Set Theory

1 Set Theory Exercise — Problem

This exercise is meant to help you refresh your knowledge of set theory and functions. Let X and Y be sets. Let $\mathcal{P}(X)$ denote the powerset of X (the set of all subsets of X). Show that there is a 1-1 correspondence (i.e., a bijection) between the sets A and B, where $A = X \to \mathcal{P}(Y)$ and $B = \mathcal{P}(X \times Y)$. Note that A is a set of functions and B is a (or can be viewed as a) set of relations. This correspondence will allow us to use functional notation for certain sets in class. This is Exercise 1.4 from page 8 of Winskel's book.

2 Set Theory Exercise — Solution 1 (Injective + Surjective)

Let us construct a function $f : A \to B$ and prove that it is injective and surjective. More precisely, the type of f is $f : (X \to \mathcal{P}(Y)) \to \mathcal{P}(X \times Y)$. We choose f as follows:

$$f(a) =_{\text{def}} \{(x, y) \mid y \in a(x)\}$$

A function f is *injective* (or *one-to-one*) if for all $a_1 \in A$ and $a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$. Let a_1 and a_2 be arbitrary elements of A, and assume $f(a_1) = f(a_2)$. Then, by definition of f:

$$\{(x, y) \mid y \in a_1(x)\} = \{(x, y) \mid y \in a_2(x)\}$$

By the *axiom of extensionality* in Set Theory, two sets are equal if they have exactly the same elements. Applied to the two sets above, we find that for any (x, y), whenever $y \in a_1(x)$, we also have $y \in a_2(x)$. Applying the *axiom of extensionality* to $a_1(x)$ and $a_2(x)$, we find that they must be equal sets (because for all y they either both contain that same y or both do not contain that same y). So for any x, $a_1(x) = a_2(x)$. Thus by the definition of *function*, a_1 and a_2 are equal functions (they agree on all arguments). Thus f is injective.

A function $f : A \to B$ is *surjective* (or *onto*) if, for every $b \in B$ there is an $a \in A$ with $f(a) = b$. To demonstrate this, let b be an arbitrary element of B. So $b \in \mathcal{P}(X \times Y)$ (by definition of B, above). So every element of b is of the form (x, y) with $x \in X$ and $y \in Y$. We now construct an a such that $f(a) = b$. By definition of f, $f(a) = \{(x, y) \mid y \in a(x)\}$. So we pick our function a by letting $a(x) = \{y \mid (x, y) \in b\}$. By substitution, $f(a) = \{(x, y) \mid y \in \{y \mid (x, y) \in b\}\}$, which simplifies to $f(a) = \{(x, y) \mid (x, y) \in b\}$. Since $f(a)$ is the set of elements that are exactly those elements found in b, by the *axiom of extensionality*, $f(a) = b$. So the function f is surjective.

Since f is injective and surjective, it is also bijective (i.e., invertible). Since there exists an invertible function $f : A \to B$, there is a 1-1 corresopndence between A and B. QED.

3 Set Theory Exercise — Solution 2 (Explicit Inverse)

In this alternate solution, we'll construct an invertible f by explicitly showing its inverse. Let f be as in the previous solution:

$$f(a) =_{\text{def}} \{(x, y) \mid y \in a(x)\}$$

We introduce a second function, g, that we will show to be the inverse of f. Since $g : B \to A$ and A is a set of functions, every $g(b)$ will be a function. We define g as follows:

$$(g(b))(x) =_{\text{def}} \{y \mid (x, y) \in b\}$$

An optional presentation of the same g in the style of the lambda calculus is:

$$g(b) =_{\text{def}} \lambda x.\ \{y \mid (x, y) \in b\}$$

However, we have not yet introduced the lambda calculus in class. Do not worry if you are not familiar with it. In either case, $g(b)$ returns a function. When that function is presented with the argument x, it returns the set $\{y \mid (x, y) \in b\}$.

By the definition of *invertible*, to show that f and g are inverses, we show that that f composed with g is the identity function: $g(f(a)) = a$. Let a be an arbitrary element of A. So a is a function mapping X to $\mathcal{P}(Y)$. To show that $g(f(a)) = a$, since they're both functions, we'll show that they behave the same way on all inputs: $(g(f(a)))(x) = a(x)$. Now we expand $g(f(a))(x)$ by definition of $f(a)$:

$$g(f(a))(x) = g(\{(x, y) \mid y \in a(x)\})(x)$$

Now we expand by definition of $(g(b))(x)$:

$$g(f(a))(x) = \{y \mid (x, y) \in \{(x, y) \mid y \in a(x)\}\}$$

By simplification we have:

$$g(f(a))(x) = \{y \mid y \in a(x)\}$$

By the *axiom of extensionality*, the set of all elements found in $a(x)$ is exactly $a(x)$ itself, so we have:

$$g(f(a))(x) = a(x)$$

So we're done: f composed with g is the identity function, so f and g are inverses, so $f : A \to B$ is invertible, so there is a 1-to-1 correspondence between A and B. QED.