Abstract Interpretation (Non-Standard Semantics)

a.k.a. "Picking The Right Abstraction"

Why analyze programs statically?

The Problem

- It is extremely useful to predict program behavior *statically* (= without running the program)
  - For optimizing compilers, program analyses, software engineering tools, finding security flaws, etc.
- The semantics we studied so far give us the precise behavior of a program
- However, precise static predictions are impossible
  - The exact semantics is not computable
- We must settle for approximate, but correct, static analyses (e.g. VC vs. WP)

One-Slide Summary

- **Abstraction interpretation** is a static analysis for soundly approximating the semantics of a program.
- While the concrete semantics refers to what actually happens when you run the program (e.g., “x*x+1” may result in multiple integers), the abstract semantics tracks only certain information about that computation (e.g., “x*x+1” will be *some positive* number, but we don't know which one).
- Special functions transfer between the abstract domain (typically a lattice) and the concrete domain.

The Plan

- We will introduce abstract interpretation by example
- Starting with a miniscule language we will build up to a fairly realistic application
- Along the way we will see most of the ideas and difficulties that arise in a big class of applications

A Tiny Language

- Consider the following language of arithmetic ("shrIMP"?)

$$e ::= n \mid e_1 * e_2$$

- The operational semantics of this language

$$n \Downarrow n$$

$$e_1 * e_2 \Downarrow = e_1\Downarrow \times e_2\Downarrow$$

- We'll take opsem as the "ground truth"
- For this language the precise semantics is computable (but in general it's not)

An Abstraction

- Assume that we are interested not in the value of the expression, but only in its sign:
  - positive (+), negative (-), or zero (0)
- We can define an abstract semantics that computes *only* the sign of the result

$$\sigma: \text{Exp} \to \{-, 0, +\}$$

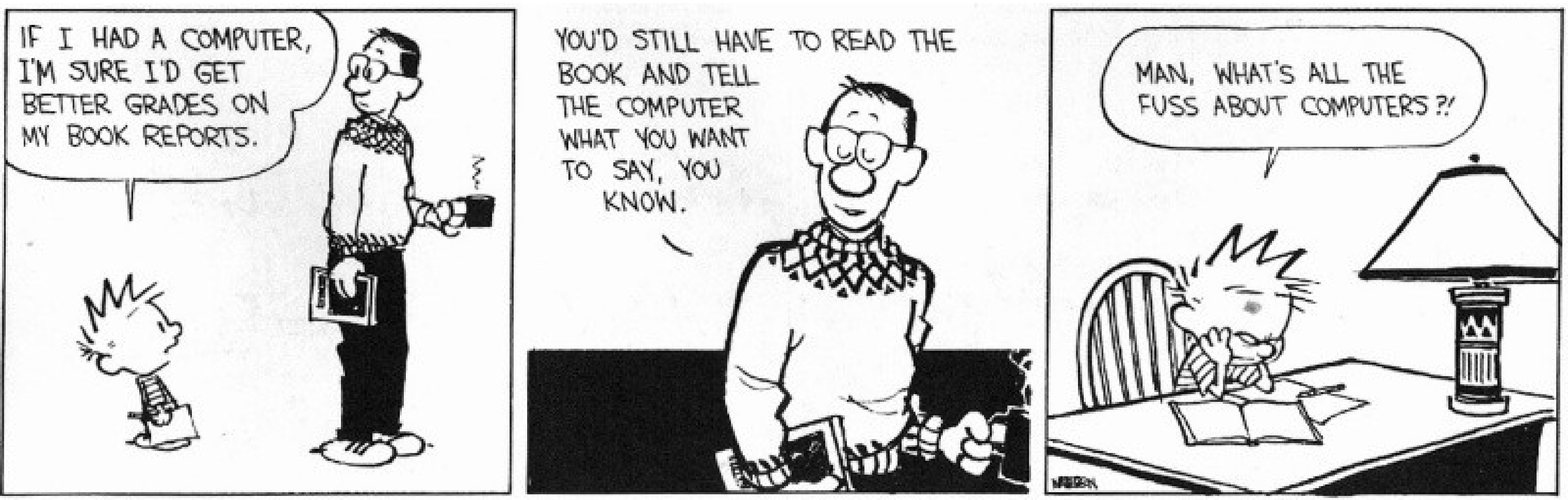
$$\sigma(n) = \text{sign}(n)$$

$$\sigma(e_1 * e_2) = \sigma(e_1) \otimes \sigma(e_2)$$

$\otimes$	-	0	+
-	+	0	-
0	0	0	0
+	-	0	+

I Saw the Sign

- Why did we want to compute the sign of an expression?
  - One reason: no one will believe you know abstract interpretation if you haven't seen the sign example :-)
- What could we be computing instead?



Correctness of Sign Abstraction

- We can show that the abstraction is correct in the sense that it predicts the sign

  $e\Downarrow > 0 \Leftrightarrow \sigma(e) = +$

  $e\Downarrow = 0 \Leftrightarrow \sigma(e) = 0$

  $e\Downarrow < 0 \Leftrightarrow \sigma(e) = -$

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  $e\Downarrow < 0 \Leftrightarrow \sigma(e) = -$

- Our semantics is abstract but precise
- Proof is by structural induction on the expression e
  - Each case repeats similar reasoning

Another View of Soundness

- Link each concrete value to an abstract one:

$$\beta : \mathbb{Z} \rightarrow \{ \text{-}, 0, + \}$$

- This is called the abstraction function (β)
  - This three-element set is the abstract domain
- Also define the concretization function (γ):

$$\gamma : \{\text{-}, 0, +\} \rightarrow \mathcal{P}(\mathbb{Z})$$

$$\gamma(+) \quad = \quad \{ n \in \mathbb{Z} \mid n > 0 \}$$

$$\gamma(0) \quad = \quad \{ 0 \}$$

$$\gamma(\text{-}) \quad = \quad \{ n \in \mathbb{Z} \mid n < 0 \}$$

Another View of Soundness 2

- Soundness can be stated succinctly

$$\forall e \in \text{Exp}.\ e\Downarrow \in \gamma(\sigma(e))$$

(the real value of the expression is among the concrete values represented by the abstract value of the expression)

- Let C be the concrete domain (e.g. $\mathbb{Z}$) and A be the abstract domain (e.g. {-, 0, +})
- Commutative diagram:

$$\begin{tikzcd}
\text{Exp} \arrow[r, "\sigma"] \arrow[d, "\Downarrow"'] & A \arrow[d, "\gamma"] \\
C \arrow[r, "\in"] & \mathcal{P}(C)
\end{tikzcd}$$

Another View of Soundness 3

- Consider the generic abstraction of an operator

$$\sigma(e_1 \text{ op } e_2) = \sigma(e_1) \underline{\text{op}} \, \sigma(e_2)$$

- This is sound iff

$$\forall a_1 \forall a_2.\ \gamma(a_1 \underline{\text{op}} a_2) \supseteq \{n_1 \text{ op } n_2 \mid n_1 \in \gamma(a_1), n_2 \in \gamma(a_2)\}$$

- e.g. $\gamma(a_1 \otimes a_2) \supseteq \{ n_1 * n_2 \mid n_1 \in \gamma(a_1), n_2 \in \gamma(a_2) \}$
- This reduces the proof of correctness to one proof for each operator

Abstract Interpretation

- This is our first example of an abstract interpretation
- We carry out computation in an abstract domain
- The abstract semantics is a sound approximation of the standard semantics
- The concretization and abstraction functions establish the connection between the two domains

Adding Unary Minus and Addition

- We extend the language to

  $e ::= n \mid e_1 * e_2 \mid - e$

- We define $\sigma(- e) = \ominus\, \sigma(e)$

	-	0	+
$\ominus$	+	0	-

- Now we add addition:

  $e ::= n \mid e_1 * e_2 \mid - e \mid e_1 + e_2$

- We define $\sigma(e_1 + e_2) = \sigma(e_1) \oplus \sigma(e_2)$

$\oplus$	-	0	+
-	-	-	?
0	-	0	+
+	?	+	+

Adding Addition

- The sign values are not closed under addition
- What should be the value of "$+ \oplus -$"?
- Start from the soundness condition:

$$\gamma(+ \oplus -) \supseteq \{ n_1 + n_2 \mid n_1 > 0,\ n_2 < 0\} = \mathbb{Z}$$

- We don't have an abstract value whose concretization includes $\mathbb{Z}$, so we add one:

  $\top$ ("top" = "don't know")

$\oplus$	-	0	+	$\top$
-	-	-	$\top$	$\top$
0	-	0	+	$\top$
+	$\top$	+	+	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

Loss of Precision

- Abstract computation may lose information:

$$[\![(1 + 2) + -3]\!] = 0$$

but: $\sigma((1+2) + -3) =$

$$(\sigma(1) \oplus \sigma(2)) \oplus \sigma(-3) =$$

$$(+ \oplus +) \oplus - = \top$$

- We lost some precision
- But this will simplify the computation of the abstract answer in cases when the precise answer is not computable

Adding Division

- Straightforward except for division by 0
  - We say that there is no answer in that case
  - $\gamma(+ \oslash 0) = \{ n \mid n = n_1 / 0 , n_1 > 0 \} = \emptyset$
- Introduce $\bot$ to be the abstraction of the $\emptyset$
  - We also use the same abstraction for non-termination!

$\bot$ = "nothing"

$\top$ = "something unknown"

$\oslash$	-	0	+	$\top$	$\bot$
-	+	0	-	$\top$	$\bot$
0	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
+	-	0	+	$\top$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Game Criticism

- This term refers to a conflict between the mechanics or dynamics of a game and its story. For example, *Bioshock* was viewed as promoting selflessness through story but selfishness through gameplay, a disconnect that pulled some players out of the game. The term is often viewed as “highbrow” or “pretentious”.

Q: Books (750 / 842)

- This 1962 Newbery Medal-winning novel by Madeleine L'Engle includes Charles Wallace, Mrs. Who, Mrs. Whatsit, Mrs. Which and the space-bending Tesseract. In 2018 it was adapted into a Disney film with Oprah Winfrey.

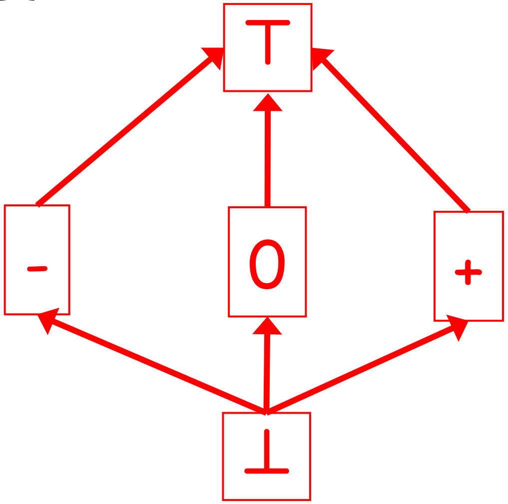
Music

- Otis Lee Jackson, Jr. is one of the most influential producers in modern hiphop. He collaborates with MF DOOM, incorporates elements from jazz, and makes heavy use of eclectic samples. Give his stage name, shared with the word-based party game in which players provide words to fill in the blanks in an unknown story.

Computer Science

- This American Turing-award winner is known for developing Speedcoding and FORTRAN (the first two high-level languages), as well creating a way to express the formal syntax of a language and using that approach to specify ALGOL. He later focused on function-level (as opposed to value-level) programming. His first major programming project calculated the positions of the Moon.

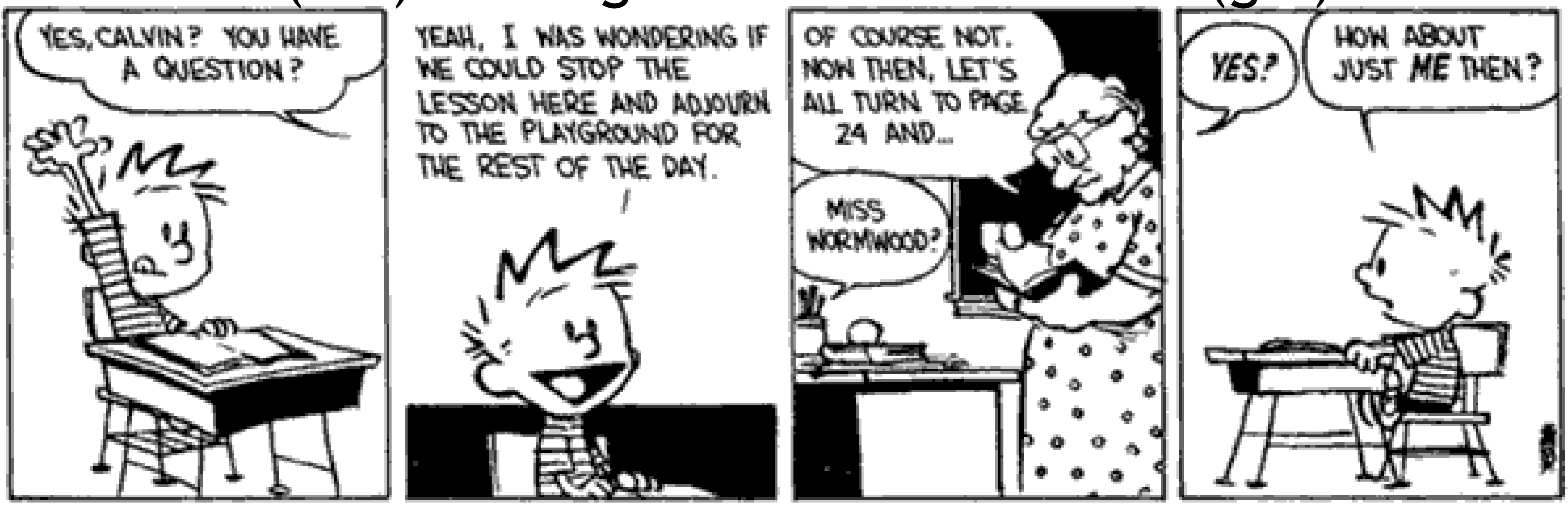
The Abstract Domain



- Our abstract domain forms a lattice
- A partial order is induced by γ

$$a_1 \leq a_2 \quad \text{iff } \gamma(a_1) \subseteq \gamma(a_2)$$

  - We say that a_1 is more precise than a_2!
- Every finite subset has a least-upper bound (lub) and a greatest-lower bound (glb)



Lattice Facts

- A lattice is <u>complete</u> when every subset has a lub and a gub
  - Even infinite subsets!
- Every finite lattice is (trivially) complete
- Every complete lattice is a complete partial order (recall: proof techniques: induction!)
  - Since a chain is a subset
- Not every CPO is a complete lattice
  - Might not even be a lattice at all

From One, Many

- We can start with the abstraction function β

  $$\beta : C \to A$$

  (maps a concrete value to the best abstract value)
  - A must be a lattice
- We can derive the concretization function γ

  $\gamma : A \to \mathcal{P}(C)$

  $\gamma(a) = \{ x \in C \mid \beta(x) \leq a \}$
- And the abstraction for sets α

  $\alpha : \mathcal{P}(C) \to A$

  $\alpha(S) = \text{lub} \{ \beta(x) \mid x \in S \}$

Example

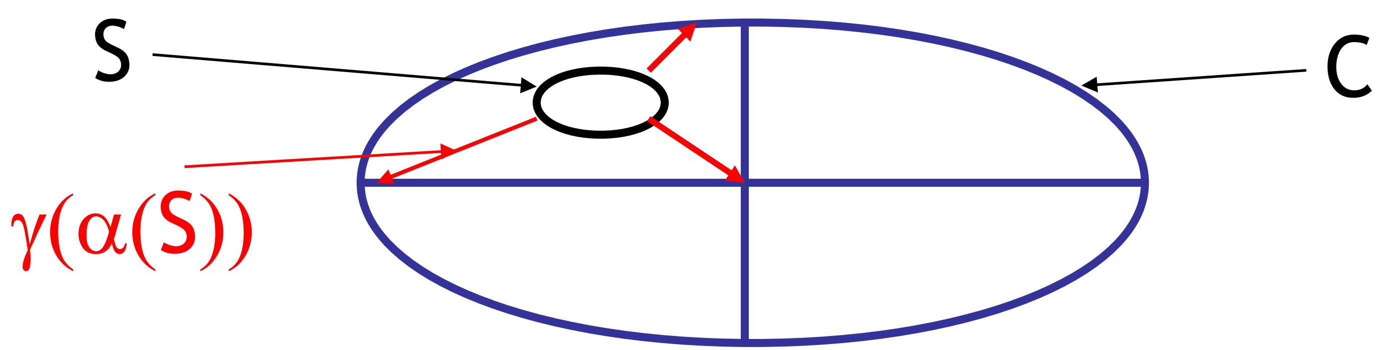
- Consider our sign lattice

$$\beta(n) = \begin{cases} + & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ - & \text{if } n < 0 \end{cases}$$

- $\alpha(S) = \text{lub } \{ \beta(x) \mid x \in S \}$
  - Example: $\alpha(\{1, 2\}) = \text{lub } \{ + \} = +$
    $\alpha(\{1, 0\}) = \text{lub } \{ +, 0 \} = \top$
    $\alpha(\{\}) = \text{lub } \emptyset = \bot$
- $\gamma(a) = \{ n \mid \beta(n) \leq a \}$
  - Example: $\gamma(+) = \{ n \mid \beta(n) \leq + \} =$
    $\{ n \mid \beta(n) = + \} = \{ n \mid n > 0 \}$
    $\gamma(\top) = \{ n \mid \beta(n) \leq \top \} = \mathbb{Z}$
    $\gamma(\bot) = \{ n \mid \beta(n) \leq \bot \} = \emptyset$

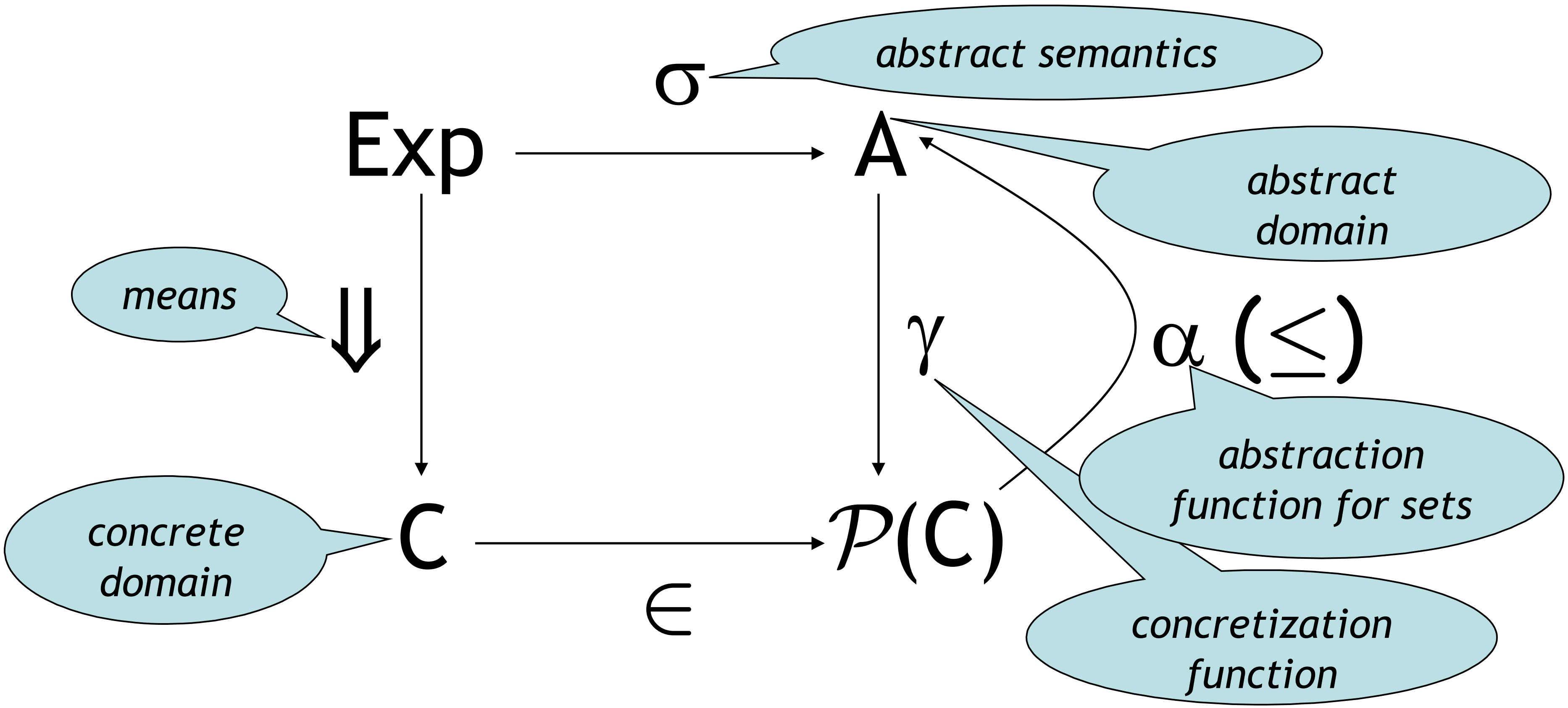
Galois Connections

- We can show that
  - γ and α are monotonic (with $\subseteq$ ordering on $\mathcal{P}(C)$)
  - $\alpha(\gamma(a)) = a$ for all $a \in A$
  - $\gamma(\alpha(S)) \supseteq S$ for all $S \in \mathcal{P}(C)$
- Such a pair of functions is called a Galois connection
  - Between the lattices A and $\mathcal{P}(C)$



Correctness Condition

- In general, abstract interpretation satisfies the following (amazingly common) diagram



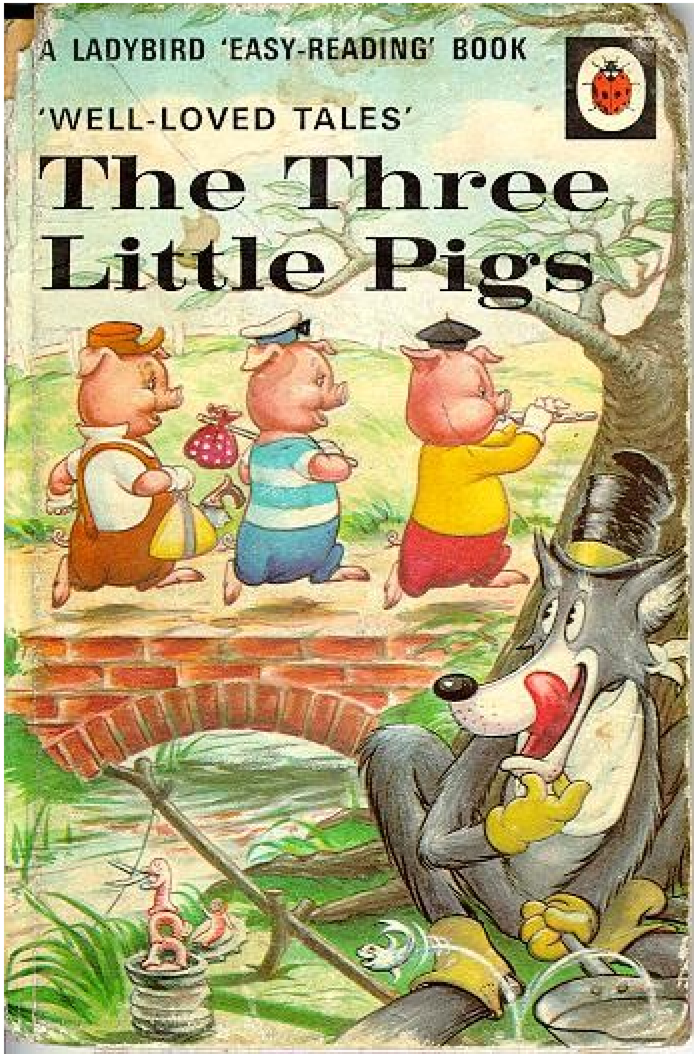
Three Little Correctness Conditions

- Three conditions define a correct abstract interpretation
- α and γ are monotonic
- α and γ form a Galois connection

  = "α and γ are almost inverses"

1. Abstraction of operations is correct

$$a_1 \ \underline{op} \ a_2 = \alpha(\gamma(a_1) \ op \ \gamma(a_2))$$



“On The Board” Questions

- What is the VC for:

$$\text{for } i = e_{low} \text{ to } e_{high} \text{ do } c \text{ done}$$

- This axiomatic rule is unsound. Why?

$$\frac{\vdash \{A \wedge p\}\ c_{then}\ \{B_{then}\} \qquad \vdash \{A \wedge \neg p\}\ c_{else}\ \{B_{else}\}}{\vdash \{A\}\ \text{if } p \text{ then } c_{then} \text{ else } c_{else}\ \{B_{then} \vee B_{else}\}}$$

Homework

- Read Cousot & Cousot Article