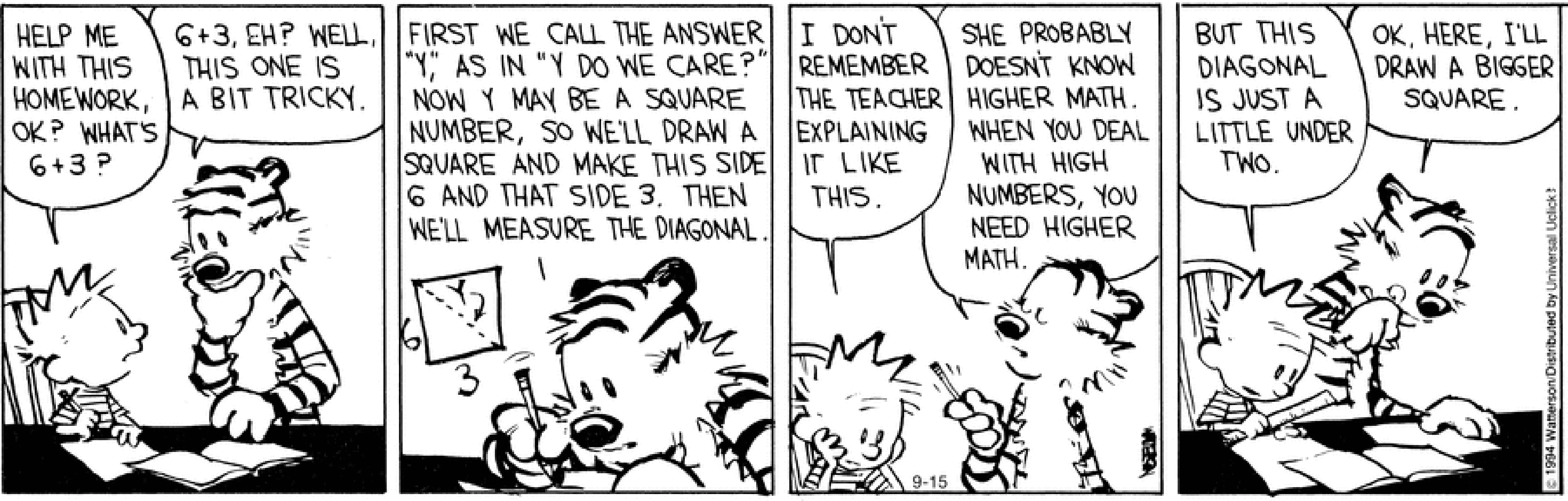
# Automated Theorem Proving: DPLL and Simplex



# One-Slide Summary

- An **automated theorem prover** is an algorithm that determines whether a mathematical or logical proposition is **valid** (**satisfiable**).
- A **satisfying** or **feasible assignment** maps variables to values that satisfy given constraints. A theorem prover typically produces a proof or a satisfying assignment (e.g., a counter-example backtrace).
- The **DPLL** algorithm uses efficient heuristics (involving "pure" or "unit" variables) to solve **Boolean Satisfiability** (SAT) quickly in practice.
- The **Simplex** algorithm uses efficient heuristics (involving visiting feasible corners) to solve **Linear Programming** (LP) quickly in practice.

# Why Bother?

- I am loathe to teach you anything that I think is a waste of your time.
- The use of "constraint solvers" or "SMT solvers" or "automated theorem provers" is becoming endemic in PL, SE and Security research, among others.
- Many high-level analyses and transformations call Chaff, Z3 or Simplify (etc.) as a black box single step.

# Recent Examples

- "VeriCon uses first-order logic to specify admissible network topologies and desired network-wide invariants, and then implements classical Floyd-Hoare-Dijkstra **deductive verification using Z3**."
  - VeriCon: Towards Verifying Controller Programs in Software-Defined Networks, PLDI 2014
- "However, the search strategy is very different: our synthesizer fills in the holes using component-based synthesis (as opposed to **using SAT/SMT solvers**)."
  - Test-Driven Synthesis, PLDI 2014
- "If the terms *l*, *m*, and *r* were of type *nat*, this **theorem is solved automatically** using Isabelle/HOL's built-in *auto* tactic."
  - Don't Sweat the Small Stuff: Formal Verification of C Code Without the Pain, PLDI 2014

# Desired Examples

- SLAM
  - Given “new = old” and “new++”, can we conclude “new = old”?
  - $(new_0 = old_0) \wedge (new_1 = new_0 + 1) \wedge (old_1 = old_0) \Rightarrow (new_1 = old_1)$
- Division By Zero
  - IMP: “print x/((x*x)+1)”
  - $(n_1 = (x * x) + 1) \Rightarrow (n_1 \neq 0)$

# Incomplete

- Unfortunately, we can't have nice things.
- **Theorem (Godel, 1931)**. No consistent system of axioms whose theorems can be listed by an algorithm is capable of proving all truths about relations of the natural numbers.
- But we can profitably restrict attention to *some* relations about numbers.

# Desired Ex…

To make progress,
we will treat “pure logic”
and “pure math”
separately.

- SLAM
  - Given “new … can we conclude “new = old”?
  - $(new_0 = old_0) \wedge (new_1 = new_0 + 1) \wedge (old_1 = old_0) \Rightarrow (new_1 = old_1)$
- Division By Zero
  - IMP: “print x/((x*x)+1)”
  - $(n_1 = (x * x) + 1) \Rightarrow (n_1 \neq 0)$

# Overall Plan

- Satisfiability

- Simple SAT Solving
- Practical Heuristics
- DPLL algorithm for SAT

Logic

- Linear programming
- Graphical Interpretation
- Simplex algorithm

Math

# Boolean Satisfiability

- Start by considering a simpler problem: propositions involving only **boolean** variables

$$
\begin{aligned}
\text{bexp} := \;& x \\
|\;\; & \text{bexp} \wedge \text{bexp} \\
|\;\; & \text{bexp} \vee \text{bexp} \\
|\;\; & \neg\, \text{bexp} \\
|\;\; & \text{bexp} \Rightarrow \text{bexp} \\
|\;\; & \text{true} \mid \text{false}
\end{aligned}
$$

- Given a bexp, return a satisfying assignment or indicate that it cannot be satisfied

# Satisfying Assignment

- A **satisfying assignment** maps boolean variables to boolean values.
- Suppose $\sigma(x)$ = true and $\sigma(y)$ = false
- $\sigma \models x$ // $\models$ = "models" or "makes
- $\sigma \models x \lor y$ // true" or "satisfies"
- $\sigma \models y \Rightarrow \neg x$
- $\sigma \not\models x \Rightarrow (x \Rightarrow y)$
- $\sigma \not\models \neg x \lor y$

# Cook-Levin Theorem

- **Theorem (Cook-Levin). The boolean satisfiability problem is NP-complete.**
- In '71, Cook published “The complexity of theorem proving procedures”. Karp followed up in '72 with “Reducibility among combinatorial problems”.
  - Cook and Karp received Turing Awards.
- SAT is in NP: verify the satisfying assignment
- SAT is NP-Hard: we can build a boolean expression that is satisfiable iff a given nondeterministic Turing machine accepts its given input in polynomial time

# Conjunctive Normal Form

- Let's make it easier (but still NP-Complete)
- A **literal** is "variable" or "negated variable"

  $x$ $\neg y$

- A **clause** is a disjunction of literals

  $(x \lor y \lor \neg z)$ $(\neg x)$

- **Conjunctive normal form** (CNF) is a conjunction of clauses

  $(x \lor y \lor \neg z) \land (\neg x \lor \neg y) \land (z)$

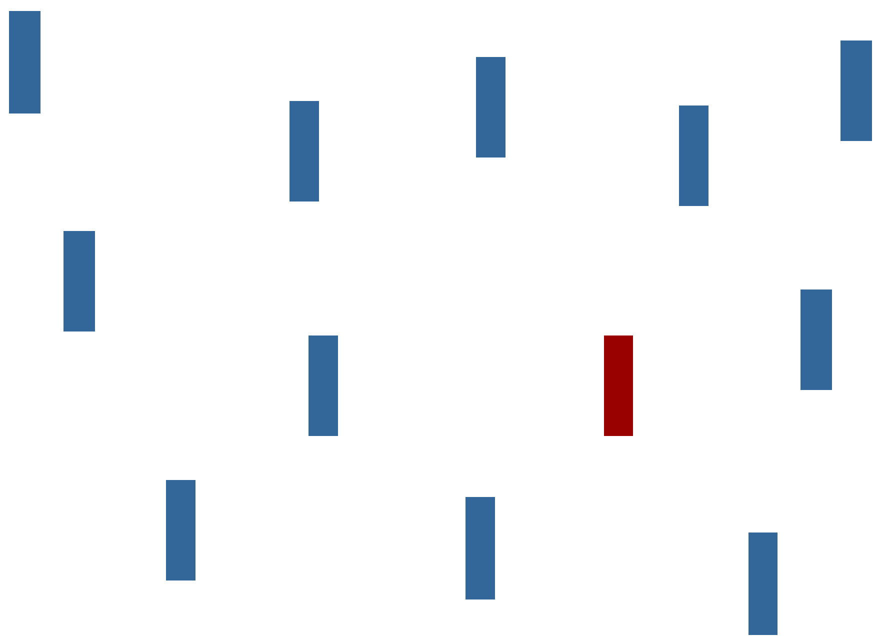
- Must satisfy all clauses at once
  - "global" constraints!

# SAT Solving Algorithms

$$\exists \sigma.\ \sigma \models (x \vee y \vee \neg z) \ \wedge \ (\neg x \vee \neg y) \ \wedge \ (z)$$

- So how do we solve it?
- Ex: $\sigma(x) = \sigma(z) = \text{true}$, $\sigma(y) = \text{false}$
- Expected running time?

# Analogy: Human Visual Search
# “Find The Red Vertical Bar”



# Human Visual Search
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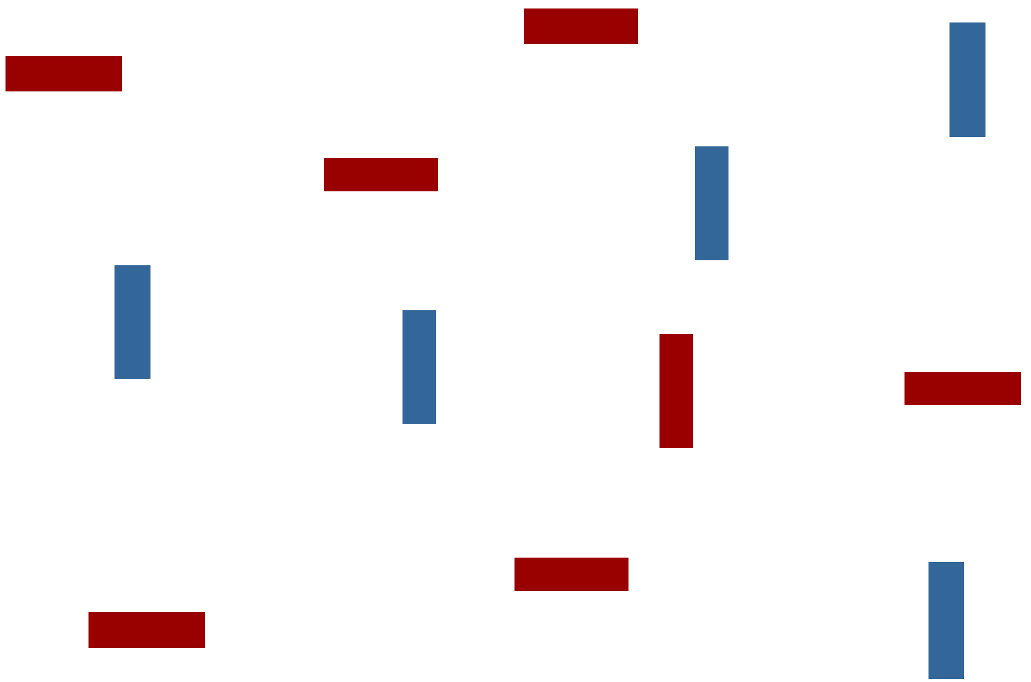
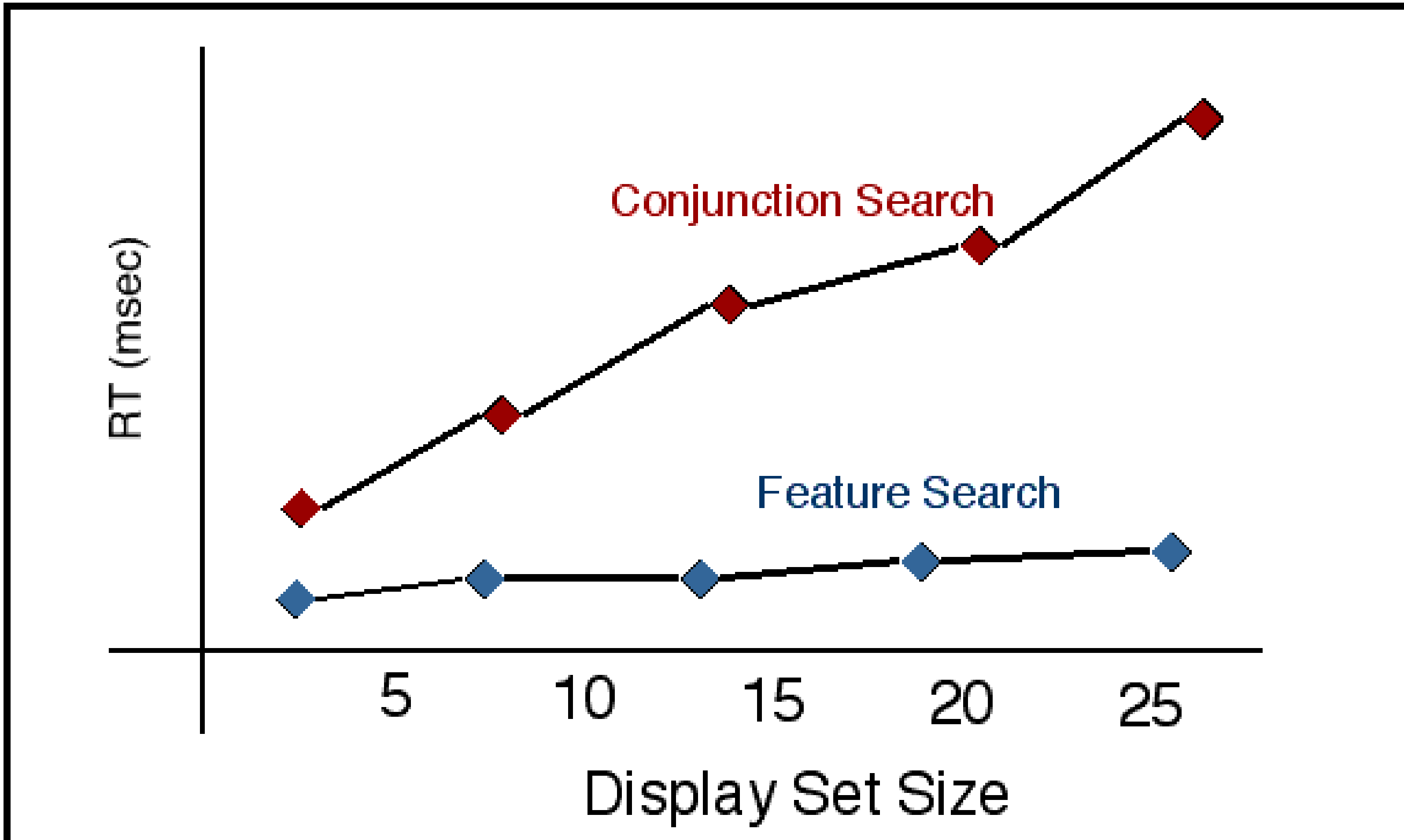


# Some Visual Features Admit O(1) Detection



# Strangers On A Train

- https://www.youtube.com/watch?v=_tVFwhoeQVM

# Think Fast: Partial Answer?

$$(\neg a \lor \neg b \lor \neg c \lor d \lor e \lor \neg f \lor g \lor \neg h \lor \neg i)$$
$$\land\ (\neg a \lor b \lor \neg c \lor d \lor \neg e \lor f \lor \neg g \lor h \lor \neg i)$$
$$\land\ (a \lor \neg b \lor \neg c \lor \neg d \lor e \lor \neg f \lor \neg g \lor \neg h \lor i)$$
$$\land\ (\neg b)$$
$$\land\ (a \lor \neg b \lor c \lor \neg d \lor e \lor \neg f \lor \neg g \lor \neg h \lor i)$$
$$\land\ (\neg a \lor b \lor \neg c \lor d \lor \neg e \lor f \lor \neg g \lor h \lor \neg i)$$

- If this instance is satisfiable, what *must* part of the satisfying assignment be?

# Think Fast: Partial Answer?

$(\neg a \vee \neg b \vee \neg c \vee d \vee e \vee \neg f \vee g \vee \neg h \vee \neg i)$

$\wedge\ (\neg a \vee b \vee \neg c \vee d \vee \neg e \vee f \vee \neg g \vee h \vee \neg i)$

$\wedge\ (a \vee \neg b \vee \neg c \vee \neg d \vee e \vee \neg f \vee \neg g \vee \neg h \vee i)$

$\wedge\ (\neg b)$

$\wedge\ (a \vee \neg b \vee c \vee \neg d \vee e \vee \neg f \vee \neg g \vee \neg h \vee i)$

$\wedge\ (\neg a \vee b \vee \neg c \vee d \vee \neg e \vee f \vee \neg g \vee h \vee \neg i)$

- If this instance is satisfiable, what *must* part of the satisfying assignment be? **b = false**

# Need For Speed 2

$$
\begin{aligned}
& (\neg a \lor c \lor \neg d \lor e \lor f \lor \neg g \lor \neg h \lor \neg i) \\
\land\ & (\neg a \lor b \lor \neg c \lor d \lor \neg e \lor f \lor g \lor h \lor i) \\
\land\ & (\neg a \lor \neg b \lor c \lor e \lor f \lor g \lor \neg h \lor i) \\
\land\ & (\neg a \lor b \lor c \lor d \lor e \lor \neg f \lor \neg g \lor h \lor \neg i) \\
\land\ & (b \lor \neg c \lor \neg d \lor e \lor \neg f \lor g \lor h \lor \neg i) \\
\land\ & (\neg a \lor b \lor c \lor d \lor \neg g \lor \neg h \lor \neg i)
\end{aligned}
$$

- If this instance is satisfiable, what *must* part of the satisfying assignment be?

# Need For Speed 2

$$(\neg a \lor c \lor \neg d \lor e \lor f \lor \neg g \lor \neg h \lor \neg i)$$
$$\land (\neg a \lor b \lor \neg c \lor d \lor \neg e \lor f \lor g \lor h \lor i)$$
$$\land (\neg a \lor \neg b \lor c \lor e \lor f \lor g \lor \neg h \lor i)$$
$$\land (\neg a \lor b \lor c \lor d \lor e \lor \neg f \lor \neg g \lor h \lor \neg i)$$
$$\land (b \lor \neg c \lor \neg d \lor e \lor \neg f \lor g \lor h \lor \neg i)$$
$$\land (\neg a \lor b \lor c \lor d \lor \neg g \lor \neg h \lor \neg i)$$

- If this instance is satisfiable, what *must* part of the satisfying assignment be? **a = false**

# Unit and Pure

- A **unit clause** contains only a single literal.
  - Ex: $(x)$ $(\neg y)$
  - Can only be satisfied by making that literal true.
  - Thus, there is no choice: just do it!
- A **pure variable** is either "always $\neg$ negated" or "never $\neg$ negated".
  - Ex: $(\neg x \vee y \vee \neg z) \wedge (\neg x \vee \neg y) \wedge (z)$
  - Can only be satisfied by making that literal true.
  - Thus, there is no choice: just do it!

# Unit Propagation

- If X is a literal in a unit clause, add X to that satisfying assignment and replace X with "true" in the input, then simplify:

1. $(\neg x \lor y \lor \neg z) \ \land \ (\neg x \lor \neg z) \ \land \ (z)$
2. identify "z" as a unit clause
3. $\sigma$ += "z = true"

# Unit Propagation

- If X is a literal in a unit clause, add X to that satisfying assignment and replace X with "true" in the input, then simplify:
  1. $(\neg x \vee y \vee \neg z) \wedge (\neg x \vee \neg z) \wedge (z)$
  2. identify "z" as a unit clause
  3. $\sigma$ += "z = true"
  4. $(\neg x \vee y \vee \neg true) \wedge (\neg x \vee \neg true) \wedge (true)$

# Unit Propagation

- If X is a literal in a unit clause, add X to that satisfying assignment and replace X with "true" in the input, then simplify:

1. $(\neg x \lor y \lor \neg z) \land (\neg x \lor \neg z) \land (z)$
2. identify "z" as a unit clause
3. σ += "z = true"
4. $(\neg x \lor y \lor \neg \text{true}) \land (\neg x \lor \neg \text{true}) \land (\text{true})$
5. $(\neg x \lor y) \qquad\qquad \land (\neg x)$

- Profit! Let's keep going ...

# Unit Propagation FTW

5. $(\neg x \vee y) \quad \wedge \quad (\neg x)$
6. Identify "$\neg x$" as a unit clause
7. σ += "$\neg x$ = true"
8. $(\text{true} \vee y) \quad \wedge \quad (\text{true})$
9. done!

$\{z, \neg x\} \models (\neg x \vee y \vee \neg z) \;\wedge\; (\neg x \text{ or } \neg z) \;\wedge\; (z)$

# Pure Variable Elimination

- If V is a variable that is always used with one polarity, add it to the satisfying assignment and replace V with "true", then simplify.

  1. $(\neg x \lor \neg y \lor \neg z) \land (x \lor \neg y \lor z)$
  2. identify "$\neg y$" as a pure literal

# Pure Variable Elimination

- If V is a variable that is always used with one polarity, add it to the satisfying assignment and replace V with "true", then simplify.
  1. $(\neg x \lor \neg y \lor \neg z) \;\land\; (x \lor \neg y \lor z)$
  2. identify "$\neg y$" as a pure literal
  3. $(\neg x \lor \text{true} \lor \neg z) \;\land\; (x \lor \text{true} \lor z)$
  4. Done.

# DPLL

- The **Davis-Putnam-Logemann-Loveland** (DPLL) algorithm is a complete decision procedure for CNF SAT based on:
  - Identify and propagate *unit* clauses
  - Identify and propagate *pure* literals
  - If all else fails, exhaustive *backtracking* search
- It builds up a partial satisfying assignment over time.

DP '60: “A Computing Procedure for Quantification Theory”

DLL '62: “A Machine Program for Theorem Proving”

# DPLL Algorithm

```
let rec dpll (c : CNF) (σ : model) : model option =
  if σ ⊨ c then                        (* polytime *)
    return Some(σ)                      (* we win! *)
  else if ( ) in c then                 (* empty clause *)
    return None                         (* unsat *)
  let u = unit_clauses_of c in
  let c, σ = fold unit_propagate (c, σ) u in
  let p = pure_literals_of c in
  let c, σ = fold pure_literal_elim (c, σ) p in
  let x = choose ((literals_of c) – (literals_of σ)) in
  return (dpll (c ∧ x) σ) or (dpll (c ∧ ¬x) σ)
```

# DPLL Example

$(x \vee \neg z) \wedge (\neg x \vee \neg y \vee z) \wedge (w) \wedge (w \vee y)$

- Unit clauses: $(w)$

$(x \vee \neg z) \wedge (\neg x \vee \neg y \vee z)$

- Pure literals: $\neg y$

$(x \vee \neg z)$

- Choose unassigned: $x$ (recursive call)

$(x \vee \neg z) \wedge (x)$

- Unit clauses: $(x)$
- Done! $\sigma=\{w, \neg y, x\}$

# SAT Conclusion

- DPLL is commonly used by award-winning SAT solvers such as Chaff and MiniSAT
- Not explained here: how you “choose” an unassigned literal for the recursive call
  - This “branching literal” is the subject of many papers on heuristics
- Very recent: specialize a MiniSAT solver to a particular problem class

  Justyna Petke, Mark Harman, William B. Langdon, Westley Weimer: **Using Genetic Improvement & Code Transplants to Specialise a C++ Program to a Problem Class.** European Conference on Genetic Programming (EuroGP) 2014 (silver human competitive award)

# Q: Computer Science

- This American mathematician and scientist developed the simplex algorithm for solving linear programming problems. In 1939 he arrived late to a graduate stats class at UC Berkeley where Professor Neyman had written two famously unsolved problems on the board. The student thought the problems “seemed a little harder than usual” but a few days later handed in complete solutions, believing them to be homework problems overdue. This real-life story inspired the introductory scene in *Good Will Hunting*.

# Linear Programming

- Example Goal:
  - Find X such that $X > 5 \wedge X < 10 \wedge 2X = 16$
- Let $x_1 \ldots x_n$ be real-valued variables
- A satisfying assignment (or **feasible solution**) is a mapping from variables to reals satisfying all available constraints
- Given a set of linear constraints and a linear objective function to maximize, **Linear Programming** (LP) finds a feasibile solution that maximizes the objective function.

# Linear Programming Instance

- Maximize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$
- Subject to
$$a_{11}x_1 + a_{12}x_2 + \ldots \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots \leq b_2$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots \leq b_n$$
$$x_1 \geq 0, \ldots, x_n \geq 0$$

- Don't “need” the objective function
- Don't “need” $x_1 \geq 0$

# 2D Running Example

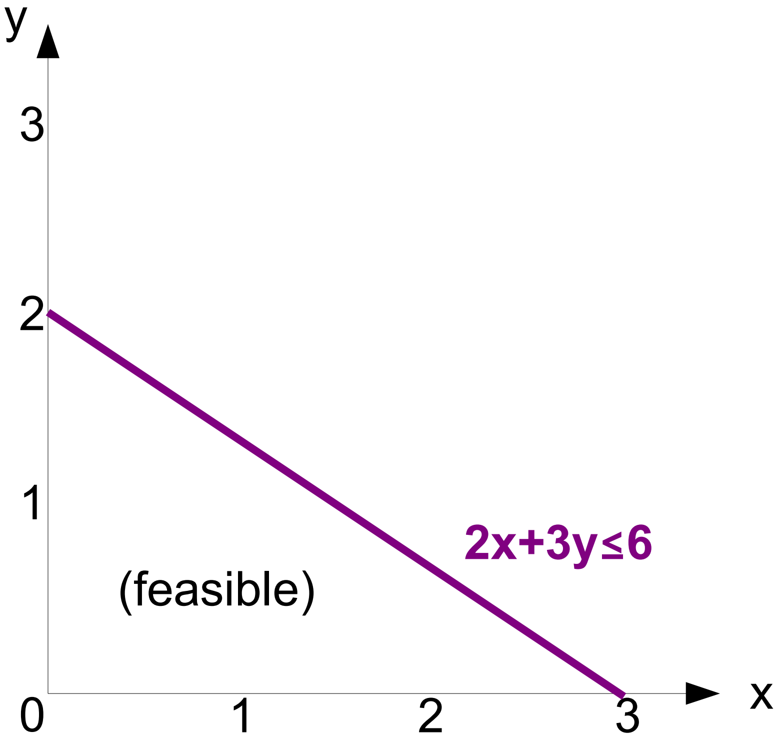
- Maximize $4x + 3y$
- Subject to

$$
\begin{array}{rrll}
2x & +3y & \le 6 & \quad (1) \\
 & 2y & \le 5 & \quad (2) \\
2x & +1y & \le 4 & \quad (3) \\
\multicolumn{3}{l}{x \ge 0,\ y \ge 0} &
\end{array}
$$

- Feasible: (1,1) or (0,0)
- Infeasible: (1,-1) or (1,2)

# Key Insight

- Each linear constraint (e.g., $2x+3y \leq 6$) corresponds to a **half-plane**
  - A feasible half-plane and an infeasible one



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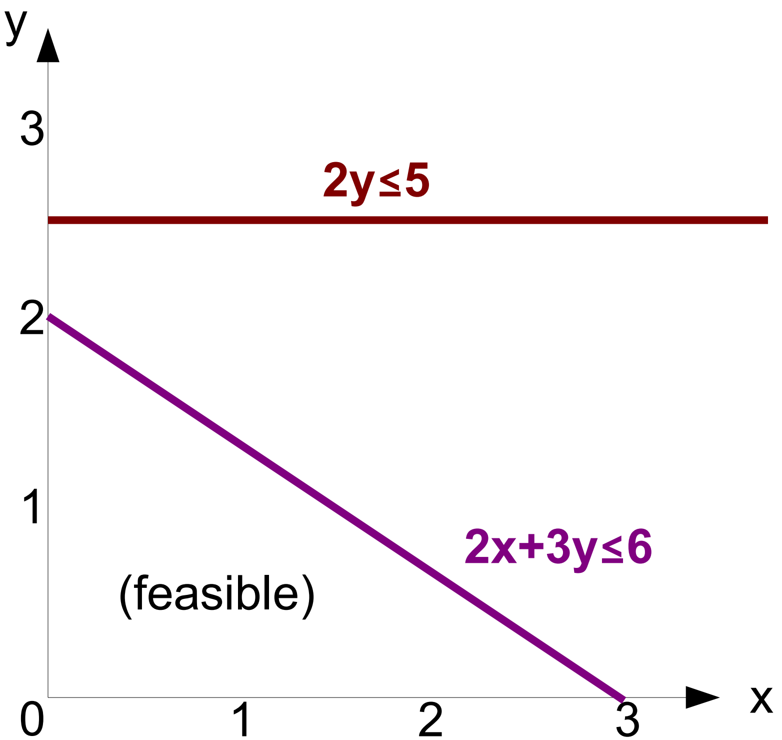

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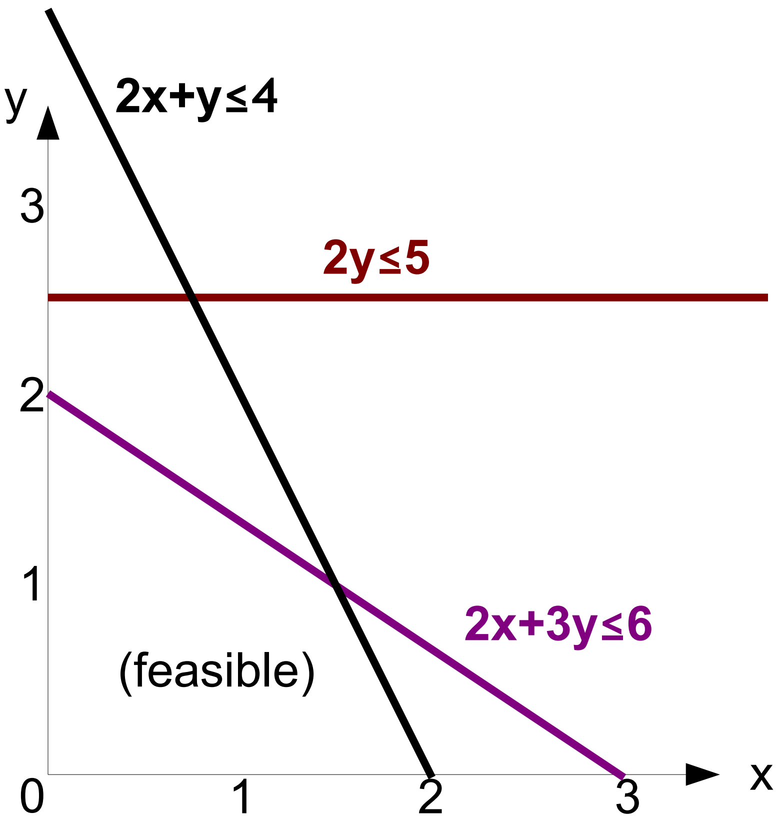
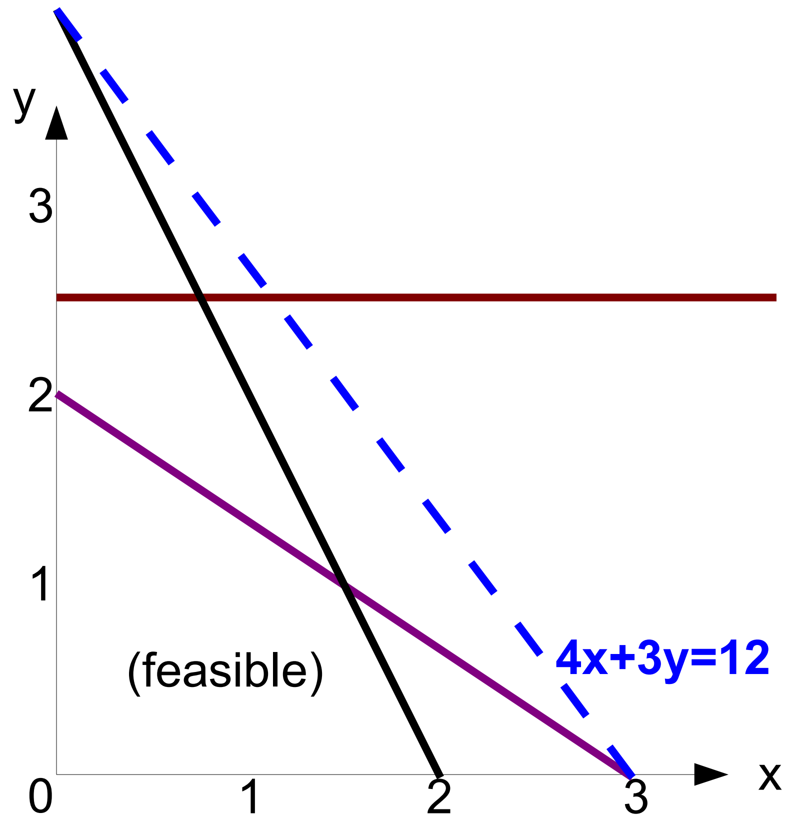


# Feasible Region

- The region that is on the "correct" side of all of the lines is the **feasible region**
- If non-empty, it is always a **convex** polygon
  - Convex, for our purposes: if A and B are points in a convex set, then the points on the line segment between A and B are also in that convex set
- Optimality: "Maximize      4x + 3y"
- For any c, 4x+3y=c is a line with the same slope
- **Corner points** of the feasible region must maximize
  - Why? Linear objective function + convex polygon

# Objective Function

- Maximize 4x+3y



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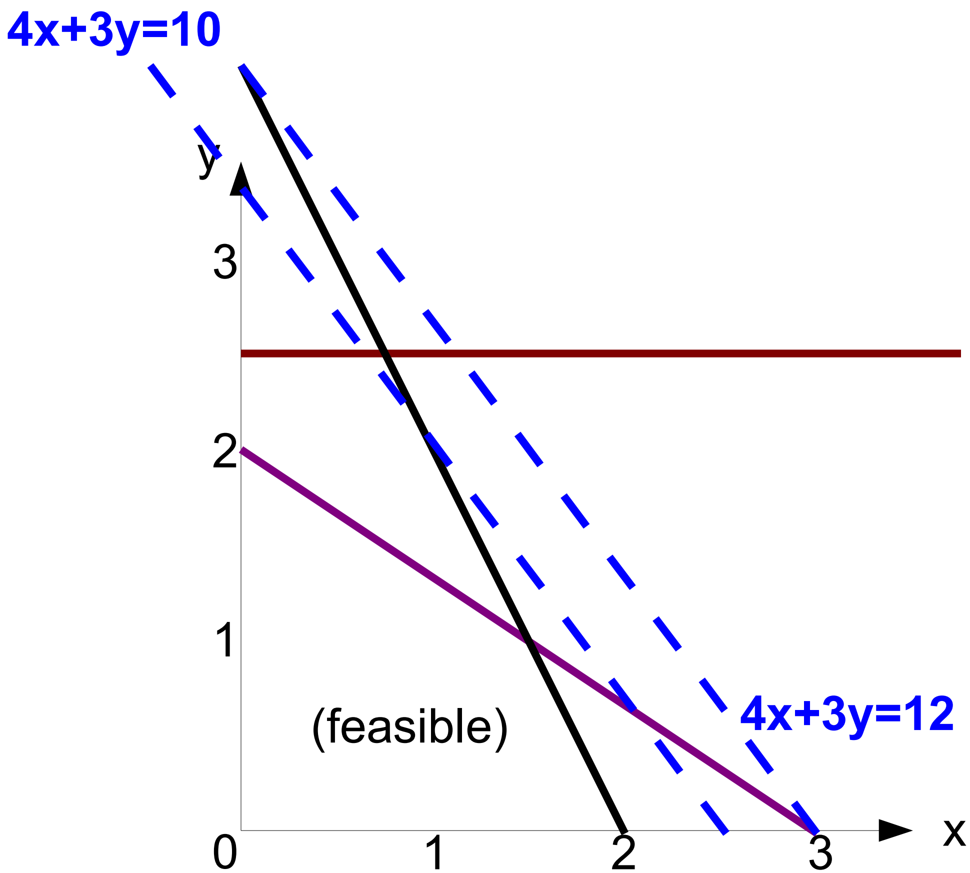
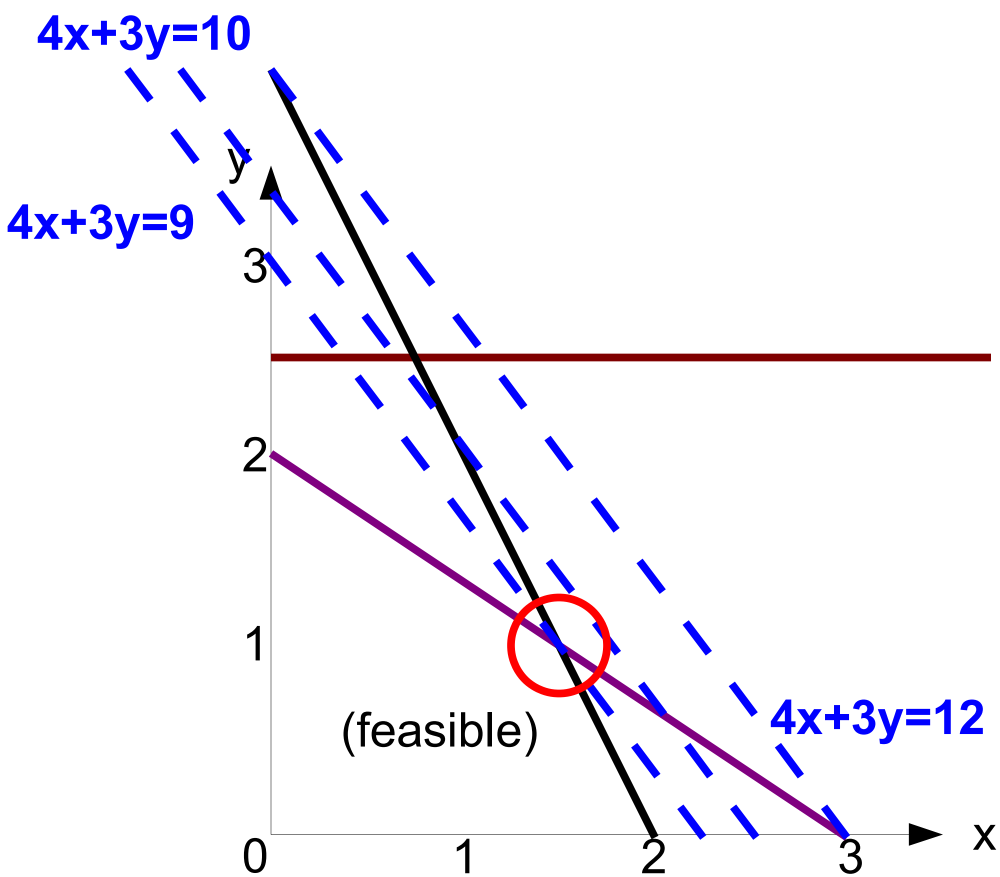


# Objective Function

- Maximize 4x+3y




Optimal Corner Point (1.5, 1)
It's the feasible point that
maximizes the objective function!

# Analogy: Rolling Pin, Pizza Dough

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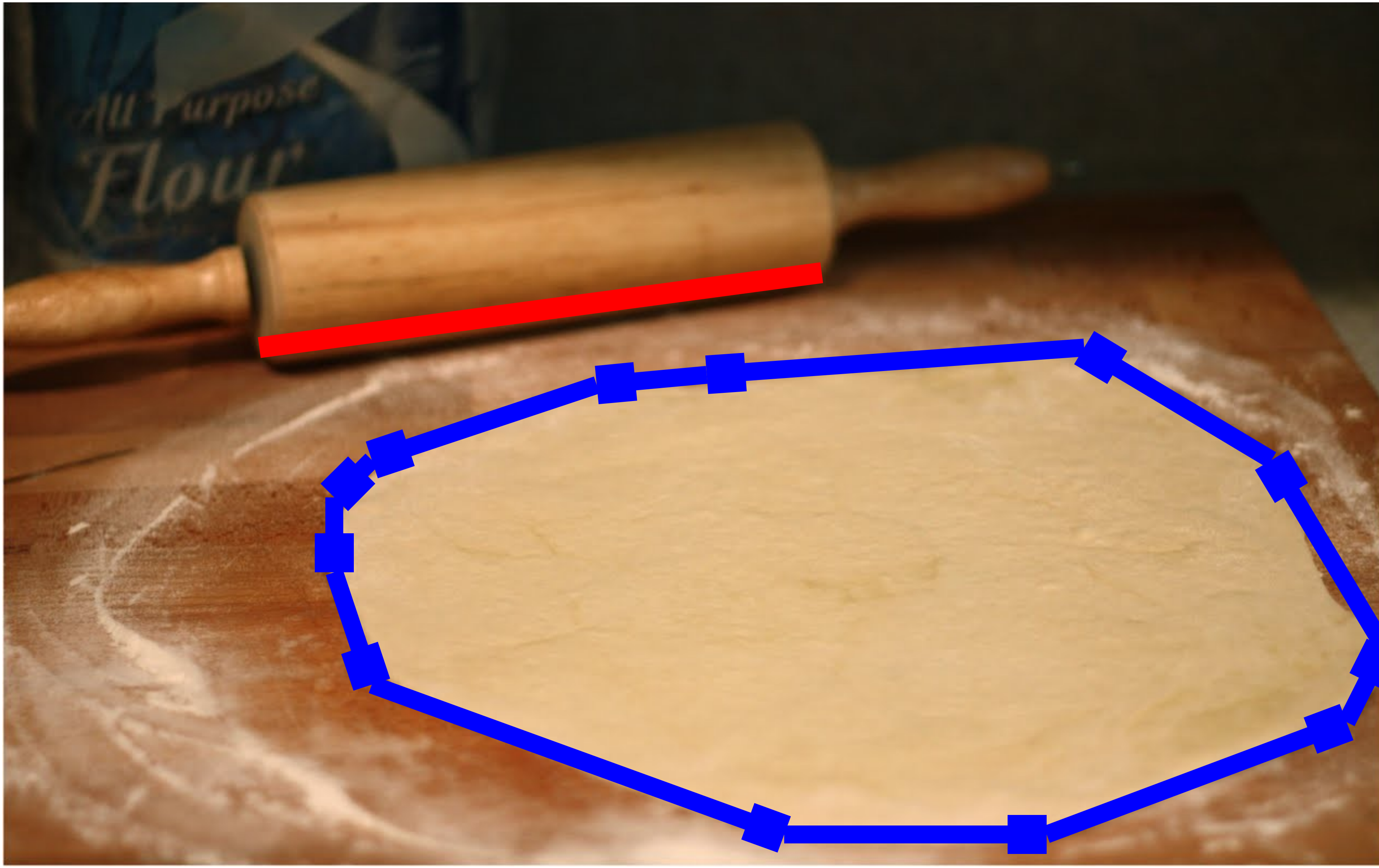


# Any Convex Pizza and Any Linear Rolling Pin Approach

# Any Convex Pizza and Any Linear Rolling Pin Approach

# Linear Programming Solver

- Three Step Process
  - Identify the coordinates of all feasible corners
  - Evaluate the objective function at each one
  - Return one that maximizes the objective function
- This totally works! We're done.

- The trick: how can we find all of the coordinates of the corners *without* drawing the picture of the graph?

# Finding Corner Points

- A **corner point** (**extreme point**) lies at the intersection of constraints.
- Recall our running example:
- Subject to

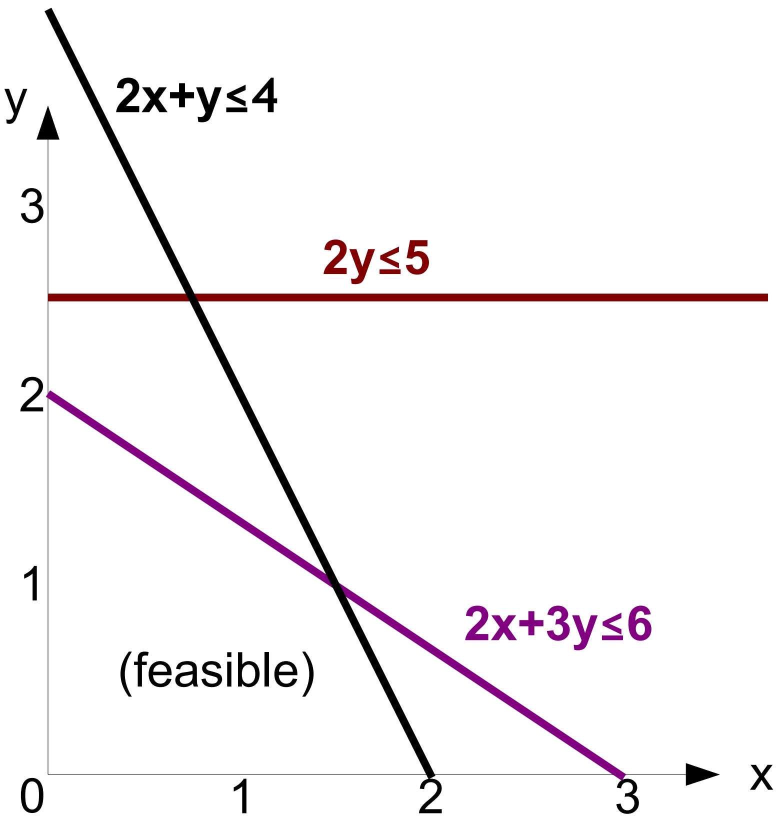
$$
\begin{array}{rrll}
2x & +3y & \le 6 & \quad (1) \\
 & 2y & \le 5 & \quad (2) \\
2x & +1y & \le 4 & \quad (3) \\
\end{array}
$$

$$x \ge 0,\ y \ge 0$$

- Take just *(1)* and *(3)* as **defining equations**

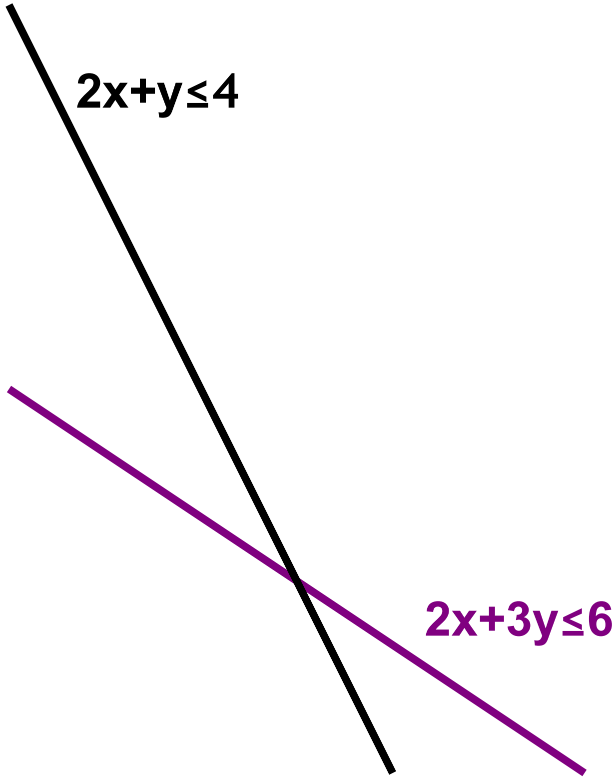
# Visually

- $2x + 3y \le 6$ and $2x + 1y \le 4$
  - Hard to see with the whole graph ...



# Visually

- $2x + 3y \le 6$ and $2x + 1y \le 4$
  - But easy if we only look at those two!

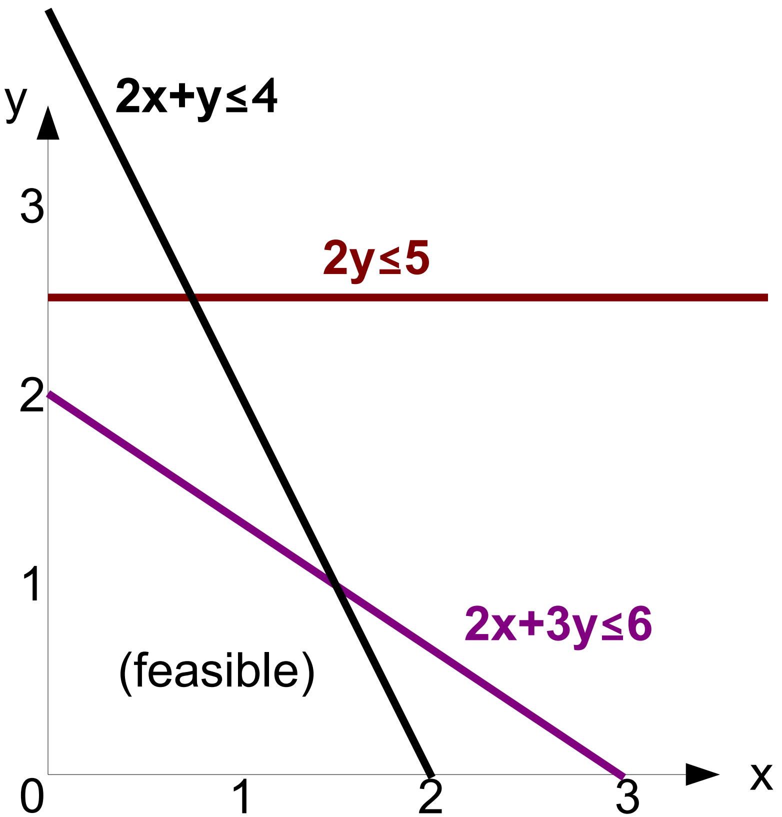


# Mathematically

- $2x + 3y \le 6$
- $2x + 1y \le 4$
- Recall linear algebra: **Gaussian Elimination**
  - Subtract the second row from the first
- $0x + 2y \le 2$
  - Yields "$y = 1$"
- Substitute "$y=1$" back in
- $2x + 3 \le 6$
  - Yields "$x = 1.5$"

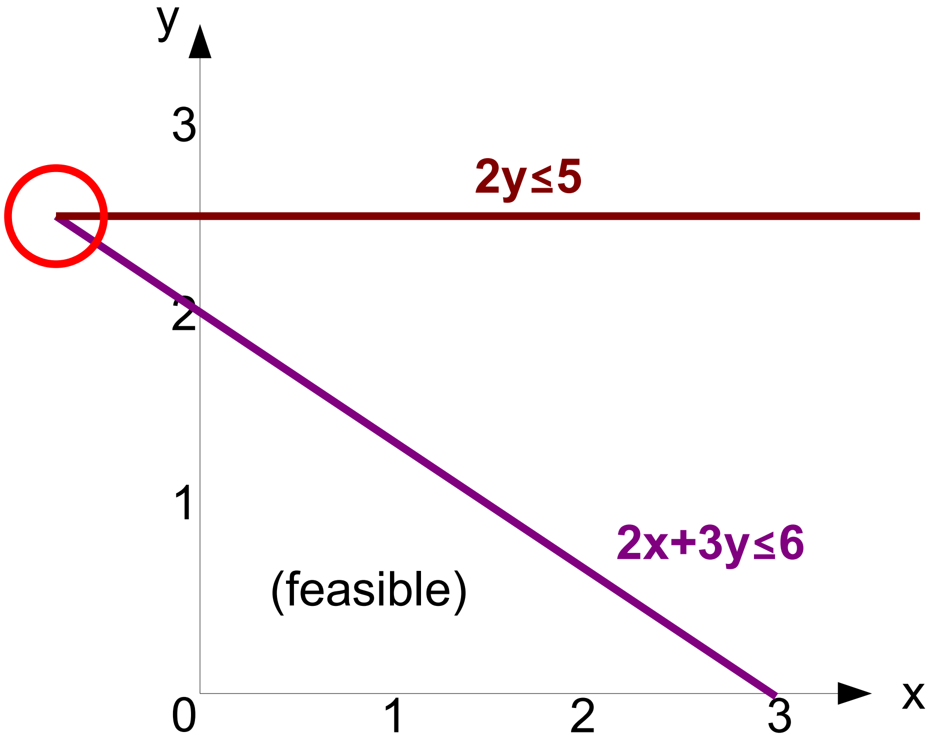
# Infeasible Corners

- $2x + 3y \le 6$ and $2y \le 5$



# Infeasible Corners

- $2x + 3y \leq 6$ and $2y \leq 5$
  - (-0.75,2.5) solves the equations but it does not satisfy our "$x \geq 0$" constraint: infeasible!



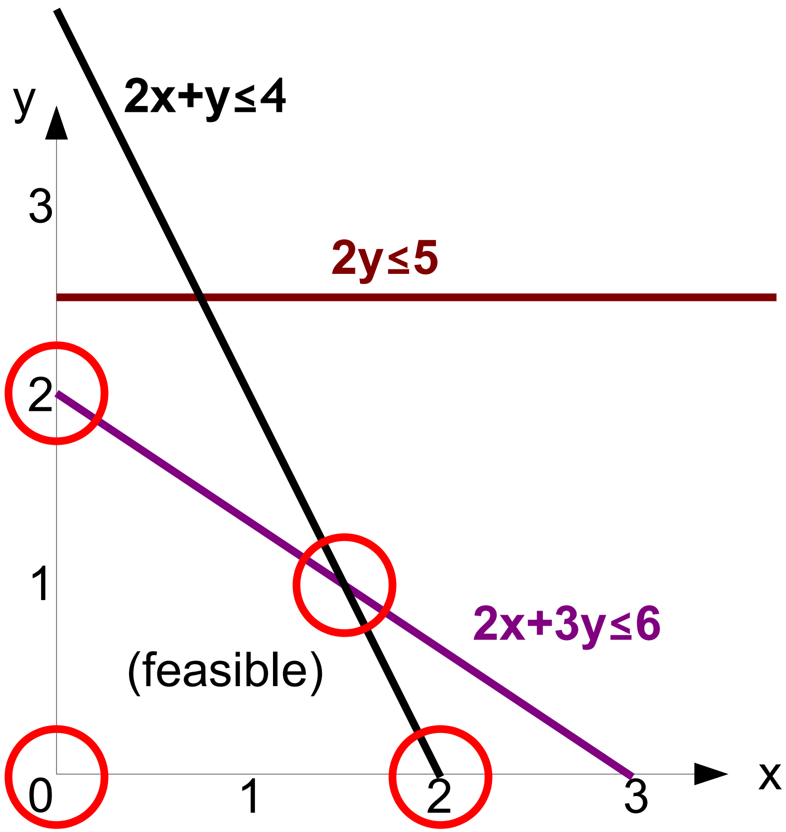
# Solving Linear Programming

- Identify the coordinates of all corners
  - Consider all pairs of constraints, solve each pair
  - Filter to retain points satisfying all constraints
- Evaluate the objective function at each point
- Return the point that maximizes

- With 5 equations, the number of pairs is "6 choose 2" = 5!/(2!3!) = 10.
  - Only 4 of those 10 are feasible.

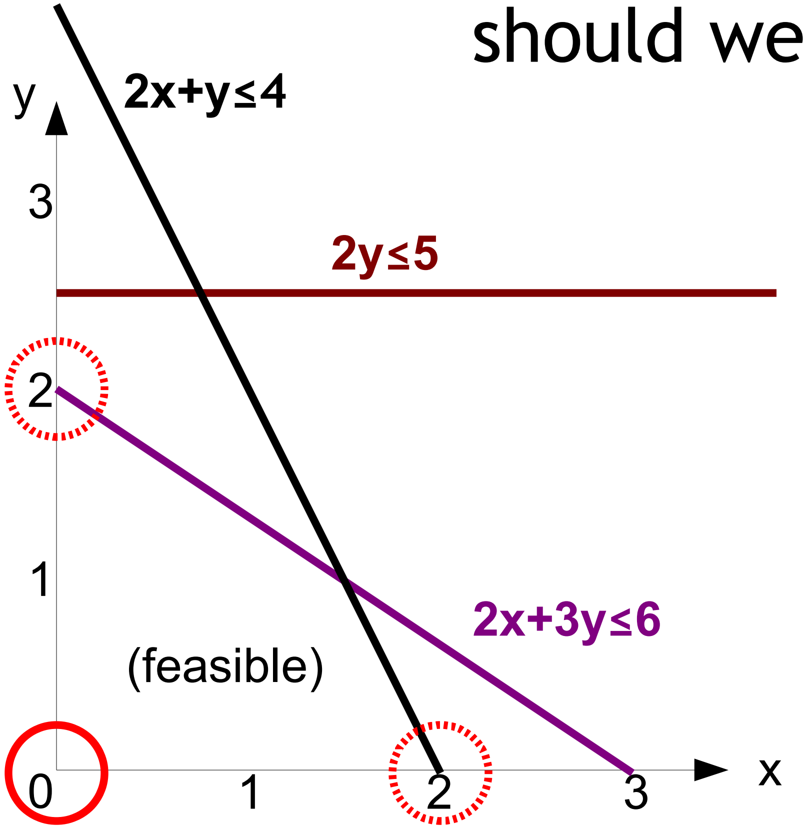
# Feasible Corners

- In our running example, there are four feasible corners



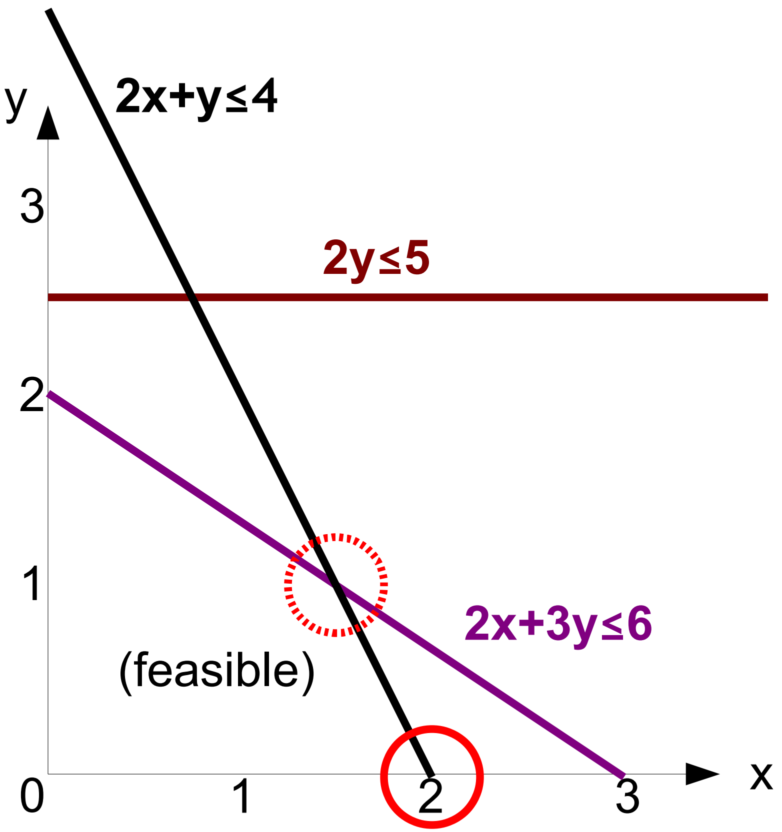
# Road Trip!

- Suppose we start in one feasible corner (0,0)
  - And we know our objective function 4x+3y
  - Do we move to corner (0,2) or (2,0) next, or should we stay here?



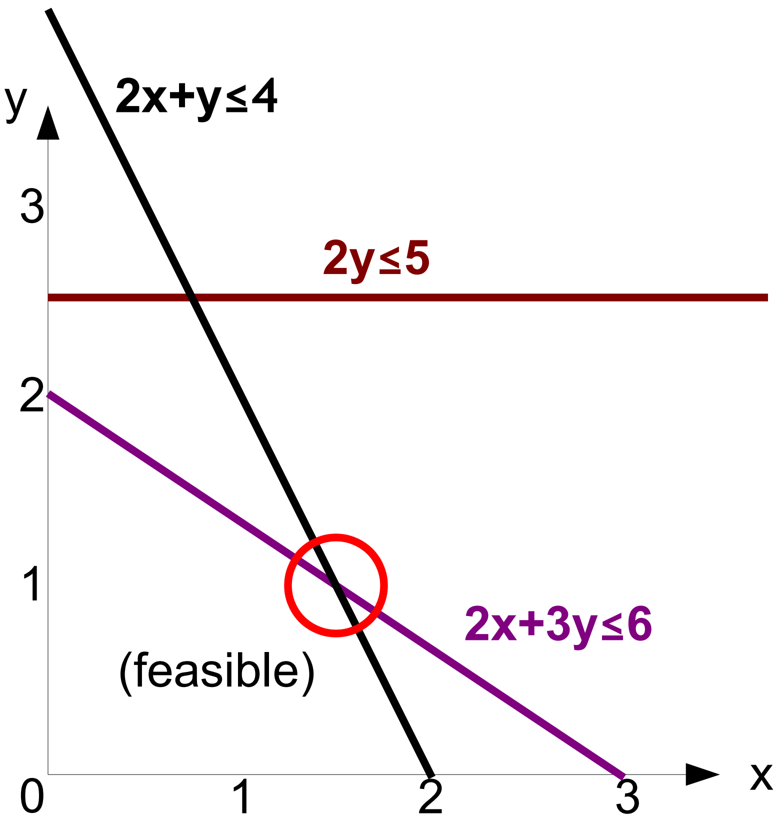
# Road Trip!

- We're now in (2,0)
  - And we know our objective function 4x+3y
  - Do we move to corner (1.5,1) or stay here?



# Road Trip!

- We're now in (1.5,1)
  - We're done! We have considered all of our neighbors and we're the best.



# Analogy: Don't Sink!

# Reach Highest Point Greedily

# Not A Counter-Example
# Why Not?

# Simplex Insight

- The **Simplex** algorithm encodes this “gradient ascent” insight: if there are many corners, we may not even need to enumerate or visit them all.
- Instead, just walk from feasible corner to adjacent feasible corner, maximizing the objective function every time.
  - It's linear and convex: you can't be “tricked” into a local maximum that's not also global.
- In a high-dimensional case, this is a huge win because there are many corners.

# Simplex Algorithm

- George Dantzig published the Simplex algorithm in 1947.
  - John von Neumann theory prize, US National Medal of Science, "one of the top 10 algorithms of the 20th century", etc.
- Phase 1: find any feasible corner
  - Ex: solve two constraints until you find one
- Phase 2: walk to best adjacent corner
  - Ex: "pivot" row operations between the "leaving" variable and the "entering" variable
- Repeat until no adjacent corner is better

# Simplex Running Time

- Despite the “gradient ascent heuristic”, the official worst-case complexity of Simplex is Exponential time
  - Open question: is there a strongly polytime algorithm for linear programming?
- Simplex is quite efficient in practice.
  - In a formal sense, “most” LP instances can be solved by Simplex in polytime. “Hard” instances are “not dense” in the set of all instances (akin to: the Integers are “not dense” in the Reals).
- 0-1 Integer Linear Programming is NP-Hard.

# Next Time

- DPLL(T) combines DPLL + Simplex into one grand unified theorem prover

# Homework

- HW2 Due for Next Time
- Reading for Monday