



Type Checking

One-Slide Summary

- A **type environment** gives types for **free variables**. You typecheck a **let-body** with an environment that has been **updated** to contain the new **let-variable**.
- If an object of type X could be used when one of type Y is acceptable then we say X is a **subtype** of Y , also written $X \leq Y$.
- A type system is **sound** if $\forall E$.
 $\text{dynamic_type}(E) \leq \text{static_type}(E)$

Lecture Outline

- Typing Rules
- Typing Environments
- “Let” Rules
- Subtyping
- Wrong Rules



Example: 1 + 2

$$\frac{\frac{}{\vdash 1 : \text{Int}} \quad \frac{}{\vdash 2 : \text{Int}}}{\vdash 1 + 2 : \text{Int}}}$$

If we can prove it, then it's true!

Soundness

- A type system is **sound** if
 - Whenever $\vdash e : T$
 - Then e evaluates to a value of type T

- We only want sound rules
 - But some sound rules are better than others:

$(i \text{ is an integer})$

$\vdash i : \text{Object}$



Type Checking Proofs

- Type checking proves facts $e : T$
 - One type rule is used for each kind of expression
- In the type rule used for a node e
 - The **hypotheses** are the proofs of types of e 's subexpressions
 - The **conclusion** is the proof of type of e itself

Rules for Constants

$$\frac{}{\vdash \text{false} : \text{Bool}} \text{[Bool]}$$
$$\frac{}{\vdash s : \text{String}} \text{[String]}$$

(s is a string constant)

Rule for New

- `new T` produces an object of type `T`
- Ignore `SELF_TYPE` for now . . .

$$\frac{}{\vdash \text{new } T : T} \text{ [New]}$$

Two More Rules



$$\frac{\vdash e : \text{Bool}}{\vdash \text{not } e : \text{Bool}} \quad [\text{Not}]$$

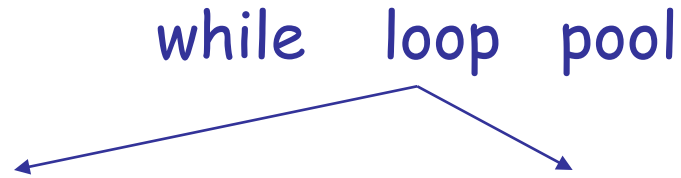
$$\vdash e_1 : \text{Bool}$$

$$\vdash e_2 : T$$

$$\frac{\vdash e_1 : \text{Bool} \quad \vdash e_2 : T}{\vdash \text{while } e_1 \text{ loop } e_2 \text{ pool} : \text{Object}} \quad [\text{Loop}]$$

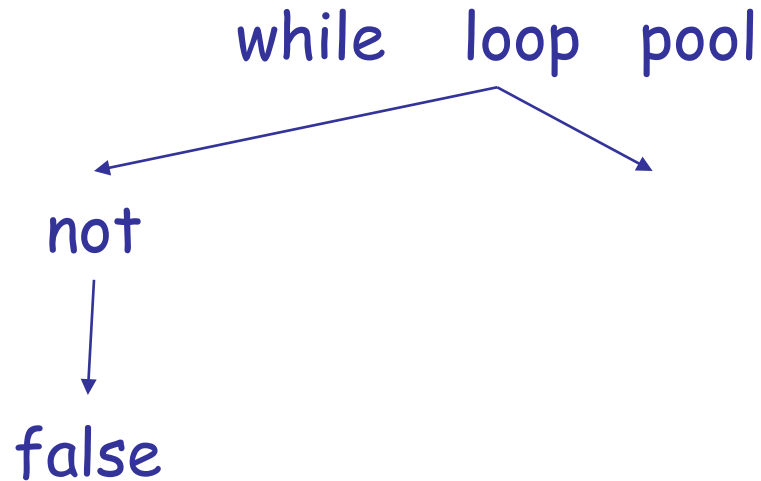
Typing: Example

- Typing for `while not false loop 1 + 2 * 3 pool`



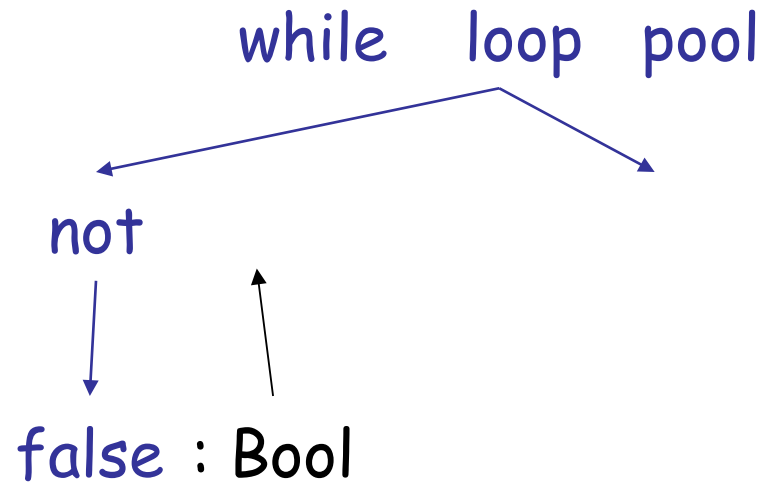
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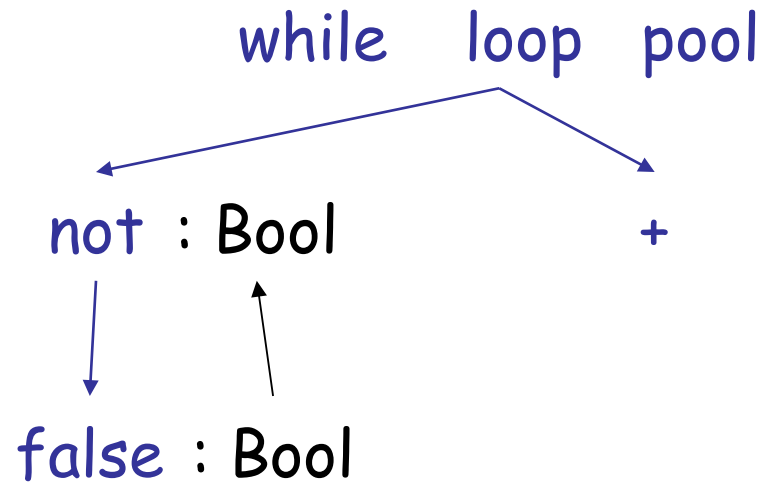
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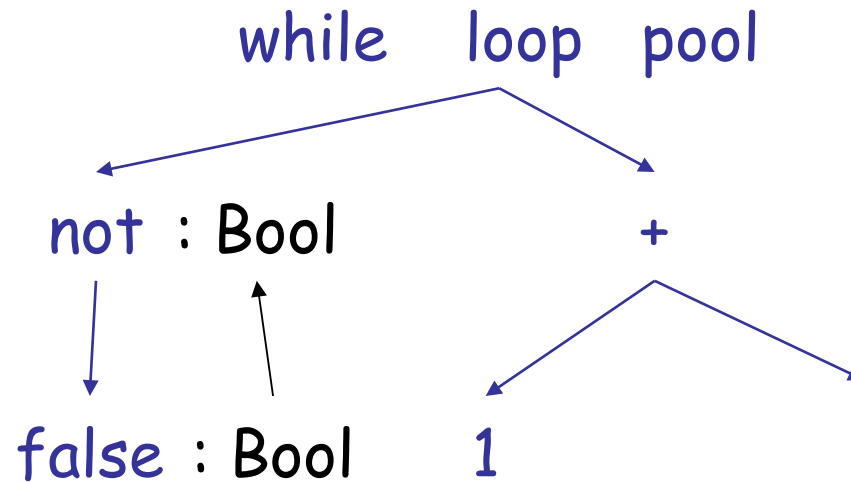
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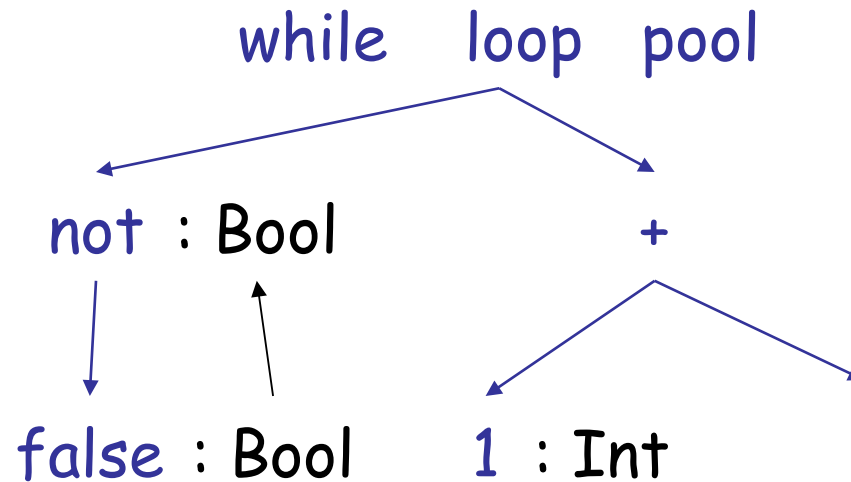
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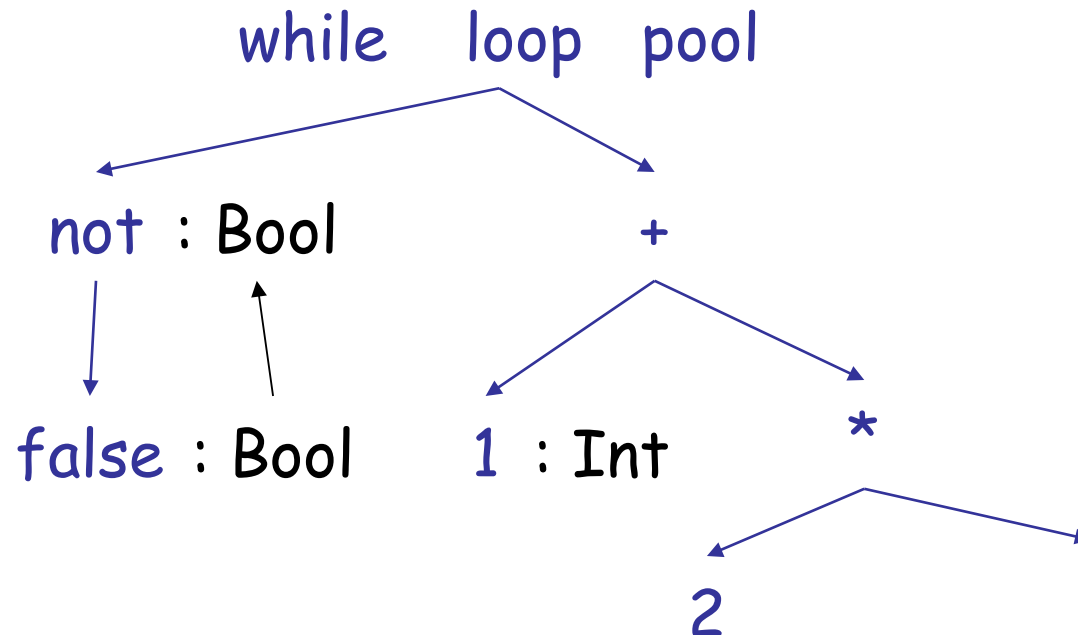
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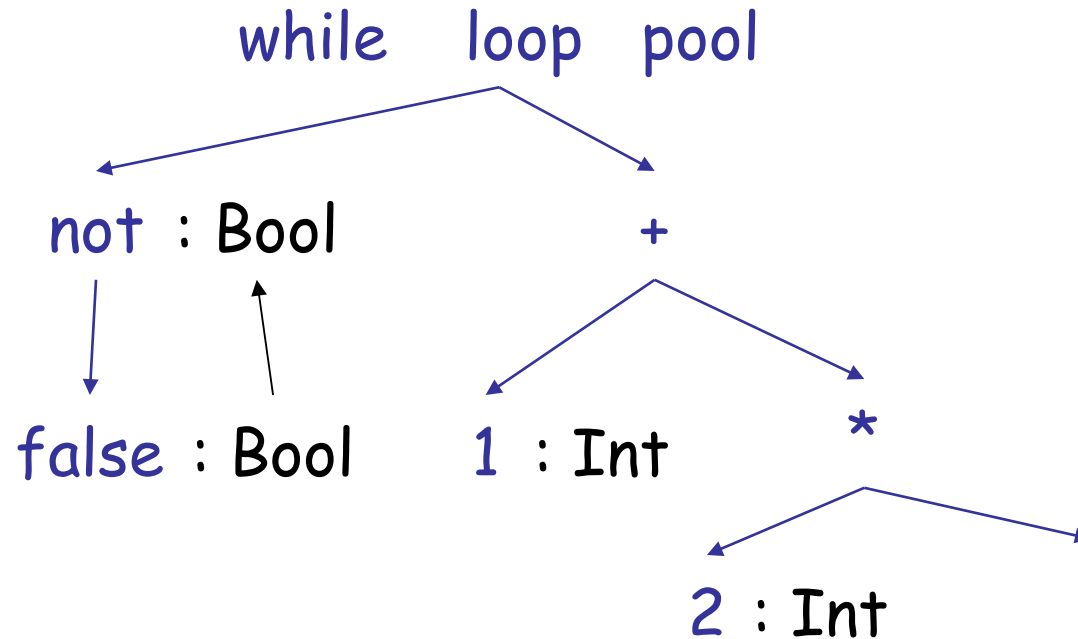
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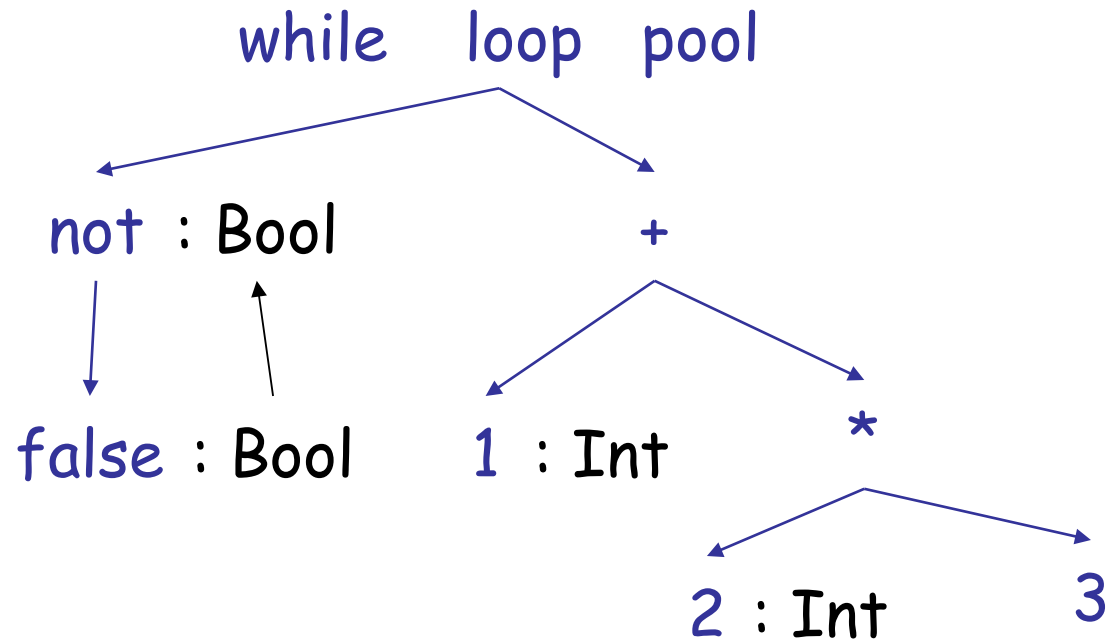
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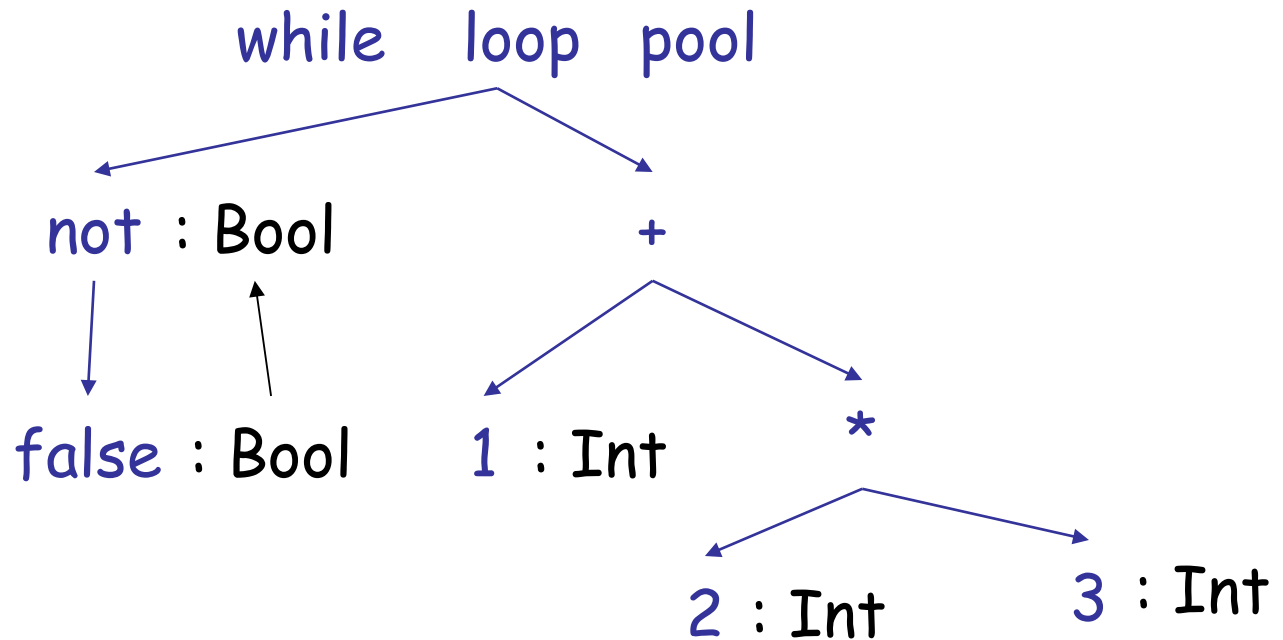
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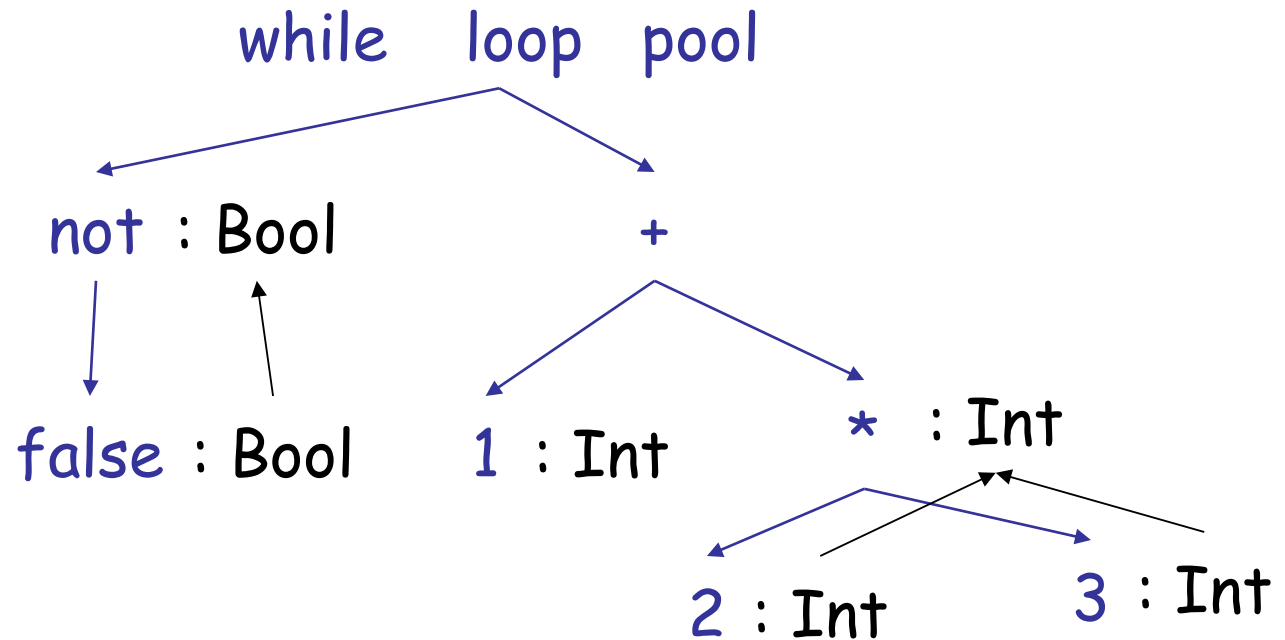
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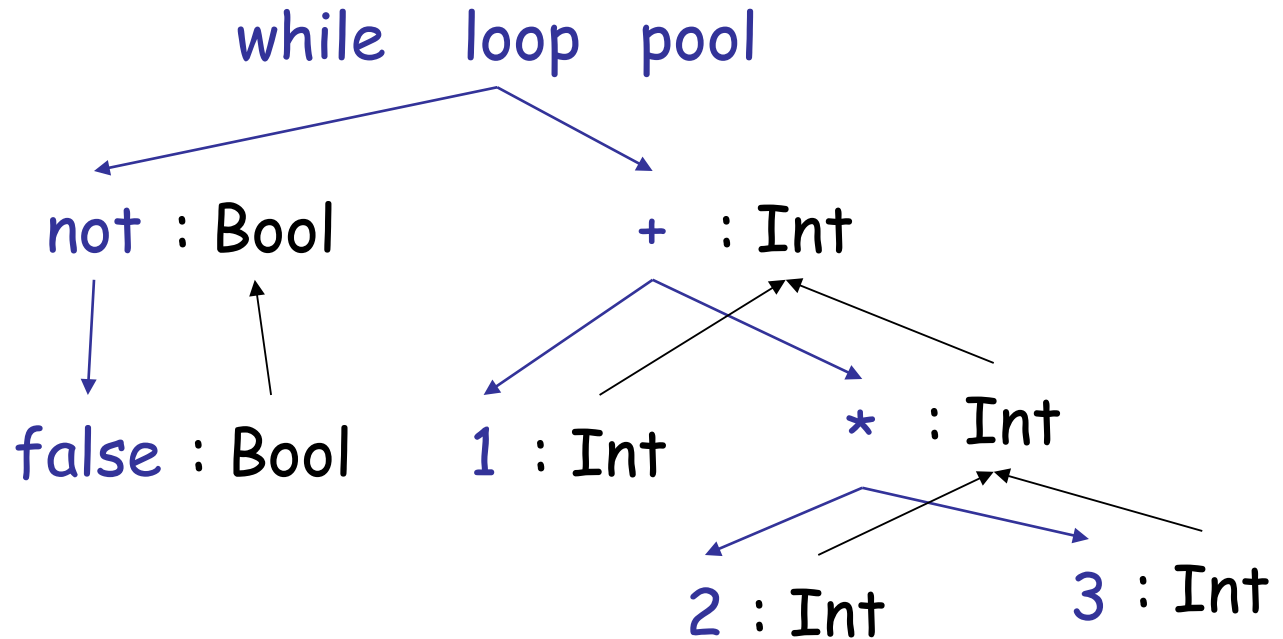
Typing: Example

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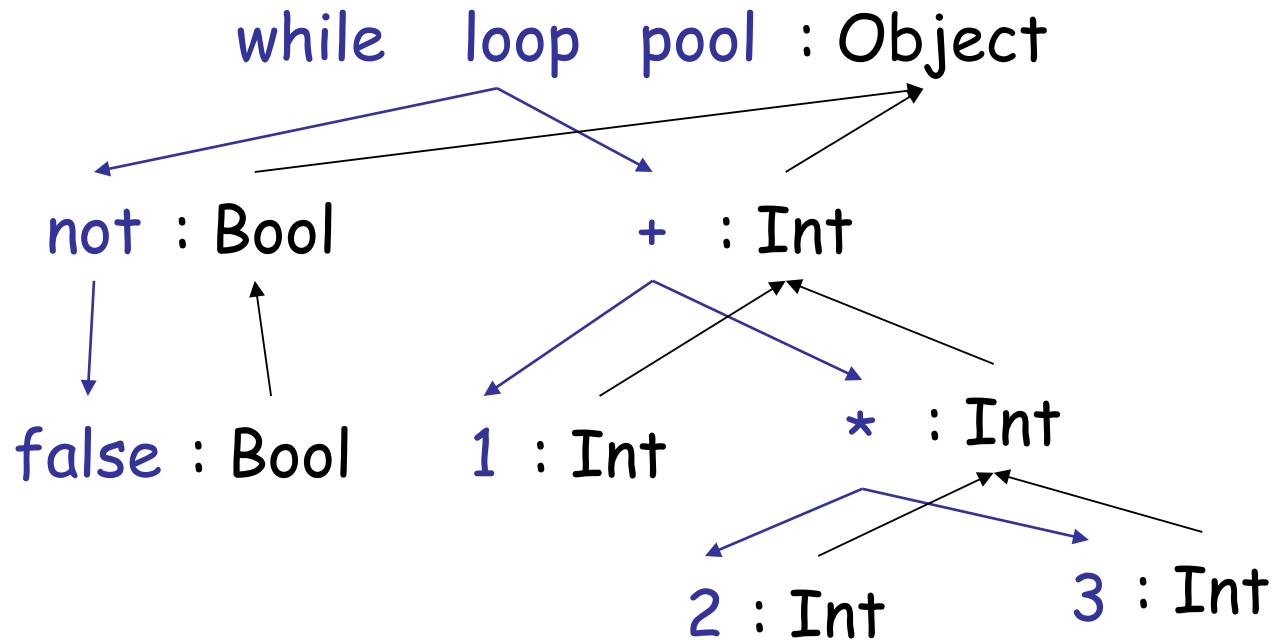
Typing: Example

- Typing for `while not false loop 1 + 2 * 3 pool`



Typing: Example

- Typing for `while not false loop 1 + 2 * 3 pool`



Typing Derivations

- The typing reasoning can be expressed as a tree:

$$\begin{array}{c}
 \frac{}{\vdash \text{false} : \text{Bool}} \quad \frac{}{\vdash 1 : \text{Int}} \quad \frac{\vdash 2 : \text{Int} \quad \vdash 3 : \text{Int}}{\vdash 2 * 3 : \text{Int}} \\
 \hline
 \vdash \text{not false} : \text{Bool} \quad \vdash 1 + 2 * 3 : \text{Int} \\
 \hline
 \vdash \text{while not false loop } 1 + 2 * 3 : \text{Object}
 \end{array}$$

- The **root** of the tree is the whole expression
- Each node is an **instance** of a typing rule
- Leaves** are the rules with no hypotheses

A Problem

- What is the type of a variable reference?

$$\frac{}{\vdash x : ?} \text{ [Var]} \quad (\text{x is an identifier})$$

- The local structural rule does *not* carry enough information to give *x* a type. Fail.



A Solution: Put more information in the rules!

- A **type environment** gives types for **free** variables
 - A **type environment** is a mapping from **Object_Identifiers** to **Types**
 - A variable is **free** in an expression if:
 - The expression contains an occurrence of the variable that refers to a declaration *outside* the expression
 - in the expression “**x**”, the variable “**x**” is free
 - in “**let x : Int in x + y**” only “**y**” is free
 - in “**x + let x : Int in x + y**” both “**x**”, “**y**” are free

Type Environments

Let \mathcal{O} be a function from **Object_Identifiers** to **Types**

The sentence $\mathcal{O} \vdash e : T$

is read: Under the assumption that variables have the types given by \mathcal{O} , it is provable that the expression e has the type T

Modified Rules

The type environment is added to the earlier rules:

$$\frac{}{0 \vdash i : \text{Int}} \text{ [Int]} \quad (\text{i is an integer})$$

$$\frac{\begin{array}{l} 0 \vdash e_1 : \text{Int} \\ 0 \vdash e_2 : \text{Int} \end{array}}{0 \vdash e_1 + e_2 : \text{Int}} \text{ [Add]}$$

New Rules

And we can write new rules:

$$\frac{}{O \vdash x : T} \text{ [Var]} \quad (O(x) = T)$$

Equivalently:

$$\frac{O(x) = T}{O \vdash x : T} \text{ [Var]}$$

Let

$$\frac{O[T_0/x] \vdash e_1 : T_1}{O \vdash \text{let } x : T_0 \text{ in } e_1 : T_1} \quad [\text{Let-No-Init}]$$

$O[T_0/x]$ means “ O modified to map x to T_0 and behaving as O on all other arguments”:

$$O[T_0/x](x) = T_0$$

$$O[T_0/x](y) = O(y)$$

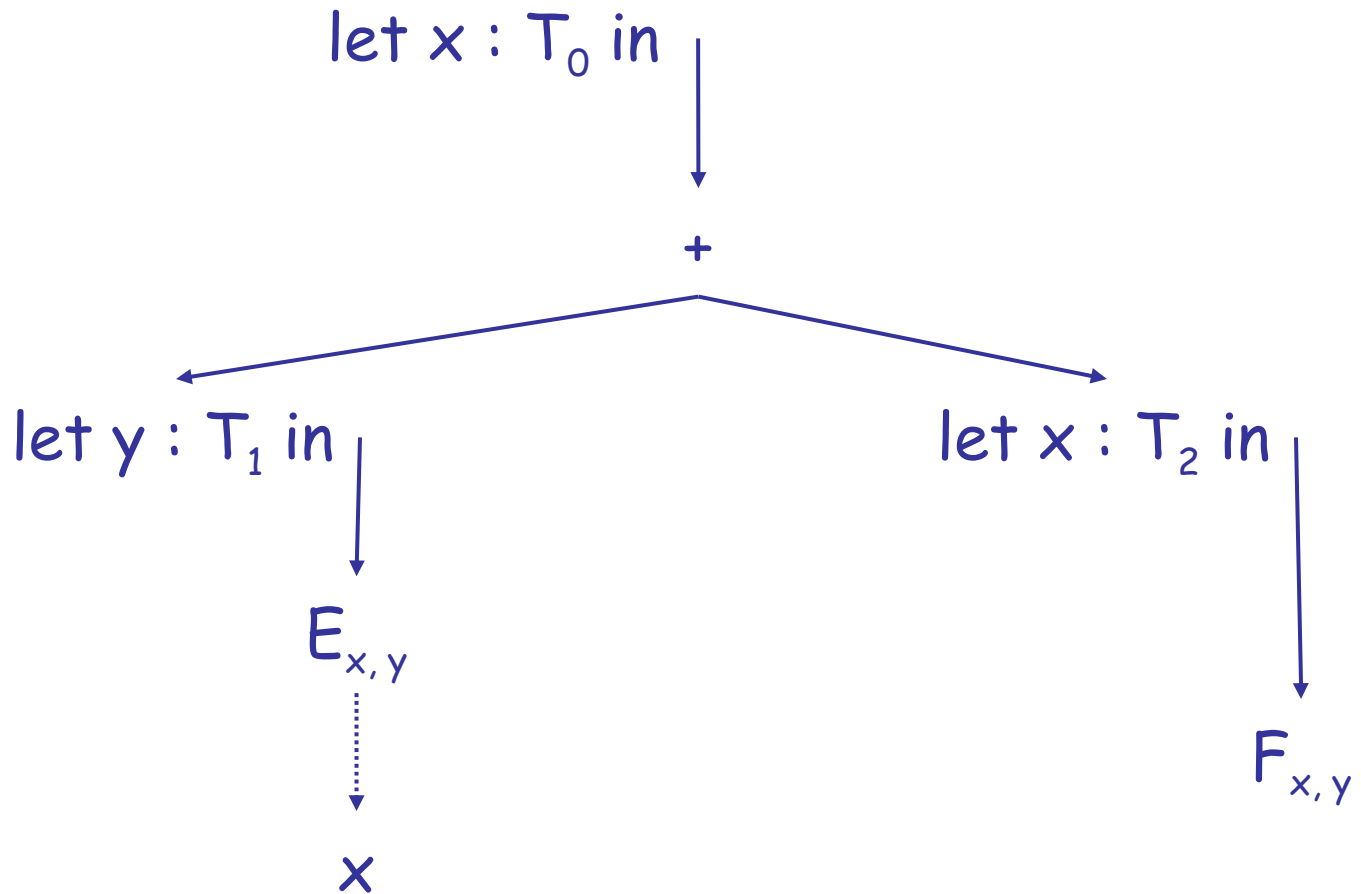
(You can write $O[x/T_0]$ on tests, etc.)

Let Example

- Consider the Cool expression
 $\text{let } x : T_0 \text{ in } (\text{let } y : T_1 \text{ in } E_{x,y}) + (\text{let } x : T_2 \text{ in } F_{x,y})$
(where $E_{x,y}$ and $F_{x,y}$ are some Cool expression that contain occurrences of “ x ” and “ y ”)
- Scope
 - of “ y ” is $E_{x,y}$
 - of outer “ x ” is $E_{x,y}$
 - of inner “ x ” is $F_{x,y}$
- This is captured precisely in the typing rule.

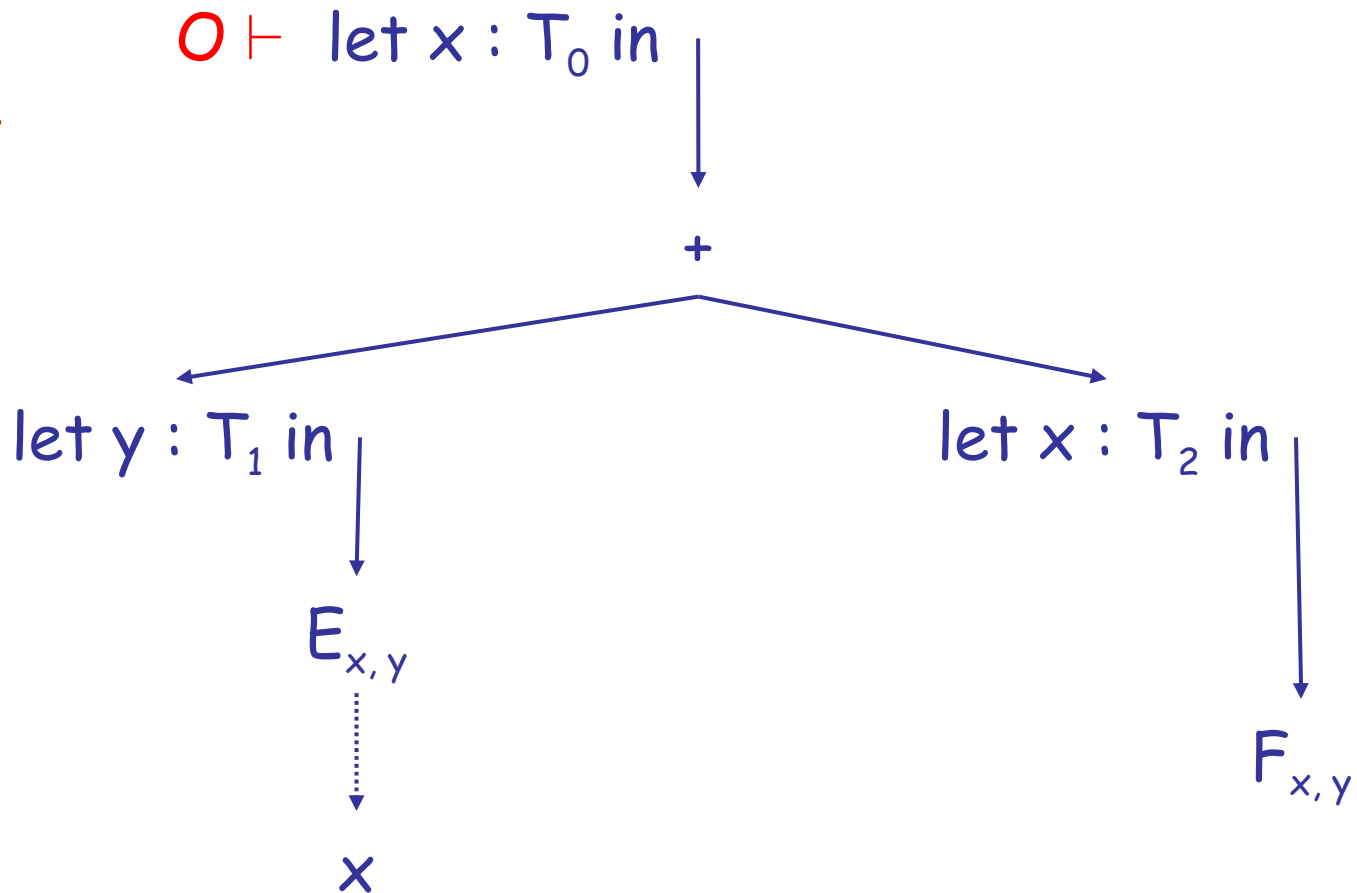
Example of Typing “let”

AST



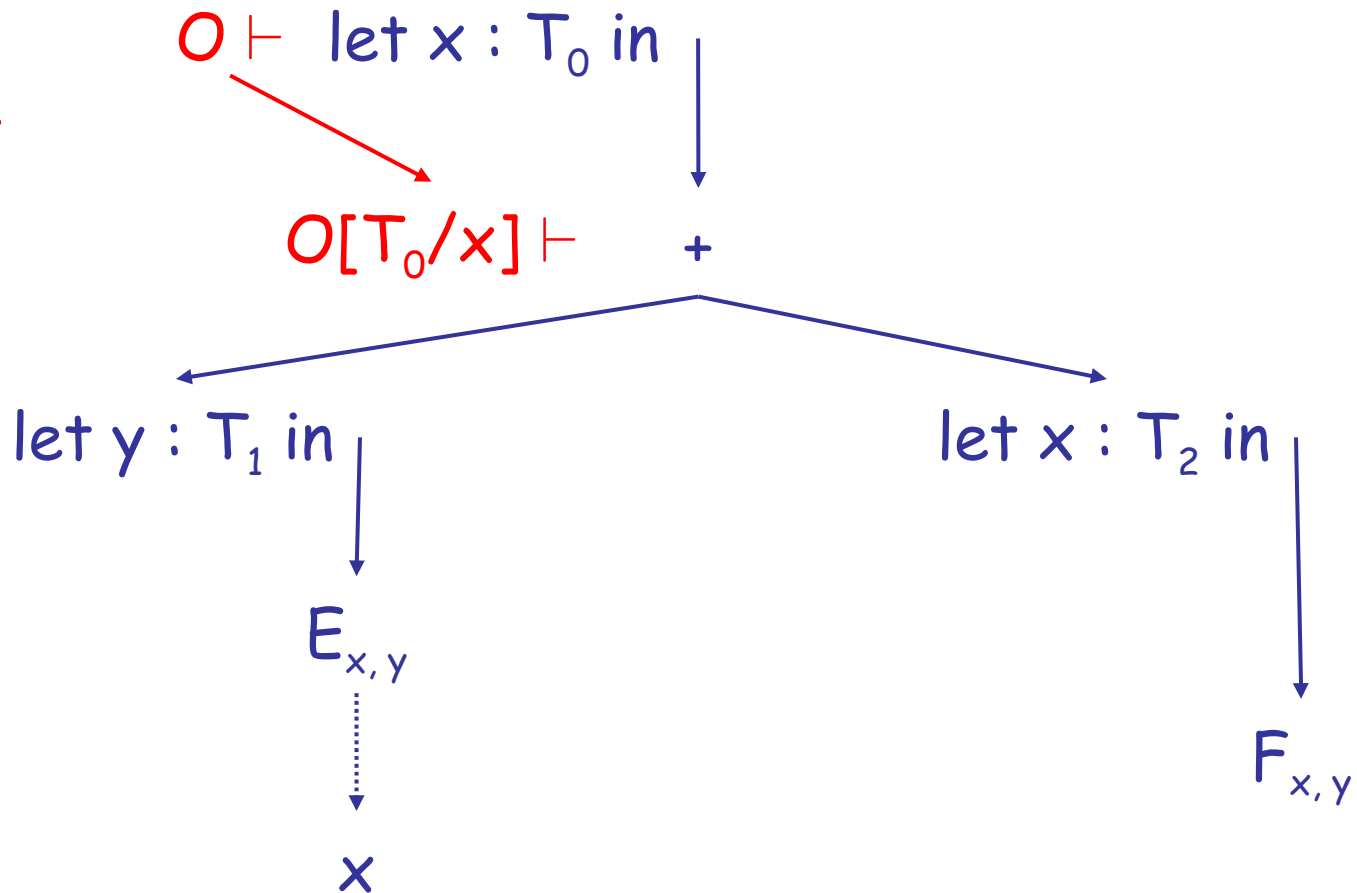
Example of Typing “let”

AST
Type env.

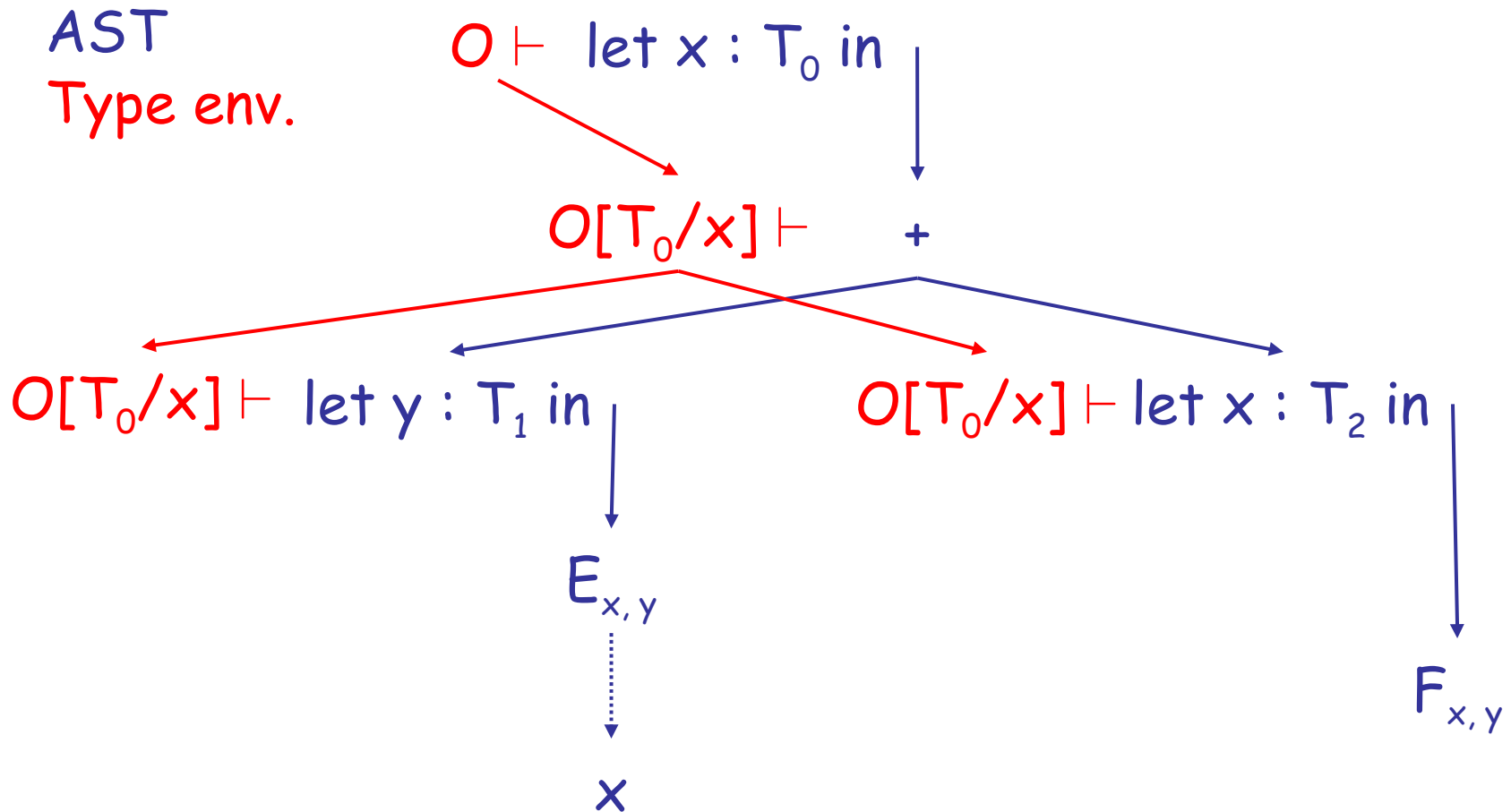


Example of Typing “let”

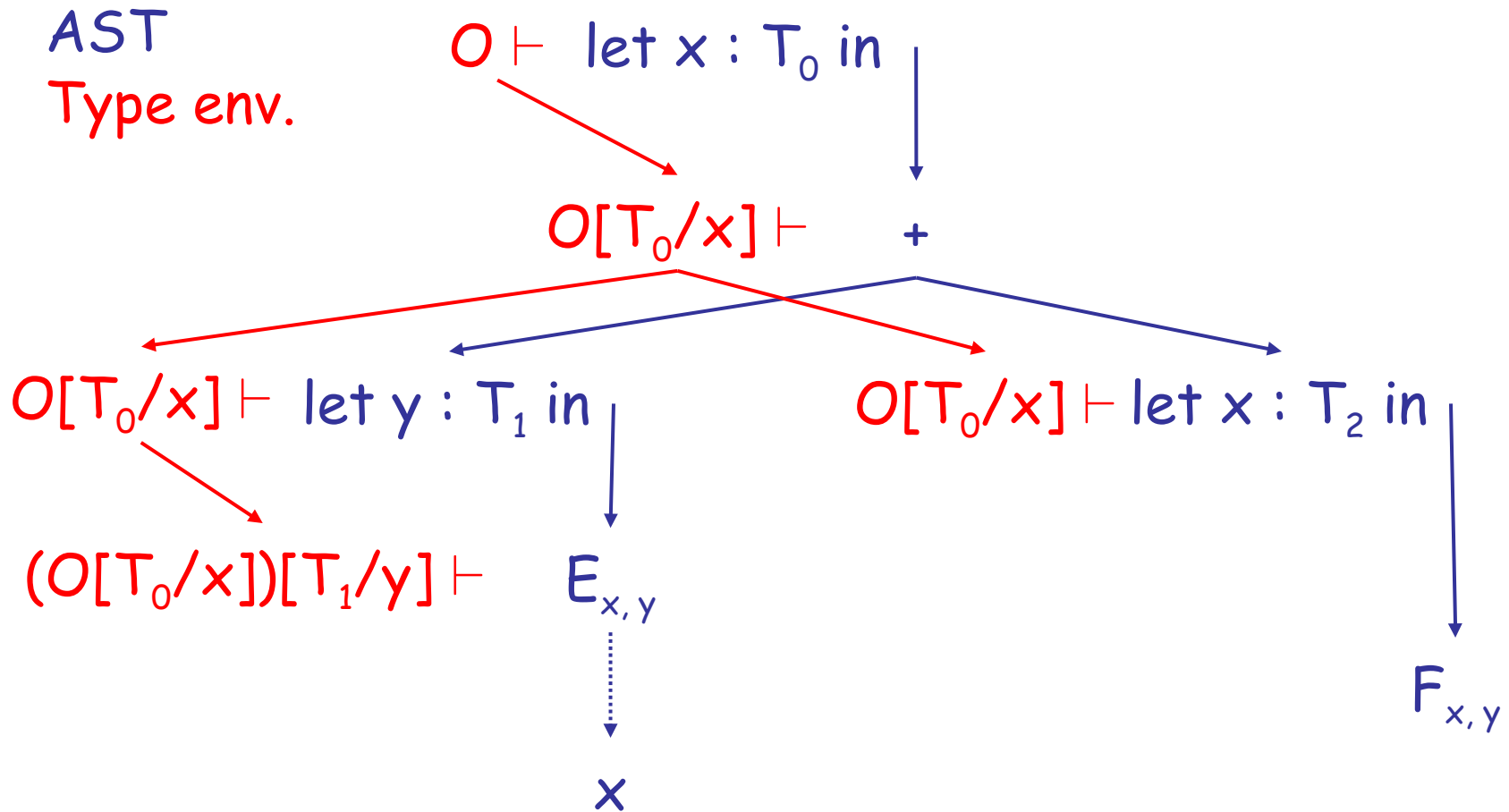
AST
Type env.



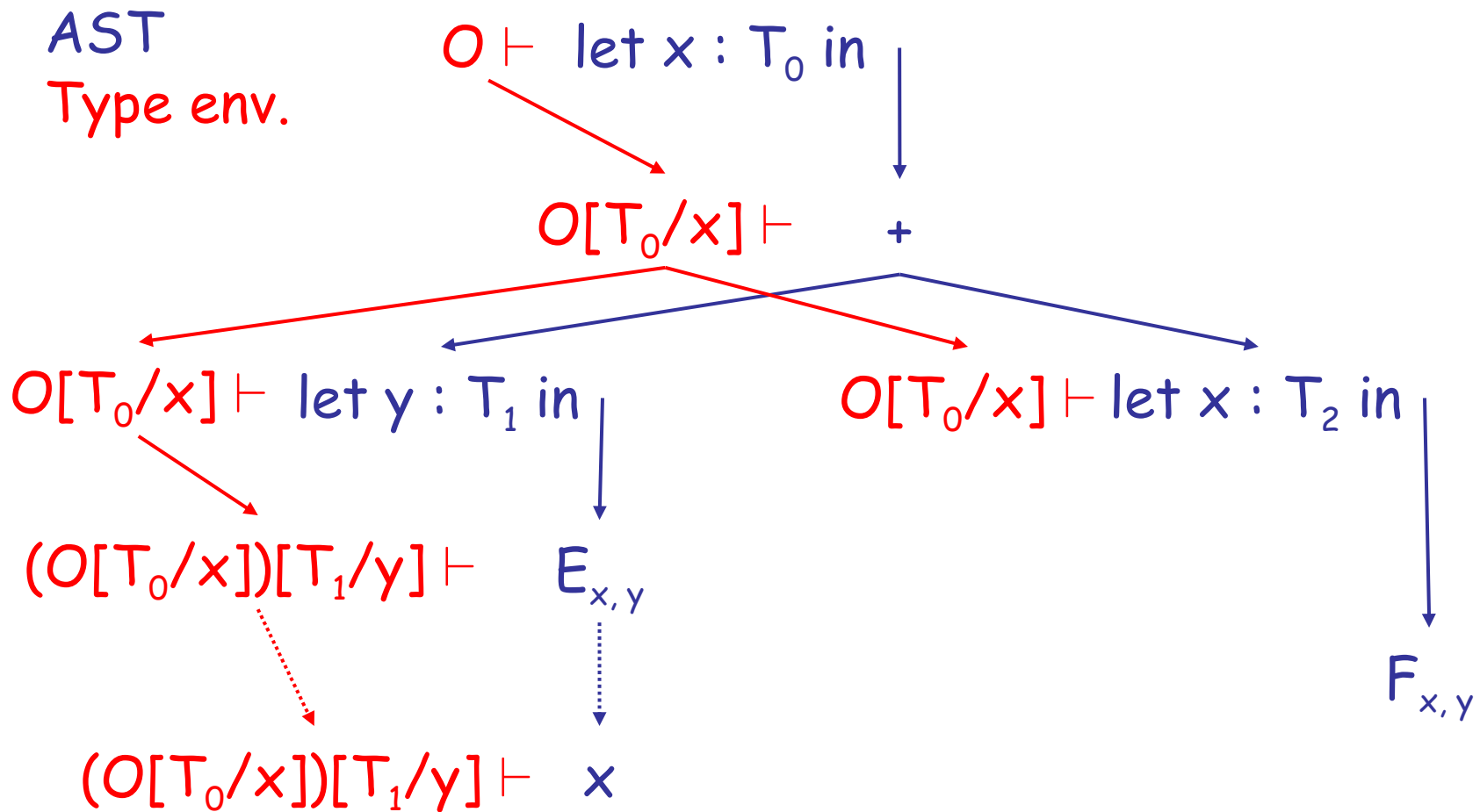
Example of Typing “let”



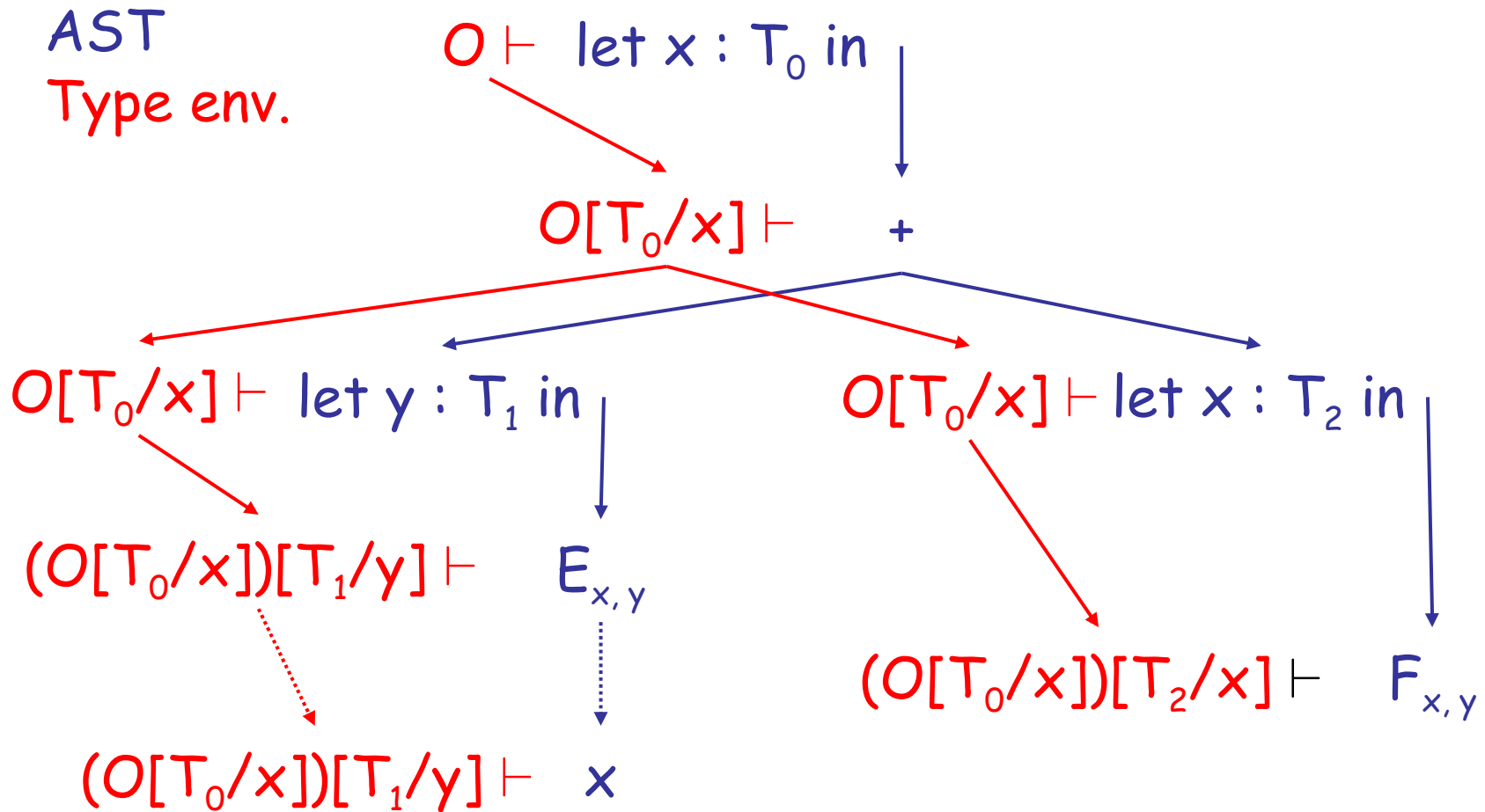
Example of Typing “let”



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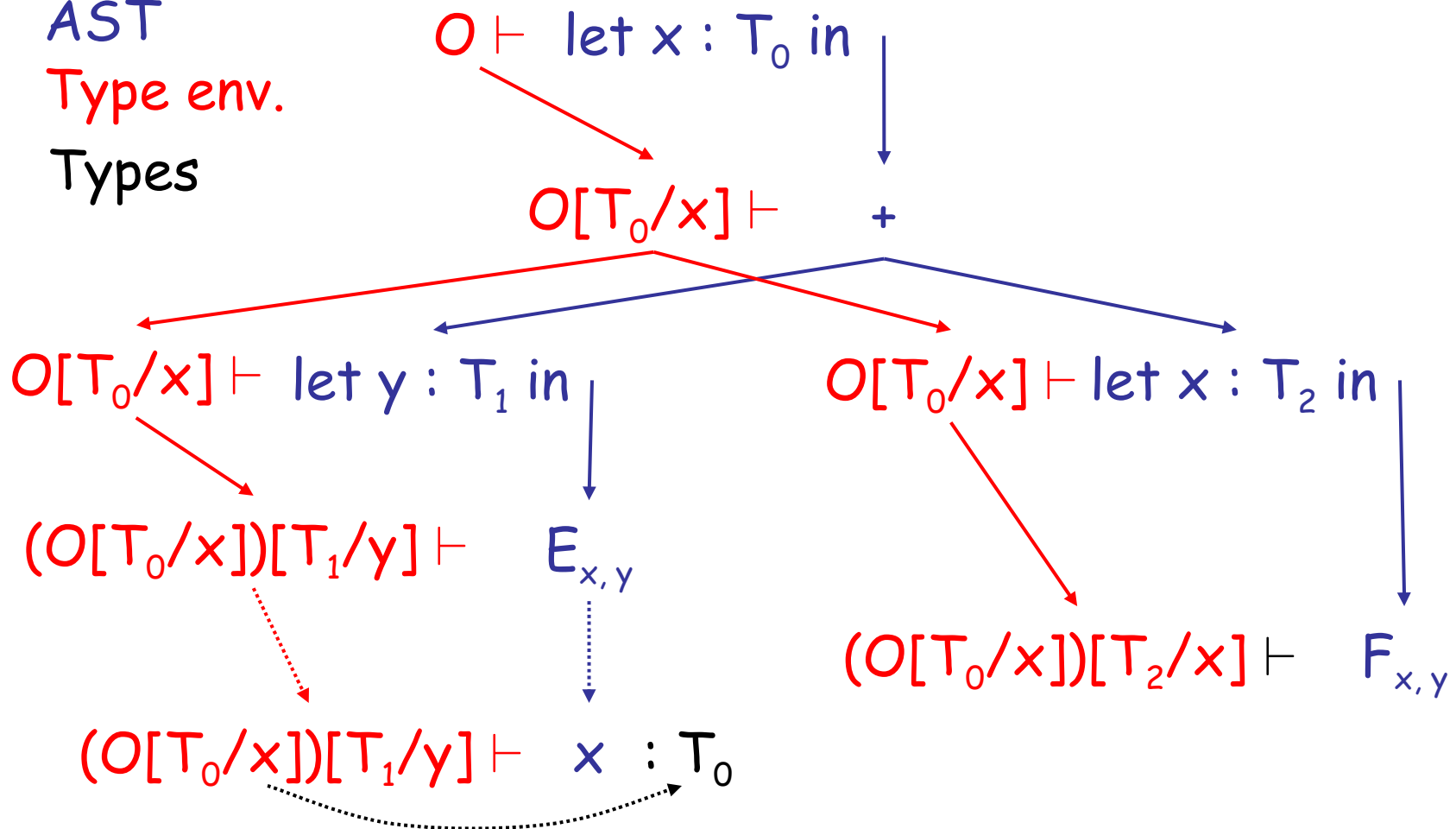


Example of Typing “let”

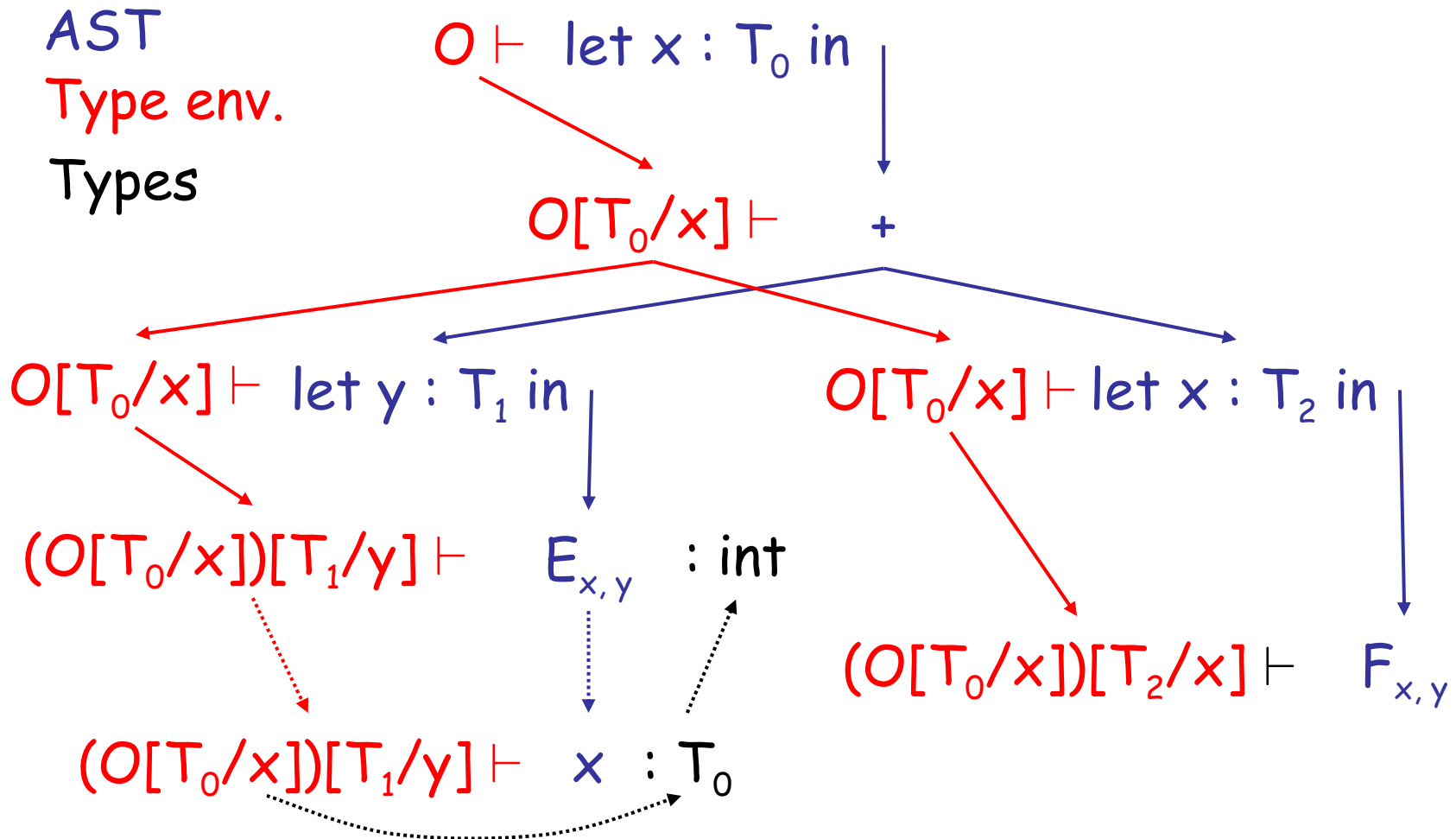
AST

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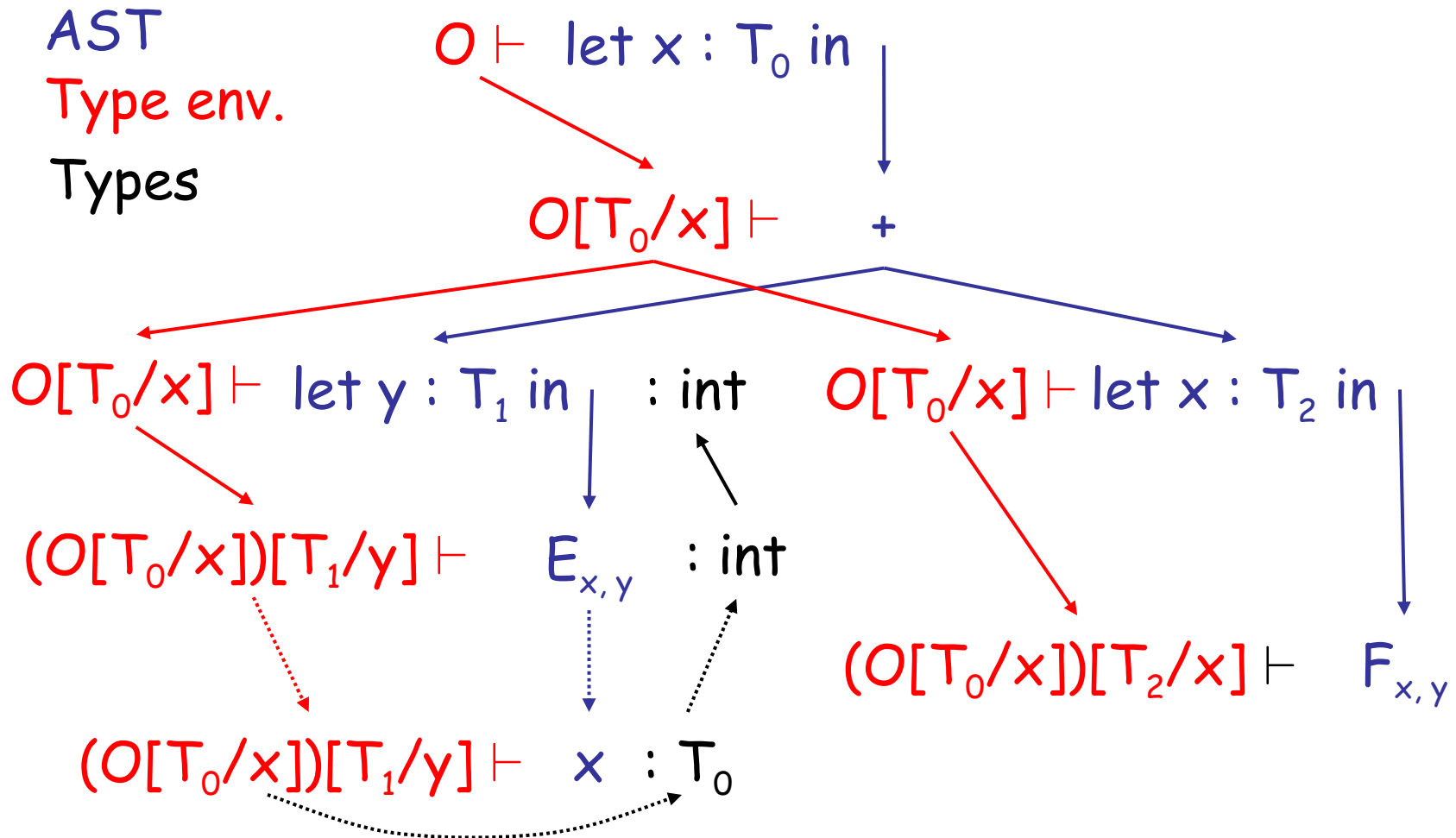
Types



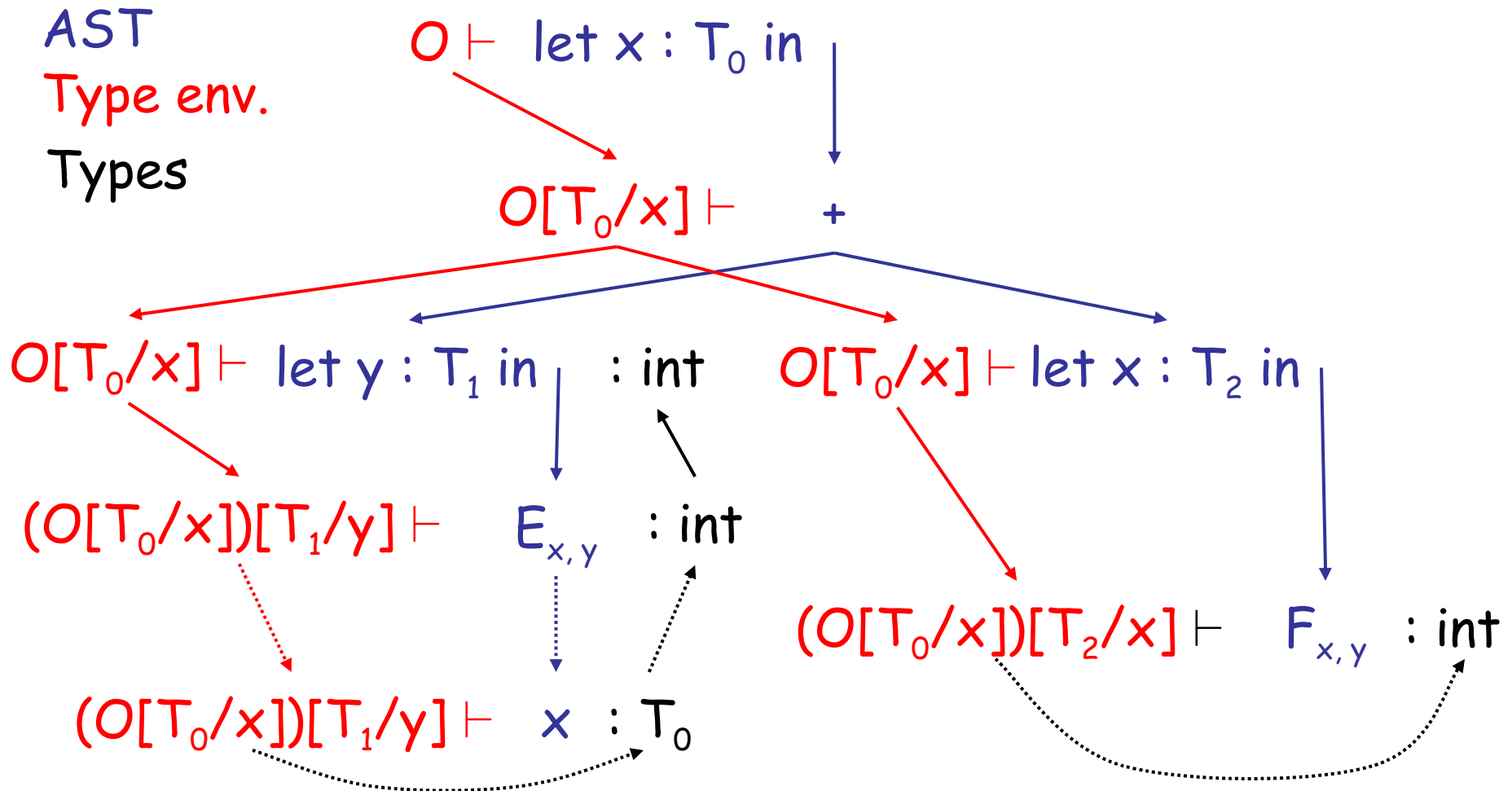
Example of Typing “let”



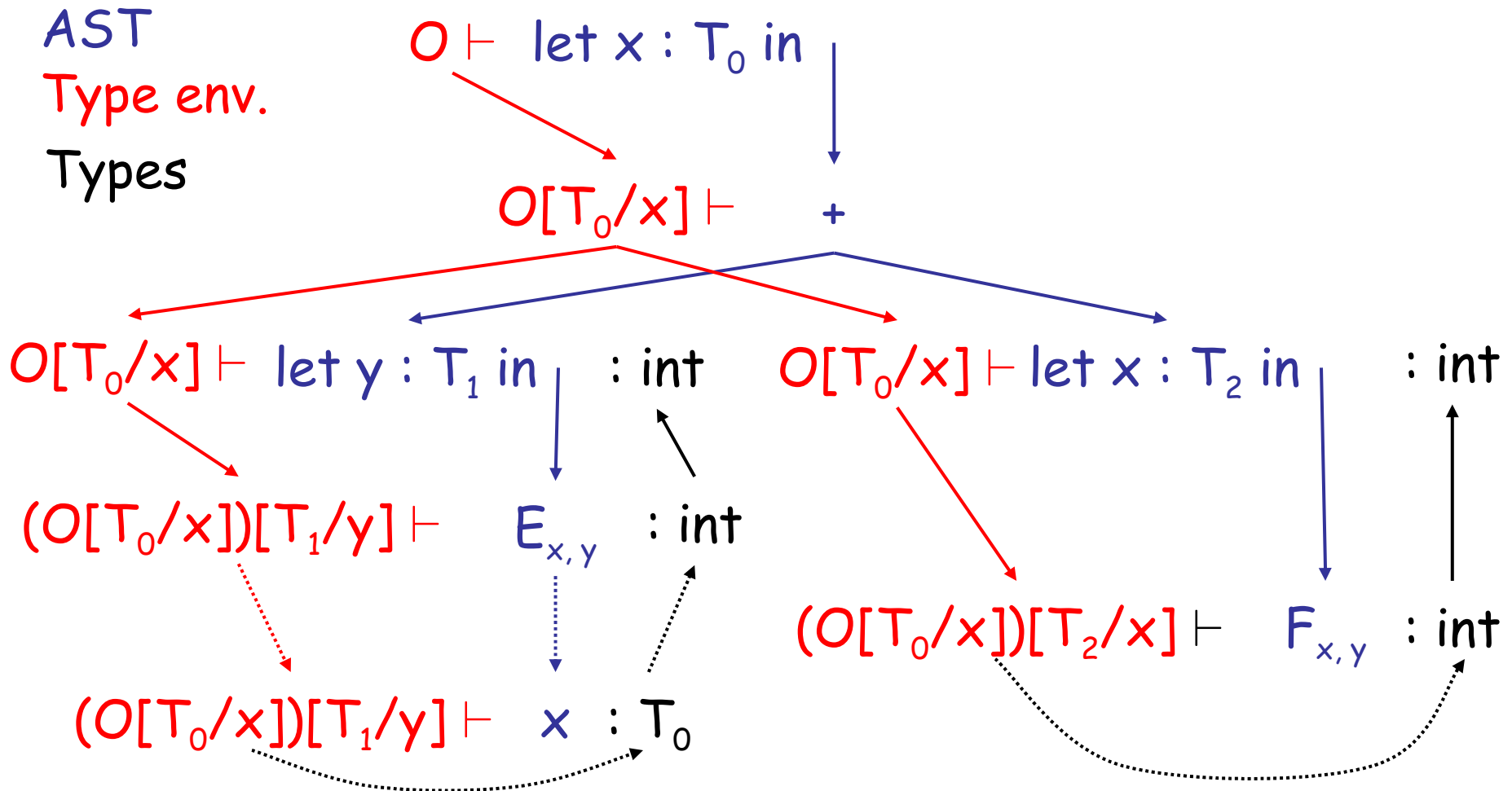
Example of Typing “let”



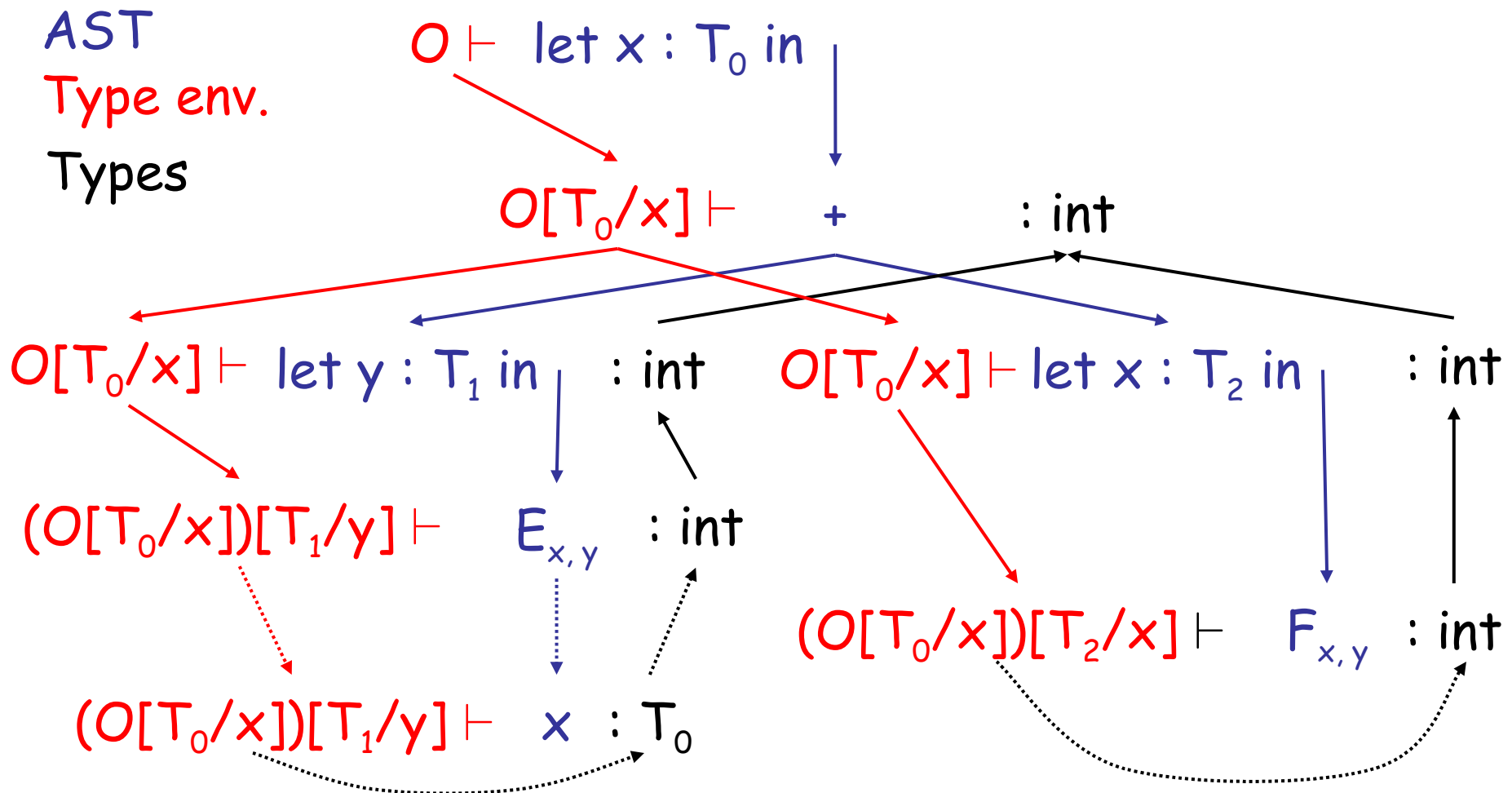
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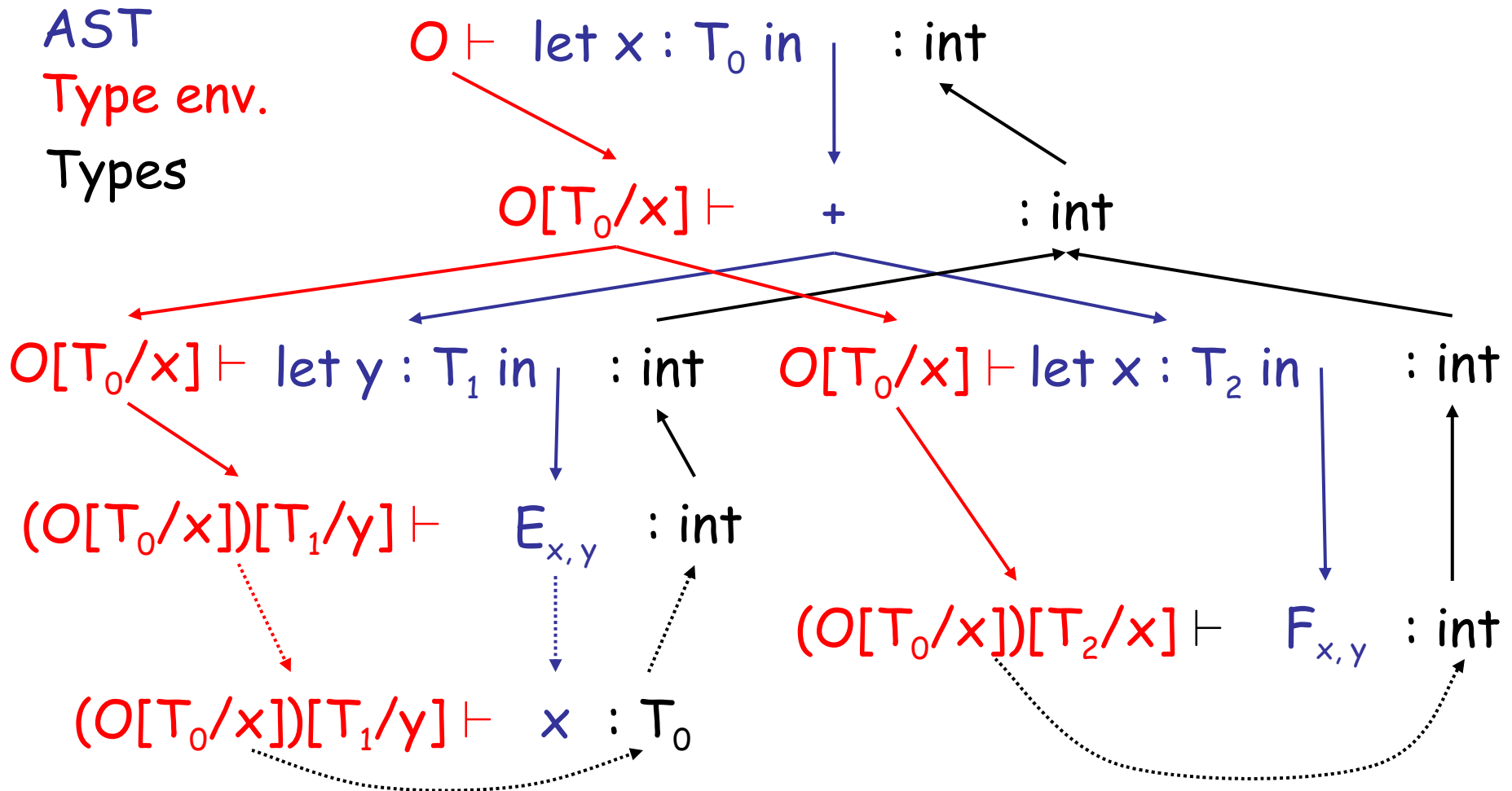
Example of Typing “let”



Example of Typing “let”



Example of Typing “let”



Notes

- The type environment gives types to the free identifiers in the current scope
- The **type environment** is **passed down** the AST from the root towards the leaves
- **Types** are computed up the AST from the leaves **towards the root**

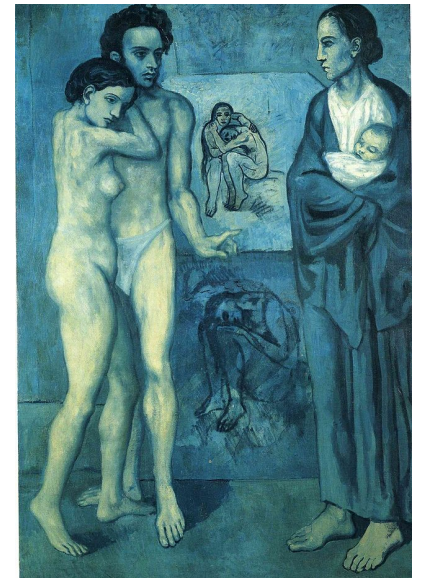
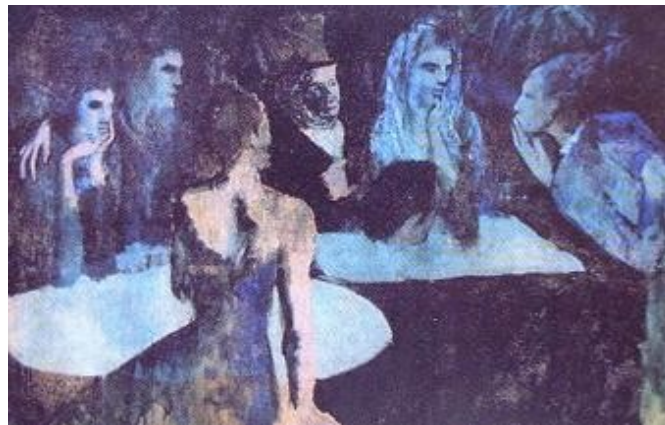
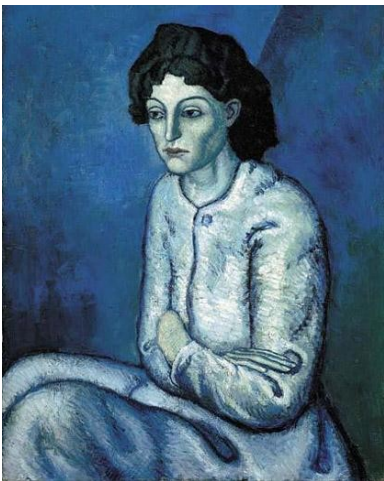
Q: General (455 / 842)

- This numerical technique for finding solutions to boundary-value problems was initially developed for use in structural analysis in the 1940's. The subject is represented by a model consisting of a number of linked simplified representations of discrete regions. It is often used to determine stress and displacement in mechanical systems.

Art Trivia

(zam4ke memorial)

- The Período Azul refers to works produced by *this artist* between 1901 and 1904. The works place a heavy emphasis on shades of blue or blue-green, with the artist sinking into depression. While difficult to sell at the time, the works are now quite popular.



Cultural Food Trivia

(jrk8pd memorial)

- Identify the culture or ethnicity associated with the following foods:
 - Latkes (potato pancakes)
 - Poutine (fries with curds and gravy)
 - Bubble tea (tapioca, fruit, tea)
 - Sosatie (lamb or mutton on skewer with spicy sauce)

Q: Movies (377 / 842)

- Identify the subject or the speaker in 2 of the following 3 **Star Wars** quotes.
 - "Aren't you a little short to be a stormtrooper?"
 - "I felt a great disturbance in the Force ... as if millions of voices suddenly cried out in terror and were suddenly silenced."
 - "I recognized your foul stench when I was brought on board."

Real-World Languages

- This Indo-European language is spoken by about 100 million people (it's the most common first language in the EU). It uses an extended Latin alphabet, inflects nouns into cases (nominative, genitive, dative and accusative), and features three genders. Verb and noun inflection allow for a flexible word order. Nobel Prize winner Hermann Hesse wrote in this language.

Let with Initialization

Now consider **let** with initialization:

$$\frac{\begin{array}{l} O \vdash e_0 : T_0 \\ O[T_0/x] \vdash e_1 : T_1 \end{array}}{O \vdash \text{let } x : T_0 \leftarrow e_0 \text{ in } e_1 : T_1} \text{ [Let-Init]}$$

This rule is weak. Why?



Let with Initialization

- Consider the example:

```
class C inherits P { ... }
```

```
...
```

```
let x : P ← new C in ...
```

```
...
```

- The previous let rule does not allow this code
 - We say that the rule is **too weak** or **incomplete**

Subtyping

- Define a relation $X \leq Y$ on classes to say that:
 - An object of type X could be used when one of type Y is acceptable, or equivalently
 - X conforms with Y
 - In Cool this means that X is a **subclass** of Y
- Define a relation \leq on classes
 - $X \leq X$
 - $X \leq Y$ if X inherits from Y
 - $X \leq Z$ if $X \leq Y$ and $Y \leq Z$

Let With Initialization (Better)

$$O \vdash e_0 : T$$

$$T \leq T_0$$

$$O[T_0/x] \vdash e_1 : T_1$$

$$O \vdash \text{let } x : T_0 \leftarrow e_0 \text{ in } e_1 : T_1 \quad \text{[Let-Init]}$$

- Both rules for let are **sound**
- But more programs type check with this new rule (it is more **complete**)

Type System Tug-of-War

- There is a tension between
 - Flexible rules that do **not constrain** programming
 - Restrictive rules that **ensure safety** of execution



Expressiveness of Static Type Systems

- A **static** type system enables a compiler to **detect** many common programming **errors**
- The cost is that some correct programs are **disallowed**
 - Some argue for dynamic type checking instead
 - Others argue for more expressive static type checking
- But more expressive type systems are also more **complex**

Dynamic And Static Types

- The **dynamic type** of an object is the class **C** that is used in the “**new C**” expression that creates the object
 - A **run-time** notion
 - Even languages that are not statically typed have the notion of dynamic type
- The **static type** of an expression is a notation that captures all possible dynamic types the expression could take
 - A **compile-time** notion

Dynamic and Static Types. (Cont.)

- In early type systems the set of static types correspond **directly** with the dynamic types
- **Soundness theorem**: for all expressions E
$$\text{dynamic_type}(E) = \text{static_type}(E)$$

(in all executions, E evaluates to values of the type inferred by the compiler)
- This gets more complicated in advanced type systems (e.g., Java, Cool)

Dynamic and Static Types in COOL

```
class A { ... }  
class B inherits A {...}  
class Main {  
  A x ← new A; ← Here, x's value has  
  ...           dynamic type A  
  x ← new B; ← Here, x's value has  
  ...           dynamic type B  
}
```

x has static type **A**

- A variable of static type **A** can hold values of static type **B**, if $B \leq A$

Dynamic and Static Types

Soundness theorem for the **Cool type system**:

$$\forall E. \text{dynamic_type}(E) \leq \text{static_type}(E)$$

Why is this Ok?

- For E , compiler uses $\text{static_type}(E)$
- All operations that can be used on an object of type C can also be used on an object of type $C' \leq C$
 - Such as fetching the value of an attribute
 - Or invoking a method on the object
- Subclasses can **only add** attributes or methods
- Methods can be redefined but with the same types!

Subtyping Example

- Consider the following Cool class definitions

```
Class A { a() : int { 0 }; }
```

```
Class B inherits A { b() : int { 1 }; }
```

- An instance of **B** has methods “a” and “b”
- An instance of **A** has method “a”
 - A type error occurs if we try to invoke method “b” on an instance of **A**

Example of Wrong Let Rule (1)

- Now consider a hypothetical **wrong** let rule:

$$\frac{O \vdash e_0 : T \quad T \leq T_0 \quad O \vdash e_1 : T_1}{O \vdash \text{let } x : T_0 \leftarrow e_0 \text{ in } e_1 : T_1}$$

- How is it different from the correct rule?



Example of Wrong Let Rule (1)

- Now consider a hypothetical **wrong** let rule:

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- How is it different from the correct rule?
- The following good program does *not* typecheck:

`let x : Int ← 0 in x + 1`

Why?

Example of Wrong Let Rule (2)

- Now consider a hypothetical **wrong** let rule:

$$\frac{O \vdash e_0 : T \quad T_0 \leq T \quad O[T_0/x] \vdash e_1 : T_1}{O \vdash \text{let } x : T_0 \leftarrow e_0 \text{ in } e_1 : T_1}$$

- How is it different from the correct rule?

Example of Wrong Let Rule (2)

- Now consider a hypothetical **wrong** let rule:

$$\frac{O \vdash e_0 : T \quad T_0 \leq T \quad O[T_0/x] \vdash e_1 : T_1}{O \vdash \text{let } x : T_0 \leftarrow e_0 \text{ in } e_1 : T_1}$$

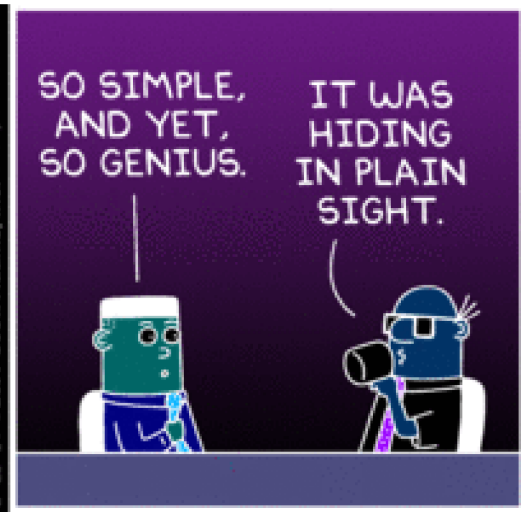
- How is it different from the correct rule?
- The following *bad* program is well typed:
$$\text{let } x : B \leftarrow \text{new } A \text{ in } x.b()$$
- Why is this program bad?

Example of Wrong Let Rule (3)

- Now consider a hypothetical **wrong** let rule:

$$\frac{O \vdash e_0 : T \quad T \leq T_0 \quad O[T/x] \vdash e_1 : T_1}{O \vdash \text{let } x : T_0 \leftarrow e_0 \text{ in } e_1 : T_1}$$

- How is it different from the correct rule?



Example of Wrong Let Rule (3)

- Now consider a hypothetical **wrong** let rule:

$$\frac{O \vdash e_0 : T \quad T \leq T_0 \quad O[T/x] \vdash e_1 : T_1}{O \vdash \text{let } x : T_0 \leftarrow e_0 \text{ in } e_1 : T_1}$$

- How is it different from the correct rule?
- The following good program is *not* well typed

let $x : A \leftarrow \text{new } B$ **in** { ... $x \leftarrow \text{new } A$; $x.a()$; }

- Why is this program not well typed?

Typing Rule Notation

- The typing rules use **very concise** notation
- They are very carefully constructed
- Virtually any change in a rule either:
 - Makes the type system **unsound**
(bad programs are accepted as well typed)
 - Or, makes the type system less usable (**incomplete**)
(good programs are rejected)
- But some good programs will be rejected anyway
 - The notion of a good program is **undecidable**

Assignment

More uses of subtyping:

$$O(\text{id}) = T_0$$

$$O \vdash e_1 : T_1$$

$$T_1 \leq T_0$$

$$\frac{}{O \vdash \text{id} \leftarrow e_1 : T_1} \text{[Assign]}$$

Initialized Attributes

- Let $O_c(x) = T$ for all attributes $x:T$ in class C
 - O_c represents the class-wide scope
- Attribute initialization is similar to **let**, except for the scope of names

$$O_c(\text{id}) = T_0$$

$$O_c \vdash e_1 : T_1$$

$$T_1 \leq T_0$$

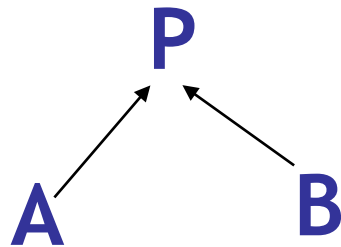
$$\frac{}{O_c \vdash \text{id} : T_0 \leftarrow e_1 ;} \text{ [Attr-Init]}$$

If-Then-Else

- Consider:
if e_0 then e_1 else e_2 fi
- The result can be either e_1 or e_2
- The dynamic type is either e_1 's or e_2 's type
- The best we can do statically is the **smallest supertype** larger than the type of e_1 and e_2

If-Then-Else example

- Consider the class hierarchy



- ... and the expression
 if ... then new A else new B fi
- Its type should allow for the dynamic type to be both A or B
 - Smallest supertype is P

Least Upper Bounds

- Define: $\text{lub}(X, Y)$ to be the **least upper bound** of X and Y . $\text{lub}(X, Y)$ is Z if
 - $X \leq Z \wedge Y \leq Z$
 Z is an upper bound
 - $X \leq Z' \wedge Y \leq Z' \Rightarrow Z \leq Z'$
 Z is least among upper bounds
- In Cool, the least upper bound of two types is their **least common ancestor** in the **inheritance tree**

If-Then-Else Revisited

$$O \vdash e_0 : \text{Bool}$$
$$O \vdash e_1 : T_1$$
$$O \vdash e_2 : T_2$$

$$O \vdash \text{if } e_0 \text{ then } e_1 \text{ else } e_2 \text{ fi} : \text{lub}(T_1, T_2)$$

[If-Then-Else]

Case

- The rule for **case** expressions takes a lub over all branches

$$O \vdash e_0 : T_0$$

$$O[T_1/x_1] \vdash e_1 : T_1'$$

...

$$O[T_n/x_n] \vdash e_n : T_n'$$

$$\frac{}{O \vdash \text{case } e_0 \text{ of } x_1:T_1 \Rightarrow e_1; \dots; x_n : T_n \Rightarrow e_n; \text{ esac} : \text{lub}(T_1', \dots, T_n')} \text{ [Case]}$$

Next Time

- Type checking method dispatch
- Type checking with `SELF_TYPE` in COOL



Homework

- PA4c Due (well, it should be)
- CA3 Due