Dependent Type Systems

(saying what you are)

Data Abstraction

(hiding what you are)

Review

- We studied a variety of type systems
- We repeatedly made the type system more expressive to enable the type checker to catch more errors
- But we have steered clear of undecidable systems
  - Thus there must still be many errors that are not caught
- Now we explore more complex type systems that bring type checking closer to program verification

Proximate Cause

- Theorem proving is quite useful and can determine if things are true or false: “your file system can segfault” or “this formula is satisfiable”
- However, we also want theorem provers to provide checkable proofs to back up what they decide
- Fortunately, “proof checking is equivalent to type checking in a dependent type system”

A **dependent type** is a type that depends on a value.

Dependent Types

- Say that we have the functions

  zero : nat $\rightarrow$ vector (creates vector of requested length)

  dotprod : vector $\rightarrow$ vector $\rightarrow$ real (dot product)

- The types do not prevent using dotprod on vectors of different length

```
let v1 = zero 5 in
let v2 = zero 15 in
print (dotprod v1 v2)
```

Dependent Types

- Say that we have the functions

  zero : nat → vector (creates vector of requested length)

  dotprod : vector → vector → real (dot product)

- The types do not prevent using dotprod on **vectors of different length**
  - If they did, we could catch more bugs!
- Idea: Make "vector" a type family annotated by a natural number

  "vector n" is the type of vectors of length n

  dotprod: vector n → vector n → real   (where is n bound?)

  zero : nat → vector ?

Need a way to refer to the *value* of the first argument in the type!

Dependent Type Notation

- How to write the type of zero : nat $\rightarrow$ vector ?
- Given two sets A and B verify the isomorphism

$$A \rightarrow B \simeq \Pi_{x \in A} B$$

  - The latter is the cartesian product of B with itself as many times as there are elements in A
  - Also written as Πx:A.B (x plays no role so far!)
  - But now we can make B depend on x!
- Definition: Πx:A.B is the type of functions with argument in A and with the result type B (possibly depending on the value of the argument x in A)
  - We write "zero : Πx:nat. vector x"
  - Special case when x$\notin$B we abbreviate as A $\rightarrow$ B
  - We play "fast and loose" with the binding of Π

Dependent Typing Rules

$$\frac{\Gamma, x : \tau_2 \vdash e : \tau}{\Gamma \vdash \lambda x : \tau_2 . e : \Pi x : \tau_2 . \tau} \qquad \frac{\Gamma \vdash e_1 : \Pi x : \tau_2 . \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 \; e_2 : [e_2/x]\tau}$$

- Note that expressions are now part of types
- Have types like “vector 5” and “vector (2 + 3)”
- We need type equivalence

$$\frac{\Gamma \vdash e : \tau \quad \Gamma \vdash \tau \equiv \tau'}{\Gamma \vdash e : \tau'}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2}{\Gamma \vdash \texttt{vector}\; e_1 \equiv \texttt{vector}\; e_2}$$

Dependent Types and Program Specifications

- Types act as specifications
- With dependent types we can specify *any property!*
- For example, define the following types:

  "eq e" - the type of values equal to "e".

  Also named "sng e" (the singleton type)

  "ge e" – the type of values larger or equal to "e"

  "lt e" – the type of values smaller than "e"

  "and τ_1 τ_2" – the type of values having both type τ_1 and τ_2
- Need appropriate typing rules for the new types
- The precondition for vector-accessing (cf. HW5)
  - read: Πn:nat.vector n $\to$ (and (ge 0) (lt n)) $\to$ int
- The type checker must do program verification

Dependent Type Commentary

- Type checking with Π types can be *as hard as full program verification*
- Type equivalence can be undecidable
  - If types are dependent on expressions drawn from a powerful language ("powerful" = "arithmetic")
  - Then even type checking will be undecidable
- Dependent types play an important role in the formalization of logics
  - Started with Per Martin-Lof
  - Proof checking via type checking
  - Proof-carrying code uses a dependent type checker to check proofs
  - There are program specification tools based on Π types

Dependent Sum Types

- We want to pack a vector with its length
  - e = (n, v) where "v : vector n"
  - The type of an element of a pair depends on the *value* of another element
  - This is another form of dependency
  - The type of e is "nat × vector ?"
- Given two sets A and B verify the isomorphism

$$\mathbf{A \times B \simeq \Sigma_{x \in A} B}$$

  - The latter is the *disjoint* union of B with itself as many times as there are elements in A
  - Also written as $\Sigma x{:}A.B$ (x here plays no role)
  - But now we can make B depend on x!

Dependent Sum Types

- Definition: $\Sigma x{:}A.B$ is the type of pairs with first element of type A and second element of type B *(possibly depending on the **value** of first element x)*
  - Now we can write $e : \Sigma x{:}nat.\ vector\ x$
- Old functions that compute the length of a vector
  - vlength : $\Pi n{:}nat.vector\ n \to nat$
    - (the result is not constrained)
  - slength : $\Pi n{:}nat.vector\ n \to sng\ n$
    - "sng n" is a dependent type that contains only n
    - called the singleton type (recall from 3 slides ago …)
- What if the vector is packed with its length?
  - pvlength : $\Sigma n{:}nat.vector\ n \to nat$
  - pslength : $\Sigma n{:}nat.vector\ n \to sng\ n$

Dependent Sum Types Static Semantics

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : [e_1/x]\tau_2}{\Gamma \vdash (e_1, e_2) : \Sigma x : \tau_1.\tau_2}$$

$$\frac{\Gamma \vdash e : \Sigma x : \tau_1.\tau_2}{\Gamma \vdash \texttt{snd}\ e : [\texttt{fst}\ e/x]\tau_2}$$

- Note how this rule reduces to the usual rules for tuples when there is no dependency

- The evaluation rules are unchanged

Weimeric Commentary

- Dependant types seem obscure: why care?
- Grand Unified Theory
  - Type Checking = Verification (= Model Checking = Proof Checking = Abstract Interpretation …)
- CCured Project
  - Rumor has it this project was successful
  - The whole thing is dependant sum types
    - SEQ = (pointer + lower bound + upper bound)
    - FSEQ = (pointer + upper bound)
    - WILD = (pointer + lower bound + upper bound + rtti)

Proof Generation

- We want our theorem prover to **emit proofs**
  - No need to trust the prover
  - Can find bugs in the prover
  - Can be used for proof-carrying code
  - Can be used to extract invariants
  - Can be used to extract models (e.g., in SLAM)
- Implements the soundness argument
  - On every run, a soundness proof is constructed

Proof Representation

- Proofs are trees
  - Leaves are hypotheses/axioms
  - Internal nodes are inference rules
- Axiom: "true introduction"
  - Constant: truei : pf
  - pf is the type of proofs
- Inference: "conjunction introduction"
  - Constant: andi : pf → pf → pf
- Inference: "conjunction elimination"
  - Constant: andel : pf → Pf
- Problem:
  - "andel truei : pf" but does not represent a valid proof
  - Need a more powerful *type system that checks content*

$$\frac{}{\vdash \text{true}}\ \textit{truei}$$

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\ \textit{andi}$$

$$\frac{\vdash A \wedge B}{\vdash A}\ \textit{andel}$$

Proofs and Dependent Types

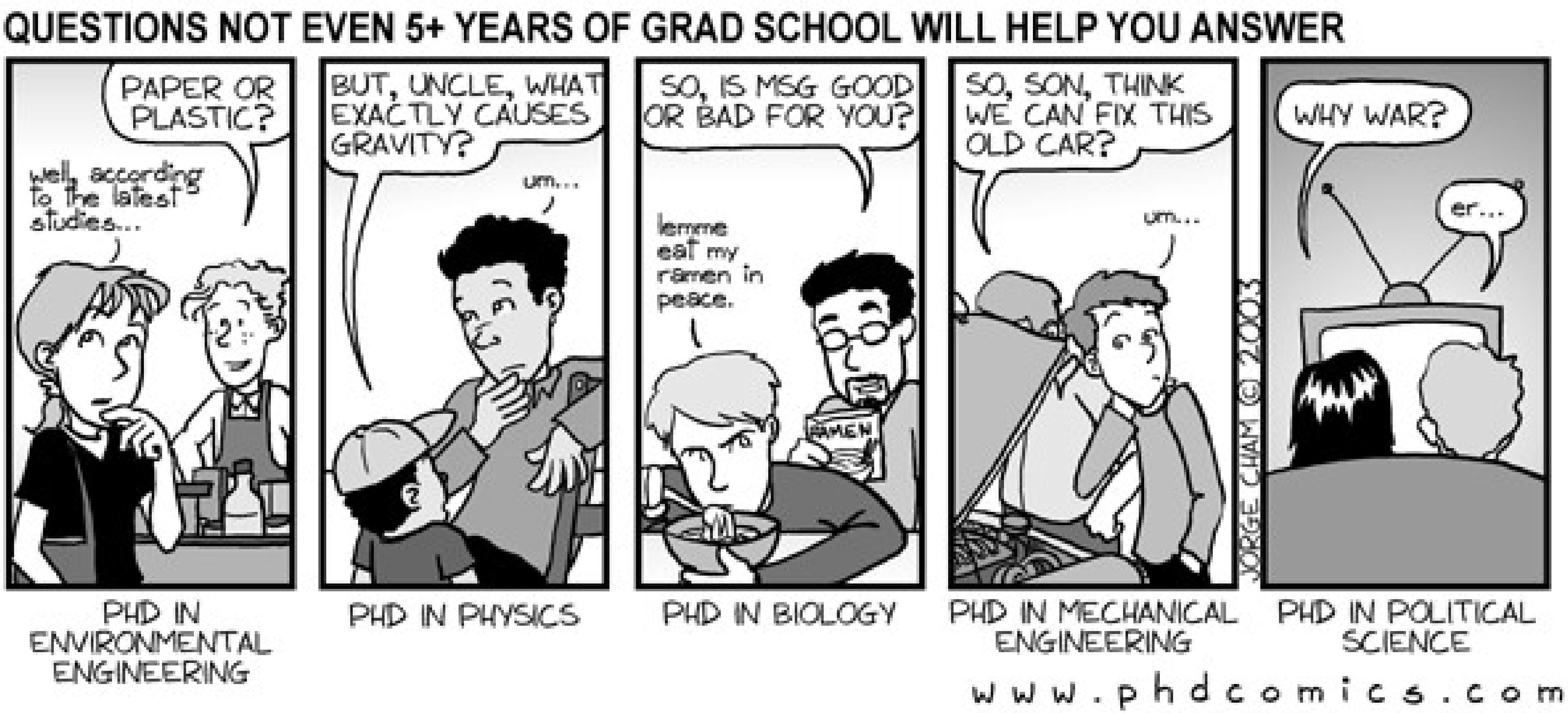
- Make pf a family of types indexed by formulas
  - f : Type (type of encodings of formulas)
  - e : Type (type of encodings of expressions)
  - pf : f → Type (the type of proofs indexed by formulas: it is a proof ***that f is true***)
- Examples:
  - true : f
  - and : f → f → f
  - truei : pf true
  - andi : pf A → pf B → pf (and A B)
  - **andi : ΠA:f. ΠB:f. pf A → pf B → pf (and A B)**
  - **(ΠA:f.X means "forall A of type f, dependent type X uses value A")**

Proof Checking

- Validate proof trees by type-checking them
- Given a proof tree X claiming to prove $A \wedge B$
- Must check X : pf (and A B)
- We use "expression tree equality", so
  - andel (andi "1+2=3" "x=y") does *not* have type pf (3=3)
  - This is already a proof system! If the proof-supplier wants to use the fact that $1+2=3 \Leftrightarrow 3=3$, she can include a proof of it somewhere!
- Thus Type Checking = Proof Checking
  - And it's quite easily *decidable*! □

Types for Data Abstraction

What's inside the implementation?
We don't know!



Q: Games (540 / 842)

- This seminal 1991 turn-based strategy computer game by Sid Meier of Microprose spawned an entire genre about micromanaging exploration, expansion and conflict.

Q: Books (754 / 842)

- Name the factory owner, the workers, and the newly-developed form of unending suckable candy in the 1964 children's book that features the title character finding a golden ticket and visiting the title chocolate factory.

Turing Award Winners

- This Ron-bearing golden trio is known best not for defeating Voldemort but for making one of the first practical public-key cryptosystems. A user publishes a public key based on two large, secret prime numbers. Anyone can use the public key to encrypt a message, but (ideally) only someone who knows the two secret prime numbers can decrypt it.

An **abstract data type** has a public name, a hidden representation, and operations to create, combine, and observe values of the abstraction.

Data Abstraction

- Ability to hide (abstract) concrete implementation details
- Modularity builds on data abstraction
- Improves program structure and minimizes dependencies
- One of the most influential developments of the 1970's
- Key element for much of the success of object orientation in the 1980's

Example of Abstraction

- Cartesian points (gotta love it!)
- Introduce the "abstype" language construct:

abstype point implements
mk : real × real → point
xc : point → real
yc : point → real
is
< point = real × real,
mk = λx. x,
xc = fst,
yc = snd >

- Shows a concrete implementation
- Allows the rest of the program to access the implementation *through an abstract interface*
- Only the interface need to be publicized
- Allows separate compilation

Data Abstraction

- It is useful to separate the creation of the abstract type and its use (newsflash …)
- Extend the syntax (t = imp, σ = interface):

  Terms ::= … | < t = τ, e : σ > | open e_a as t, x : σ in e_b

  Types ::= … | $\exists$t. σ
- The expression <t=τ, e : σ> takes the concrete implementation e and "packs it" as a value of an abstract type
  - Alternative notation: "pack e as $\exists$t. σ with t = τ"
  - "existential types" – used to model the stack, etc.
- The "open" expression allows e_b to access the abstract type expression e_a using the name x, the unknown type of the concrete implementation "t" and the interface σ

Example with Abstraction

- $C = \{mk = \lambda x.x,\ xc = fst,\ yc = snd\ \}$ is a *concrete implementation* of points as real × real
- We want to hide the type of the representation

  σ is the following type:

  { mk : real × real → point,
  xc : point → real, yc : point → real}
- Note that C : [real×real/point]σ
- A = <point=real×real, C : σ> is an expression of the abstract type $\exists$point.σ
- We want clients to access only the second component of A and just use the abstract name "point" for the first component:

  open A as point, P : σ in … P.xc(P.mk(1.0, 2.0)) …

Typing Rules for Existential Types

- We add the following typing rules:

$$\frac{\Gamma \vdash [\tau/t]e : [\tau/t]\sigma}{\Gamma \vdash \langle\, t = \tau, e : \sigma \,\rangle : \exists t.\sigma}$$

$$\frac{\Gamma \vdash e_a : \exists t.\sigma \qquad \Gamma, t, p : \sigma \vdash e_b : \tau}{\Gamma \vdash \texttt{open}\ e_a\ \texttt{as}\ t, p : \sigma\ \texttt{in}\ e_b : \tau} \qquad t \notin FV(\Gamma \cup \tau)$$

- The restriction in the rule for "open" ensures that t does not escape its scope

Evaluation Rules for Abstract Types

- We add a new form of value

  v ::= … | <t=τ, v : σ>

  - This is just like v but with some type decorations that make it have an existential type

$$\frac{e_a \Downarrow \langle\, t = \tau, v : \sigma \,\rangle \qquad [v/x][\tau/t]e_b \Downarrow v'}{\texttt{open}\ e_a\ \texttt{as}\ t, x : \sigma\ \texttt{in}\ e_b \Downarrow v'}$$

- At the time e_b is evaluated, abstract-type variables are replaced with concrete values
  - If we ignore the type issues "open e_a as t, x : σ in e_b" is like "let x : σ = e_a in e_b"
  - Difference: e_b *cannot know statically* what is the concrete type of x so it cannot take advantage of it

Abstract Types as a Specification Mechanism

- Just like polymorphism, existential types are mostly a type checking mechanism
- A function of type $\forall t.\ t\ List \rightarrow int$ does not know *statically* what is the type of the list elements. Therefore no operations are allowed on them
  - But it will have at run-time the actual value of t
  - "There are no type variables at run-time"
- Same goes for existentials
- These type mechanisms are a very powerful (and widely used!) form of static checking
  - Recall Wadler's "Theorems for Free"

Data Abstraction and the Real World

- Example: file descriptors
- Solution 1:
  - Represent file descriptors as “int” and export the interface {open:string→int, read:int→ data}
- An untrusted client of the interface calls “read”
- How can we know that “read” is invoked with a file descriptor that was obtained from “open”? Anyone?

Data Abstraction and the Real World

- Example: file descriptors
- Solution 1:
  - Represent file descriptors as "int" and export the interface {open:string→int, read:int→ data}
- An untrusted client of the interface calls "read"
- How can we know that "read" is invoked with a file descriptor that was obtained from "open"?
  - We must keep track of all integers that represent file descriptors
  - We design the interface such that all such integers are small integers and we can essentially keep a bitmap
  - This becomes expensive with more complex (e.g. pointer-based) representations

Data Abstraction, Static Checking

- Solution 2: Use the same representation but *export an abstraction* of it.
  - ∃fd. File   or
  - ∃fd. {open : string → fd, read : fd → data}
  - A possible value:
  - Fd = < fd = int, { open = …, read = …} : File> : ∃fd. File
- Now the *untrusted* client e

  open Fd as fd, x : File in e
- At run-time "e" can see that file descriptors are integers
  - But cannot cast 187 as a file descriptor.
  - Static checking with no run-time costs!
  - Catch: you must be able to type check e!

Modularity

- A module is a program fragment along with ***visibility constraints***
- Visibility of functions and data
  - Specify the function interface but hide its implementation
- Visibility of type definitions
  - More complicated because the type might appear in specifications of the visible functions and data
  - Can use data abstraction to handle this
- A module is represented as a type component and an implementation component

  $<t = \tau, e : \sigma>$ (where t can occur in e and σ)

  - even though the specification (σ) refers to the implementation type we can still hide the latter
  - But there are problems ...

Problems with Existentialists

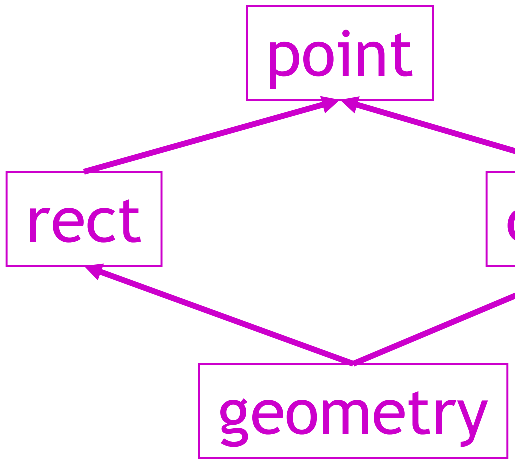
- Existentialist types
  - Assert that truth is subjectivity
  - Oppose the rational tradition and positivism
  - Are subject to an “absurd” universe
- Problems:
  - "In so far as Existentialism is a philosophical doctrine, it remains an idealistic doctrine: it hypothesizes specific historical conditions of human existence into ontological and metaphysical characteristics. Existentialism thus becomes part of the very ideology which it attacks, and its radicalism is illusory." (Herbert Marcuse, "Sartre's Existentialism", p. 161)

Problems with Existentials

- Existential types
  - Allow representation (type) hiding
  - Allow separate compilation. Need to know only the type of a module to compile its client
  - First-class modules. They can be selected at run-time. (cf. OO interface subtyping)
- Problems:
  - Closed scope. Must open an existential before using it!
  - Poor support for module hierarchies

Problems with Existentials (Cont.)

- There is an inherent tension between handling modules in isolation (good for separate compilation, interchangeability) and the need to integrate them



(the arrow means "depends on")

- Solution 1: open "point" at top level
  - Inversion of program structure
  - The most basic construct has the widest scope

Give Up Abstraction?

- Solution 2: incorporate point in rect and circle

  R = < point = …, <rect = point × point, …> … >

  C = < point = …, <circle = point × real, …> … >

- When we open R and C we get *two distinct notions of point*!
  - And we will *not* be able to combine them
- Another option is to allow the type checker to see the representation type
  - and thus give up representation hiding

Strong Sums

- New way to open a package

  Terms $e ::= \ldots \mid \text{Ops}(e)$

  Types $\tau ::= \ldots \Sigma t.\tau \mid \text{Typ}(e)$

  - Use Typ and Ops to decompose the module
  - Operationally, they are just like "fst" and "snd"

  $\Sigma t.\tau$ is the dependent sum type

  - It is like $\exists t.\tau$ except we can look at the type

$$\frac{\Gamma \vdash e : \Sigma t.\tau}{\Gamma \vdash \texttt{Ops}(e) : \tau[\texttt{Typ}(e)/t]}$$

Modularity with Strong Sums

- Consider the R and C defined as before:

  $Pt = \langle point = real \times real, \ldots\rangle : \Sigma point.\ \tau_P$

  $R = \langle point = Typ(Pt),$

  $\quad \langle rect = point \times point, \ldots\rangle : \Sigma rect.\ \tau_R$

  $C = \langle point = Typ(Pt),$

  $\quad \langle circle = point \times real, \ldots\rangle : \Sigma circle.\ \tau_C$

- Since we use strong-sums the type checker sees that the two point types are the same

Modules with Strong Sums

- ML's module system is based on strong sums

Problems:

- Poorer data abstraction
- Expressions appear in types (Typ(e))
  - Types might not be known until at run time
  - Lost separate compilation
  - Trouble if e has side-effects (but we can use a value restriction – e.g., "IntSet.t")
- Second-class modules (because of value restriction)
- We can combine existentials with strong sums
  - Translucent sums: partially visible

Homework

- Project!
  - You have ~22 days (including holidays) to complete it.
  - Need help? Stop by my office or send email.