Quickest Sorting and Double Deltas

One-Slide Summary

- **Insert-sort** is $\Theta(n^2)$ worst case (reverse list), but is $\Theta(n)$ best case (sorted list).
- A recursive function that divides its input in **half** each time is often in $\Theta(\log n)$.
- If we could divide our input list in half rapidly, we could do a **quicker sort**: $\Theta(n \log n)$.
- **Sorted binary trees** are an efficient data structure for maintaining sorted sets.
- British codebreakers used **cribs** (guesses), brute force, and **analysis** to break the Lorenz cipher. Guessed wheel settings were likely to be correct if they resulted in a message with the right linguistic properties for German (e.g., repeated letters).

Outline

- Insert-sort
- Going half-sies
- Sorted binary trees
- Quicker-sort
- WWII Codebreaking

Pick Up Graded Problem Sets (from Mohsin)!

How much work is insert-sort?

```
def insert_sort(lst, cf):
  if not lst: return []
  return insert_one(lst[0], insert_sort(lst[1:], cf))

def insert_one(elt, lst, cf):
  if not lst: return [elt]
  if cf(elt, lst[0]): return [elt] + lst
  return [lst[0]] + insert_one(elt, lst[1:], cf)
```

How many times does insert-sort evaluate insert-one?

n times (once for each element)

running time of insert-one is in $\Theta(n)$

insert-sort has running time in $\Theta(n^2)$ where n is the number of elements in the input list

best-first-sort vs. insert-sort

- Both are $\Theta(n^2)$ worst case (reverse list)
- Both are $\Theta(n^2)$ when sorting a randomly ordered list
  - But insert-sort is about twice as fast
- insert-sort is $\Theta(n)$ best case (ordered input list)

Can we do better?

insert_one(88, [1,2,3,5,6,22,63,77,89,90], ascending)

Suppose we had procedures

first_half(lst)

second_half(lst)

that *quickly* divided the list in two halves?

quicker-insert using halves

```
def quicker_insert(elt, lst, cf):
  if not lst: return [elt]     # just like insert_one
  if len(lst) == 1:            # handle 1 element by hand
    return [elt]+lst   if cf(elt, lst[0])   else lst+[elt]
  front = first_half(lst)
  back = second_half(lst)
  if cf(elt, back[0]):         # insert into front half
    return quicker_insert(elt, front, cf) + back
  else:                        # insert into back half
    return front + quicker_insert(elt, back, cf)
```

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Evaluating quicker-sort

```
>>> quicker_insert(3, [1,2,4,5,7,8,9,10], ascend)
Front = [1,2,4,5]
Back = [7,8,9,10]
Is 3 < 7? Yes. So
return quicker_insert(3,[1,2,4,5],ascend) + [7,8,9,10]
Front = [1,2]
Back = [4,5]
Is 3 < 4? Yes. So
return quicker_insert(3,[1,2],ascend) + [4,5]
Front = [1]
Back = [2]
Is 3 < 2? No. So
Return [1] + quicker_insert(3,[2],ascend)
One element. Compare 3 and 2, return [2,3]

So final result is:
[1] + [2,3] + [4,5] + [7,8,9,10]
```

```
def quicker_insert(elt, lst, cf):
  if not lst: return [elt]              # just like insert_one
  if len(lst) == 1:                     # handle 1 element
    return [elt]+lst   if cf(elt, lst[0])   else lst+[elt]
  front = first_half(lst)
  back = second_half(lst)
  if cf(elt, back[0]):     # insert into front half
    return quicker_insert(elt, front, cf) + back
  else:                    # insert into back half
    return front + quicker_insert(elt, back, cf)
```

Every time we call quicker-insert, the length of the list is approximately **halved**!

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How much work is quicker-sort?

Each time we call quicker-insert, the size of lst halves. So doubling the size of the list only increases the number of calls by 1.

```
def quicker_insert(elt, lst, cf):
  if not lst: return [elt]              # just like insert_one
  if len(lst) == 1:                     # handle 1 element
    return [elt]+lst   if cf(elt, lst[0])   else lst+[elt]
  front = first_half(lst)
  back = second_half(lst)
  if cf(elt, back[0]):     # insert into front half
    return quicker_insert(elt, front, cf) + back
  else:                    # insert into back half
    return front + quicker_insert(elt, back, cf)
```

List Size	# quicker_insert applications
1	1
2	2
4	3
8	4
16	5
32	6

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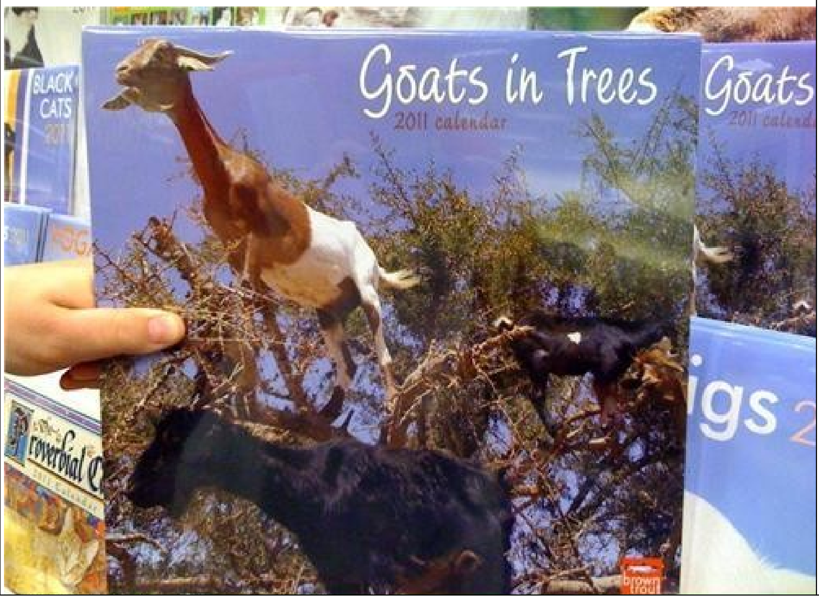
Liberal Arts Trivia: ?

- The argan tree, found primarily in Morocco, has a knobby, twisted trunk that allows these animals to climb it easily. The animals eat the fruit, which has an indigestible nut inside, which is collected by farmers and used to make argan oil: handy in cooking and cosmetics, but pricey at $45 per 500 ml.



Liberal Arts Trivia: Scandinavian Studies

- This capital of and largest city in Denmark is situated on the islands of Zealand and Amager. It is the birthplace of Neils Bohr, Søren Kierkegaard, and Victor Borge. The city's origin as a harbor and a place of commerce is reflected in its name. Its original designation, from which the contemporary Danish name is derived, was Køpmannæhafn, "merchants' harbor". The English name for the city is derived from its (similar) Low German name.

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Remembering Logarithms

$\log_b n = x$ means $b^x = n$

What is $\log_2 1024$?

What is $\log_{10} 1024$?

Is $\log_{10} n$ in $\Theta(\log_2 n)$?

Changing Bases

$$\log_b n = \underbrace{(1/\log_k b)}_{\text{If } k \text{ and } b \text{ are constants, this is constant}} \log_k n$$

$$\Theta(\log_2 n) \equiv \Theta(\log_{10} n) \equiv \Theta(\log n)$$

No need to include a constant base within asymptotic operators.

Number of Applications

Assuming the list is well-balanced, the number of applications of quicker-insert is in $\Theta(\log n)$ where n is the number of elements in the input list.

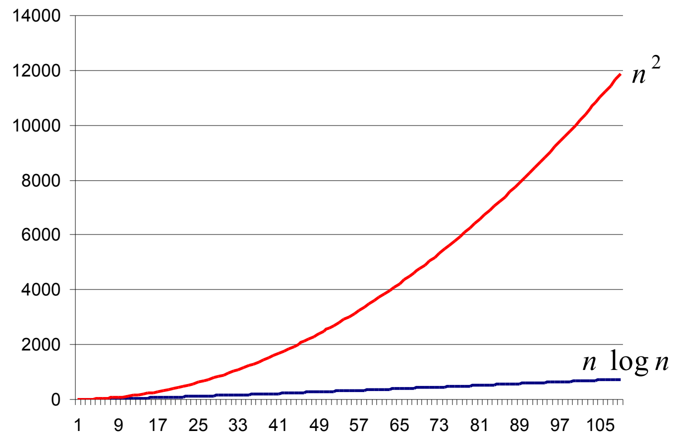
quicker-sort ?

```
def quicker_sort(lst, cf):
  if not lst: return [ ]
  return quicker_insert(
    lst[0],
    quicker_sort(lst[1:], cf),
    cf)
```

```
def quicker_insert(elt, lst, cf):
  if not lst: return [elt]
  if len(lst) == 1:    # handle 1 element by hand
    return [elt]+lst   if cf(elt, lst[0])   else lst+[elt]
  front = first_half(lst)
  back = second_half(lst)
  if cf(elt, back[0]): # insert into front half
    return quicker_insert(elt, front, cf) + back
  else:                # insert into back half
    return front + quicker_insert(elt, back, cf)
```

quicker_sort using halves would have running time in $\Theta(n \log n)$ ***if*** we have first_half, second_half, and append (e.g., [1,2,3] **+** [4,5,6]) procedures that run in constant time.

Orders of Growth



Is there a fast first-half procedure?

- No! (at least not on lists)
- To produce the first half of a list length n, we need to walk down the first $n/2$ elements
- So, first-half on lists has running time in $\Theta(n/2) = \Theta(n)$

Making it faster

We need to either:

1. Reduce the number of applications of insert-one in insert-sort

   Impossible – need to consider each element

2. Reduce the number of applications of quicker-insert in quicker-insert

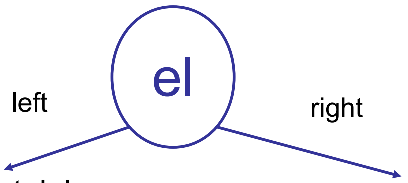
   Unlikely… each application already halves the list

3. Reduce the time for each application of quicker-insert

   Need to make first-half, second-half and append faster than $\Theta(n)$

"Nothing yet. ...How about you, Newton?"

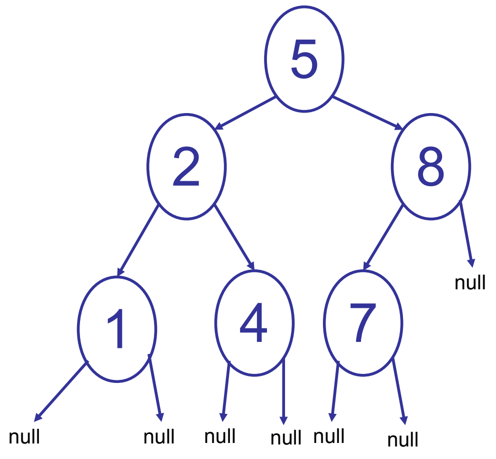
Sorted Binary Trees



A tree containing all elements *x* such that cf(*x*,el) is true

A tree containing all elements *x* such that cf(*x*,el) is false

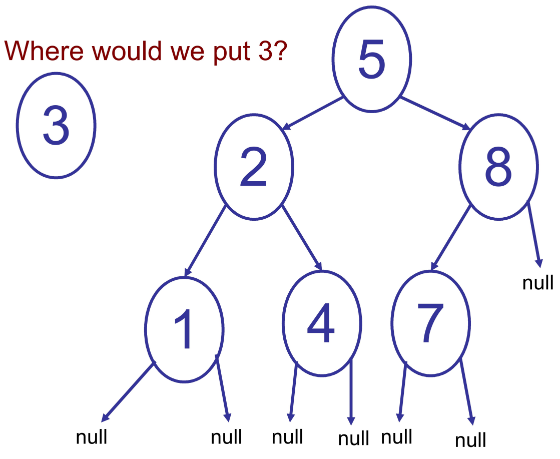
Tree Example



cf is **<**

Tree Example

Where would we put 3?



cf is **<**

Representing Trees

```
def make_tree(left, el, right):
  return [el, left, right]
```

left and right are trees (**[ ]** is a tree)

```
def get_element(tree):
  return tree[0]
```

tree must be a non-null tree

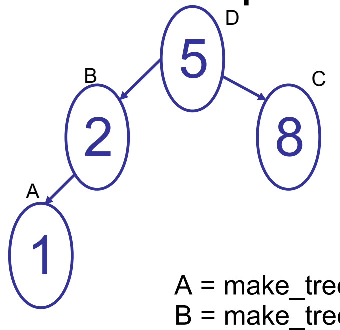
```
def get_left(tree):
  return tree[1]
```

tree must be a non-null tree

```
def get_right(tree):
  return tree[2]
```

tree must be a non-null tree

Representing Trees



```
A = make_tree([], 1, [])
B = make_tree(A, 2, [])
C = make_tree([], 8, [])
D = make_tree(B, 5, C)
```

insert-one-tree

```
def insert_one_tree(cf, new_elt, tree):
  if not tree:
    return make_tree([], new_elt, [])
  element_here = get_element(tree)
  if cf(new_elt, element_here):
    return make_tree(
     insert_one_tree(cf, new_elt, get_left(tree)),
     element_here,
     get_right(tree))
  else:
    return make_tree(
      get_left(tree),
      element_here,
      insert_one_tree(cf, new_elt, get_right(tree)))
```

If the tree is null, make a new tree with new_elt as its element and no left or right trees.

Otherwise, decide if elt should be in the left or right subtree. Insert it into that subtree, but leave the other subtree unchanged.

How much work is insert-one-tree?

```
def insert_one_tree(cf, new_elt, tree):
  if not tree:
    return make_tree([], new_elt, [])
  element_here = get_element(tree)
  if cf(new_elt, element_here):
    return make_tree(
     insert_one_tree(cf, new_elt, get_left(tree)),
     element_here, get_right(tree))
  else:
    return make_tree(
     get_left(tree), element_here,
     insert_one_tree(cf, new_elt, get_right(tree)))
```

Each time we call insert-one-tree, the size of the tree approximately halves (if it is well balanced).

Each application is constant time.

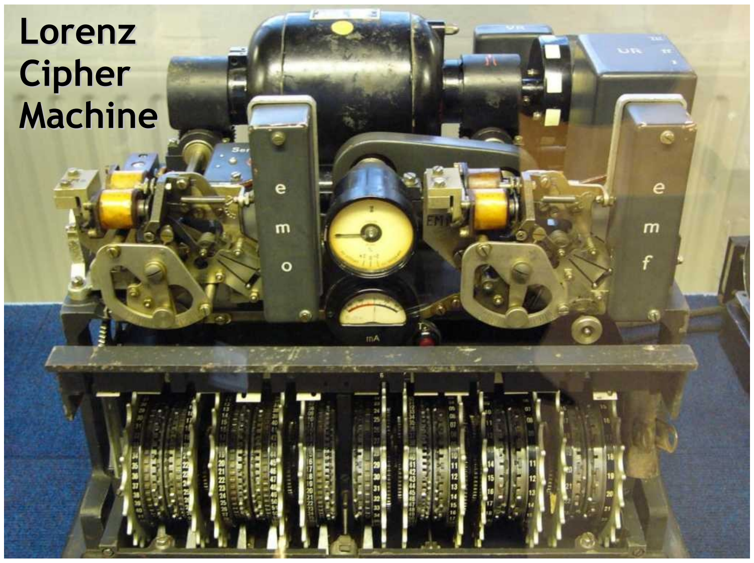
The running time of insert-one-tree is in $\Theta(\log n)$ where n is the number of elements in the input tree, which must be well-balanced.

quicker-insert-one

```
def quicker_insert_one(cf, lst):
  if not lst: return [ ]
  return insert_one_tree(cf,
    lst[0],
    quicker_insert_one(cf, lst[1:]))
```

No change (other than using insert_one_tree)…but evaluates to a tree not a list!

Practice: You should be able to write a procedure that takes a TREE as input and prints out all of its elements (e.g., “from left to right”).



Liberal Arts Trivia: Classics

- This ancient Greek epic poem, traditionally attributed to Homer, is widely believed to be the oldest extant work of Western literature. It describes the events of the final year of the Trojan War. The plot follows Achilles and his anger at Agamemnon, king of Mycenae. It is written in dactylic hexameter and comprises 15,693 lines of verse. It begins:
  - μῆνιν ἄειδε θεὰ Πηληϊάδεω Ἀχιλῆος
  - οὐλομένην, ἣ μυρί' Ἀχαιοῖς ἄλγε' ἔθηκεν

Liberal Arts Trivia: Chemistry

- This violet variety of quartz, often used in jewelry, takes its name from the ancient Greek (a ("not") and methustos ("intoxicated")), a reference to the belief that it protected its own from drunkenness; ancient Greeks and Romans made drinking vessels of it to prevent intoxication.

Liberal Arts Trivia: Literature

- Name the author of the Age of Innocence (1920). The novel describes the upper class in New York city in the 1870s and questions the mores and assumptions of society. The title is an ironic comment on the polished outward manners of New York society, when compared to its inward machinations. The authors was the first woman to win the Pulitzer Prize for Literature.

Lorenz Wheels

12 wheels
501 pins total (set to control wheels)

Work to break in $\Theta(p^w)$ so real Lorenz is $41^{12}/5^3 \sim$ 1 quintillion (10^{18}) times harder!

Code Breaking Intuition

- Suppose we are using a simple letter substitution cipher (i.e., replace every A with Q, etc.)
- You intercept these two messages:
  - PF1120: Vagebqhpgvba gb Pbzchgvat: Rkcybengvbaf va Ynathntr, Ybtvp, naq Znpuvarf
  - PF1120: Vageb gb Pbzc: Rkcyber Ynat., Ybtvp, naq Znpu.
- What does the first one say? What hints did you have?

Breaking Fish

- Gov't Communications HQ learned about first Fish link (Tunny) in May 1941
  - British codebreakers used "Fish" to refer to German teleprinter traffic
  - Intercepted unencrypted Baudot-encoded test messages
- August 30, 1941: Big Break!
  - Operator retransmits failed message with same starting configuration
  - Gets lazy and uses some abbreviations, makes some mistakes
    - SPRUCHNUMMER/SPRUCHNR (Serial Number)

"Two Time" Pad

- Allies have intercepted:

  C1 = M1 ⊕ **K1**

  C2 = M2 ⊕ **K1**

  Same key used for both (same starting configuration)

⊕ means XOR

- Breaking message:

$$\begin{aligned} C1 \oplus C2 &= (M1 \oplus \mathbf{K1}) \oplus (M2 \oplus \mathbf{K1}) \\ &= (M1 \oplus M2) \oplus (\mathbf{K1} \oplus \mathbf{K1}) \\ &= M1 \oplus M2 \end{aligned}$$

"Cribs"

- Know: C1, C2 (intercepted ciphertext)

  C1 ⊕ C2 = M1 ⊕ M2
- Don't know M1 or M2
  - But, can make some guesses (cribs)
    - SPRUCHNUMMER
    - Sometimes allies moved ships, sent out bombers to help the cryptographers get good cribs
- Given guess for M1, calculate M2

  M2 = C1 ⊕ C2 ⊕ M1
- Once guesses that work for M1 and M2

  K1 = M1 ⊕ C1 = M2 ⊕ C2

Reverse Engineering Lorenz

- From the 2 intercepted messages, Col. John Tiltman worked on guessing cribs to find M1 and M2: 4000 letter messages, found 4000 letter key K1
- Bill Tutte (recent Chemistry graduate) given task of determining machine structure
  - Already knew it was 2 sets of 5 wheels and 2 wheels of unknown function
  - Six months later new machine structure likely to generate K1

Intercepting Traffic

- Set up listening post to intercept traffic from 12 Lorenz (Fish) links
  - Different links between conquered capitals
  - Slightly different coding procedures, and different configurations
- 600 people worked on intercepting traffic

Breaking Traffic

- Knew machine structure, but a different initial configuration was used for each message
- Need to determine wheel setting:
  - Initial position of each of the 12 wheels
  - 1271 possible starting positions
  - Needed to try them fast enough to decrypt message while it was still strategically valuable

This is what you did for PS4 (except with fewer wheels)

Recognizing a Good Guess

- Intercepted Message (divided into 5 channels for each Baudot code bit)

  $Z_c = z_0 z_1 z_2 z_3 z_4 z_5 z_6 z_7 \ldots$

  $z_{c,i}$ = *i*th bit of ciphertext is (*i*th bit of message) ⊕ with (*i*th bit of key)

  key comes from all of the wheels (e.g., S-wheel, ...)
- Look for statistical properties
  - How many of the $z_{c,i}$'s are 0? ½ (not useful)
  - How many of ($z_{c,i+1} \oplus z_{c,i}$) are 0? ½

Double Delta

X is random part of key (i.e., K-wheel)
S is not-truly-random part from S wheels

$$\Delta Z_{c,i} = Z_{c,i} \oplus Z_{c,i+1}$$

Combine two channels:

$$\Delta Z_{1,i} \oplus \Delta Z_{2,i} = \Delta M_{1,i} \oplus \Delta M_{2,i} \quad > ½ \text{ Yippee!}$$

$$\oplus \Delta X_{1,i} \oplus \Delta X_{2,i} \quad = ½ \text{ (key)}$$

$$\oplus \Delta S_{1,i} \oplus \Delta S_{2,i} \quad > ½ \text{ Yippee!}$$

Why is $\Delta M_{1,i} \oplus \Delta M_{2,i} > ½$

Message is in German, more likely following letter is a repetition than random

Why is $\Delta S_{1,i} \oplus \Delta S_{2,i} > ½$

S-wheels only turn when M-wheel is 1

Actual Advantage: Linguistics

- Probability of repeating letters

  Prob[$\Delta M_{1,i} \oplus \Delta M_{2,i} = 0$] ~ 0.614

  3.3% of German digraphs are repeating

- Probability of repeating S-keys

  Prob[$\Delta S_{1,i} \oplus \Delta S_{2,i} = 0$] ~ 0.73

Prob[$\Delta Z_{1,i} \oplus \Delta Z_{2,i} \oplus \Delta X_{1,i} \oplus \Delta X_{2,i} = 0$]

$= \underbrace{0.614 * 0.73}_{\Delta \text{ M and S are 0}} + \underbrace{(1-0.614) * (1-0.73)}_{\Delta \text{ M and S are 1}}$

= 0.55 if the wheel settings guess is correct (0.5 otherwise)

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Using the Advantage

- If the guess of **X** is correct, should see higher than ½ of the double deltas are 0
- Try guessing different configurations to find highest number of 0 double deltas
- Problem:

  # of double delta operations to try one config

  = length of Z * length of X

  = for 10,000 letter message = 12 M for each setting

  * 7 $\oplus$ per double delta

  = 89 M $\oplus$ operations

Need a fast way to compute XOR!

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Homework

- Problem Set 4
- Study for Exam 1
  - Out Soon

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