problems was initially developed for use in structural analysis in the 1940's. The subject is represented by a model consisting of a number of linked simplified representations of discrete regions. It is often used to determine stress and displacement in mechanical systems.

## Languages

- This Dravidian language dates back to at least 300 BCE and combines vowels and consonants to form over 200 compound characters. It uses multiple suffices to mark noun numbers and verb tenses (called *agglutination*). The poet C. Subramania Bharati wrote in this language against child marriage and the caste system and in favor of women's rights.
  - Example: மனிதப்பிறவியினர்

# Computer Science



- This American Turing-award winner is known for his visionary and pioneering contributions to Computer Graphics, and for Sketchpad, an early predecessor to the GUI. He created the first virtual reality display, and a graphics line clipping algorithm. His students include Alan Kay (Smalltalk), Henri Gouraud (shading), Frank Crow (anti-aliasing), and Edwin Catmull (Pixar). When asked, "How could you possibly have done the first interactive graphics program, the first non-procedural programming language, the first object oriented software system, all in one year?" He replied: "Well, I didn't know it was hard."

# Subtyping for Functions

$$\frac{\tau < \sigma \qquad \tau' < \sigma'}{\tau \to \tau' < \sigma \to \sigma'}$$

- What do you think of this rule?

SPECIAL EDITION
THE GOOD THE BAD AND THE UGLY
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# Subtyping for Functions

$$\frac{\tau < \sigma \qquad \tau' < \sigma'}{\tau \to \tau' < \sigma \to \sigma'}$$

- This rule is **unsound**
  - Let $\Gamma$ = f : int $\to$ bool (and assume int < real)
  - We show using the above rule that $\Gamma \vdash$ f 5.0 : bool
  - But this is wrong since 5.0 is *not a valid argument* of f

$$\dfrac{\dfrac{\Gamma \vdash f : \texttt{int} \to \texttt{bool} \qquad \dfrac{\texttt{int} < \texttt{real} \qquad \texttt{bool} < \texttt{bool}}{\texttt{int} \to \texttt{bool} < \texttt{real} \to \texttt{bool}}}{\Gamma \vdash f : \texttt{real} \to \texttt{bool}} \qquad \Gamma \vdash 5.0 : \texttt{real}}{\Gamma \vdash f\ 5.0 : \texttt{bool}}$$

# Correct Function Subtyping

$$\frac{\sigma < \tau \qquad \tau' < \sigma'}{\tau \to \tau' < \sigma \to \sigma'}$$

- We say that $\to$ is **covariant** in the result type and **contravariant** in the argument type
- Informal correctness argument:
  - Pick **f : $\tau \to \tau'$**
  - f expects an argument of type $\tau$
  - It also accepts an argument of type $\sigma < \tau$
  - f returns a value of type $\tau'$
  - Which can also be viewed as a $\sigma'$ (since $\tau' < \sigma'$)
  - Hence f can be used as $\sigma \to \sigma'$

# More on Contravariance

- Consider the subtype relationships:

$$\begin{tikzcd}
 & \text{int} \to \text{real} \arrow[dl] \arrow[dr] & \\
\text{real} \to \text{real} \arrow[dr] & & \text{int} \to \text{int} \arrow[dl] \\
 & \text{real} \to \text{int} &
\end{tikzcd}$$

- In what sense ($f \in$ real $\to$ int) $\Rightarrow$ ($f \in$ int $\to$ int) ?
  - "real $\to$ int" has a *larger domain*!
  - (recall the set theory (arg,result) pair encoding for functions)
- This suggests that "subtype-as-subset" interpretation is not straightforward
  - We'll return to this issue (after these commercial messages …)

# Subtyping References

- Try covariance

$$\frac{\tau < \sigma}{\tau\ \texttt{ref} < \sigma\ \texttt{ref}}$$

Wrong!

  - Example: assume $\tau < \sigma$
  - The following holds (if we assume the above rule):

    $x : \sigma, y : \tau\ \mathbf{ref}, f : \tau \to \mathbf{int} \vdash y := x; f\ (!\ y)$

  - Unsound: f is called on a $\sigma$ but is defined only on $\tau$
  - Java has covariant arrays!
- If we want covariance of references we can recover type safety with a runtime check for each y := x
  - The actual type of x matches the actual type of y
  - But this is generally considered a *bad design*

# Subtyping References (Part 2)

- Contravariance?

$$\frac{\tau < \sigma}{\sigma\ \mathtt{ref} < \tau\ \mathtt{ref}}$$

Also Wrong!

  - Example: assume $\tau < \sigma$
  - The following holds (if we assume the above rule):

$$x : \sigma,\ y : \sigma\ \text{ref},\ f : \tau \to \text{int} \vdash y := x;\ f\ (!\ y)$$

  - Unsound: f is called on a $\sigma$ but is defined only on $\tau$
- References are invariant
  - *No subtyping for references* (unless we are prepared to add run-time checks)
  - hence, *arrays* should be invariant
  - hence, *mutable records* should be invariant

# Subtyping Recursive Types

- Recall **$\tau$ list = $\mu$t.(unit + $\tau\times$t)**
  - We would like $\tau$ list < $\sigma$ list whenever $\tau < \sigma$
- Covariance?

$$\frac{\tau < \sigma}{\mu t.\tau < \mu t.\sigma}$$

Wrong!

- This is *wrong if t occurs contravariantly in $\tau$*
- Take $\tau = \mu t.t \to \text{int}$ and $\sigma = \mu t.t \to \text{real}$
- Above rule says that $\tau < \sigma$
- We have $\tau \simeq \tau \to \text{int}$ and $\sigma \simeq \sigma \to \text{real}$
- $\tau < \sigma$ would mean covariant function type!
- How can we get safe subtyping for lists?

# Subtyping Recursive Types

- The correct rule

$$\frac{\left.\begin{array}{c} t < s \\ \vdots \\ \tau < \sigma \end{array}\right\}}{\mu t.\tau < \mu s.\sigma}$$

Means assume t < s and use that to prove $\tau < \sigma$

- We add as an *assumption* that the type variables stand for types with the desired subtype relationship
  - Before we assumed they stood for the *same* type!
- Verify that now subtyping works properly for lists
- There is no subtyping between $\mu$t.t$\to$int and $\mu$t.t$\to$ real (recall: $\dfrac{\tau < \sigma}{\mu t.\tau < \mu t.\sigma}$ Wrong!

## Conversion Interpretation

- The subset interpretation of types leads to an abstract modeling of the operational behavior
  - e.g., we say int < real even though an int could not be directly used as a real in the concrete x86 implementation (cf. IEEE 754 bit patterns)
  - The int needs to be converted to a real
- We can get closer to the "machine" with a conversion interpretation of subtyping
  - We say that $\tau < \sigma$ when there is a conversion function that converts values of type $\tau$ to values of type $\sigma$
  - Conversions also help explain issues such as contravariance
  - But: must be careful with conversions

# Conversions

- Examples:
  - nat < int with conversion $\lambda x.x$
  - int < real with conversion 2's comp $\rightarrow$ IEEE
- The subset interpretation is a *special case* when all conversions are *identity functions*
- Write "$\tau < \sigma \Rightarrow C(\tau, \sigma)$" to say that $C(\tau,\sigma)$ is the conversion function from subtype $\tau$ to $\sigma$
  - If $C(\tau, \sigma)$ is expressed in $F_1$ then $C(\tau,\sigma) : \tau \rightarrow \sigma$

# Issues with Conversions

- Consider the expression "printreal 1" typed as follows:

$$\dfrac{\texttt{printreal} : \texttt{real} \rightarrow \texttt{unit} \qquad \dfrac{1 : \texttt{int} \quad \texttt{int} < \texttt{real}}{1 : \texttt{real}}}{\texttt{printreal}\ 1 : \texttt{unit}}$$

we convert 1 to real: printreal (C(int,real) 1)

- But we can also have another type derivation:

$$\dfrac{\dfrac{\texttt{printreal} : \texttt{real} \rightarrow \texttt{unit} \quad \texttt{real} \rightarrow \texttt{unit} < \texttt{int} \rightarrow \texttt{unit}}{\texttt{printreal} : \texttt{int} \rightarrow \texttt{unit}} \qquad 1 : \texttt{int}}{\texttt{printreal}\ 1 : \texttt{unit}}$$

with conversion "(C(real -> unit, int -> unit) printreal) 1"

- Which one is right? What do they mean?

# Introducing Conversions

- We can compile a language with subtyping into one without subtyping by introducing conversions
- The process is similar to type checking

$$\Gamma \vdash e : \tau \Rightarrow \underline{e}$$

  - Expression e has type $\tau$ and its conversion is $\underline{e}$
- Rules for the conversion process:

$$\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \Rightarrow \underline{e_1} \quad \Gamma \vdash e_2 : \tau_2 \Rightarrow \underline{e_2}}{\Gamma \vdash e_1\ e_2 : \tau \Rightarrow \underline{e_1}\ \underline{e_2}}$$

$$\frac{\Gamma \vdash e : \tau \Rightarrow \underline{e} \quad \tau < \sigma \Rightarrow C(\tau, \sigma)}{\Gamma \vdash e : \sigma \Rightarrow C(\tau, \sigma)\underline{e}}$$

# Coherence of Conversions

- Questions and Concerns:
  - Can we build *arbitrary subtype relations* just because we can write conversion functions?
  - Is real < int just because the "floor" function is a conversion?
  - *What is the conversion* from "real→int" to "int→int"?
- What are the restrictions on conversion functions?
- A system of conversion functions is coherent if whenever we have $\tau < \tau' < \sigma$ then
    - $C(\tau, \tau) = \lambda x.x$
    - $C(\tau,\sigma) = C(\tau', \sigma) \circ C(\tau, \tau')$ *(= composed with)*
    - Example: if b is a bool then (float)b == (float)((int)b)
  - otherwise we end up with confusing uses of subsumption

# Example of Coherence

- We want the following subtyping relations:
  - int < real ⇒ $\lambda$x:int. toIEEE x
  - real < int ⇒ $\lambda$x:real. floor x
- For this system to be coherent we need
  - C(int, real) ∘ C(real, int) = $\lambda$x.x, and
  - C(real, int) ∘ C(int, real) = $\lambda$x.x
- This requires that
  - $\forall$x : real . ( toIEEE (floor x) = x )
  - which is *not true*

# Building Conversions

- We start from conversions on basic types

$$\frac{}{\tau < \tau \Rightarrow \lambda x : \tau . x}$$

$$\frac{\tau_1 < \tau_2 \Rightarrow C(\tau_1, \tau_2) \quad \tau_2 < \tau_3 \Rightarrow C(\tau_2, \tau_3)}{\tau_1 < \tau_3 \Rightarrow C(\tau_2, \tau_3) \circ C(\tau_1, \tau_2)}$$

$$\frac{\tau_1 < \sigma_1 \Rightarrow C(\tau_1, \sigma_1) \quad \tau_2 < \sigma_2 \Rightarrow C(\tau_2, \sigma_2)}{\tau_1 \times \tau_2 < \sigma_1 \times \sigma_2 \Rightarrow \lambda x : \tau_1 \times \tau_2 . (C(\tau_1, \sigma_1)(\texttt{fst}(x)), C(\tau_2, \sigma_2)(\texttt{snd}(x)))}$$

$$\frac{}{\tau_1 \times \tau_2 < \tau_1 \Rightarrow \lambda x : \tau_1 \times \tau_2 .\ \texttt{fst}(x)}$$

$$\frac{\sigma_1 < \tau_1 \Rightarrow C(\sigma_1, \tau_1) \quad \tau_2 < \sigma_2 \Rightarrow C(\tau_2, \sigma_2)}{\tau_1 \to \tau_2 < \sigma_1 \to \sigma_2 \Rightarrow \lambda f : \tau_1 \to \tau_2 .\ \lambda x : \sigma_1 .\ C(\tau_2, \sigma_2)(f(C(\sigma_1, \tau_1)(x)))}$$

# Comments

- With the conversion view we see why we do not necessarily want to impose antisymmetry for subtyping
  - Can have multiple representations of a type
  - We want to reserve type equality for representation equality
  - $\tau < \tau'$ and also $\tau' < \tau$ (are interconvertible) but not necessarily $\tau = \tau'$
  - e.g., Modula-3 has packed and unpacked records
- We'll encounter subtyping again for object-oriented languages
  - Serious difficulties there due to recursive types

## Homework

- Homework 5, Homework 6, etc.