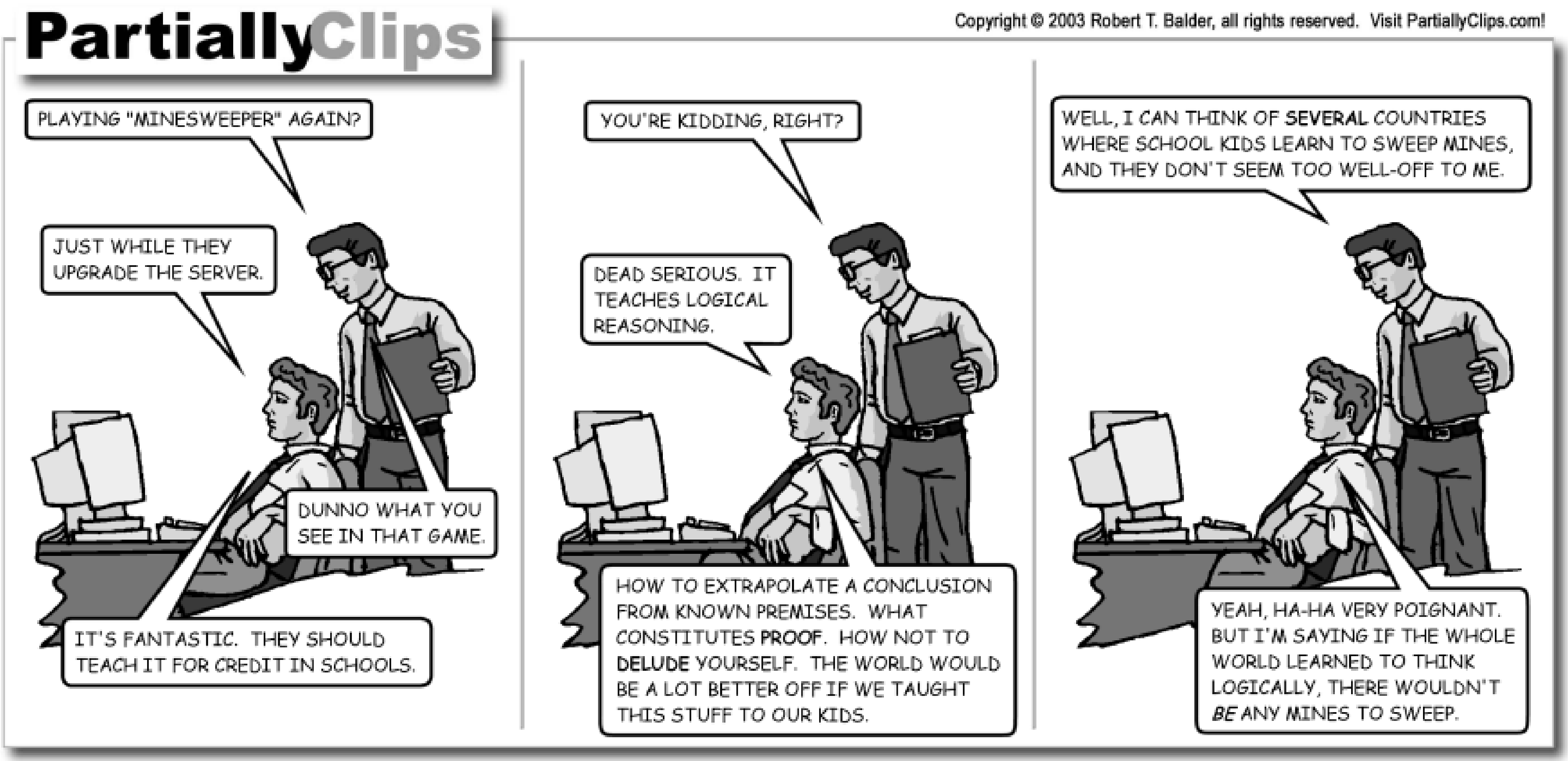
Automated Theorem Proving and Proof Checking



Cunning Theorem-Proving Plan

- There are full-semester courses on automated deduction; we will elide details.
- Logic Syntax
- Theories
- Satisfiability Procedures
- Mixed Theories
- Theorem Proving
- Proof Checking
- SAT-based Theorem Provers (cf. Engler paper)

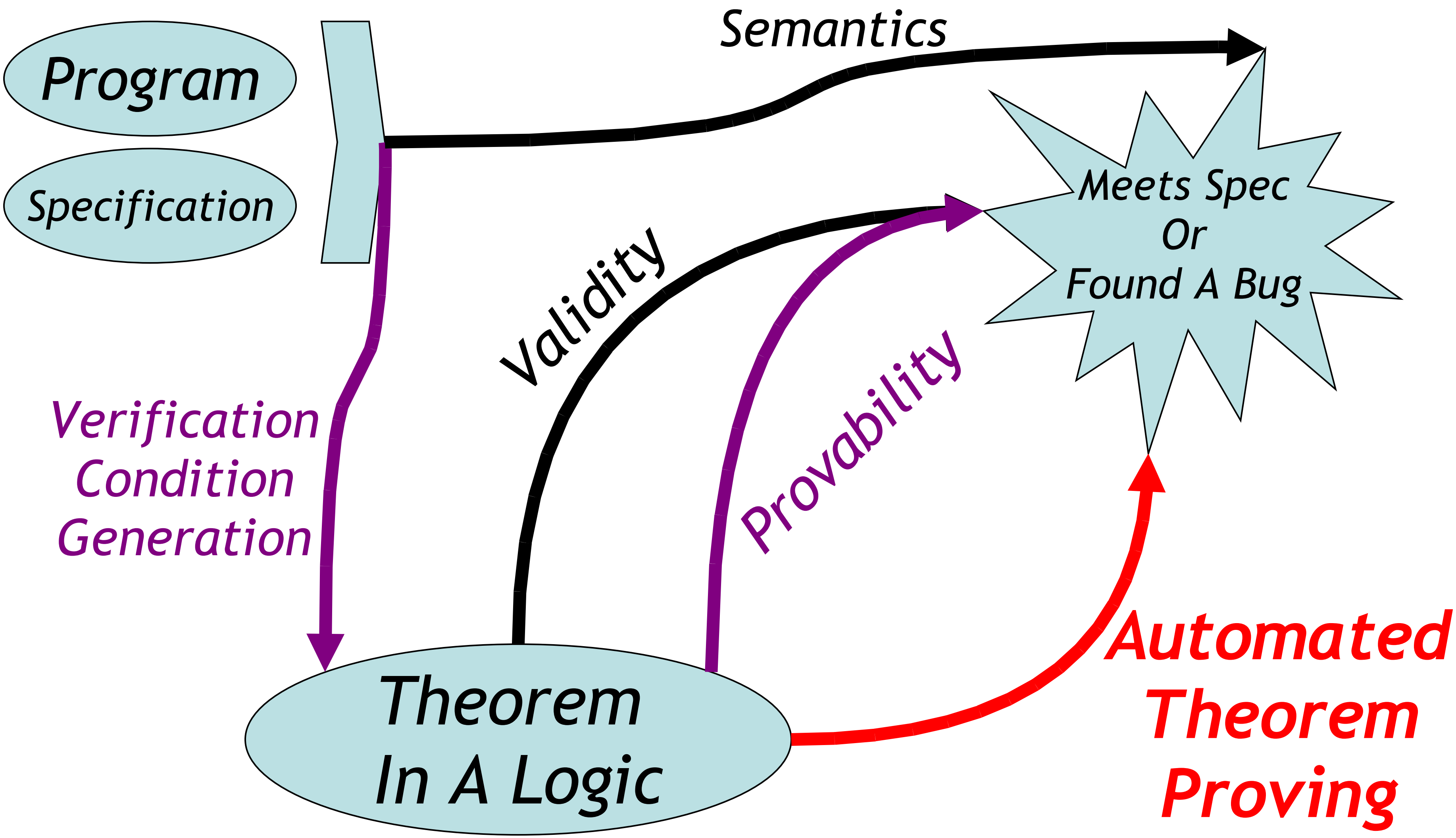
Motivation

- Can be viewed as "*decidable AI*"
  - Would be nice to have a procedure to automatically reason from premises to conclusions …
- Used to rule out the exploration of infeasible paths (model checking, dataflow)
- Used to reason about the heap (McCarthy, symbolic execution)
- Used to automatically synthesize programs from specifications (e.g. Leroy, Engler optional papers)
- Used to discover proofs of conjectures (e.g., Tarski conjecture proved by machine in 1996, efficient geometry theorem provers)
- Generally under-utilized

History

- Automated deduction is *logical deduction performed by a machine*
- Involves logic and mathematics
- One of the oldest and technically deepest fields of computer science
  - Some results are as much as 75 years old
  - “Checking a Large Routine”, Turing 1949
  - Automation efforts are about 40 years old
  - Floyd-Hoare axiomatic semantics
- Still experimental (even after 40 years)

Standard Architecture



Logic Grammar

- We'll use the following logic:

Goals: $\mathbf{G ::= L \mid true \mid G_1 \land G_2 \mid H \Rightarrow G \mid \forall x.\ G}$

Hypotheses: $\mathbf{H ::= L \mid true \mid H_1 \land H_2}$

Literals: $\mathbf{L ::= p(E_1, \ldots, E_k)}$

Expressions: $\mathbf{E ::= n \mid f(E_1, \ldots, E_m)}$

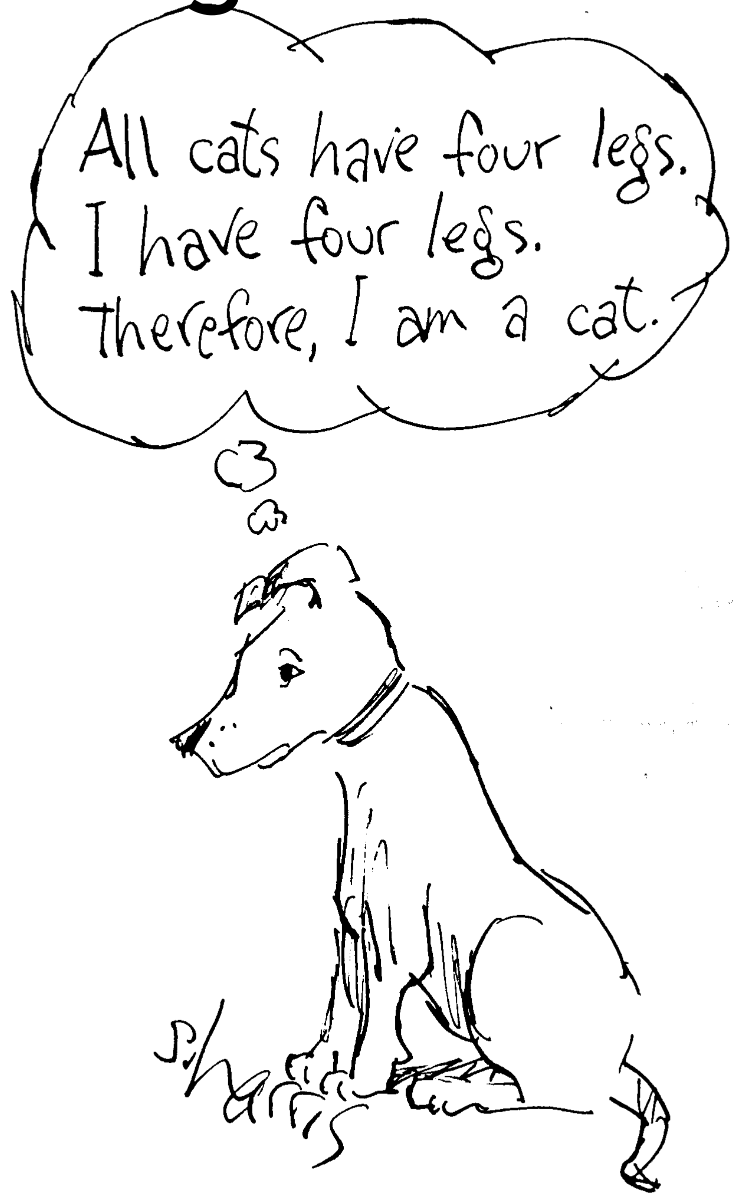
- This is a subset of first-order logic
  - Intentionally restricted: no $\lor$ so far
  - Predicate functions p: <, =, …
  - Expression functions f: +, *, sel, upd,

Theorem Proving Problem

- Write an algorithm "prove" such that:
- If prove(G) = true then $\models G$
  - Soundnes (must have)
- If $\models G$ then prove(G) = true
  - Completeness (nice to have, optional)
- prove(H,G) means prove $H \Rightarrow G$
- Architecture: Separation of Concerns
  - #1. Handle $\wedge$, $\Rightarrow$, $\forall$, =
  - #2. Handle $\leq$, *, sel, upd, =

Theorem Proving

- Want to prove true things
- Avoid proving false things
- We'll do **proof-checking** later to rule out the "cat proof" shown here
- For now, let's just get to the point where we can prove something



Basic Symbolic Theorem Prover

- Let's define **prove(H,G)** …

prove(H, true) = true

$\text{prove}(H, G_1 \wedge G_2) = \text{prove}(H, G_1)$ && $\text{prove}(H, G_2)$

$\text{prove}(H_1, H_2 \Rightarrow G) = \text{prove}(H_1 \wedge H_2, G)$

$\text{prove}(H, \forall x.\ G) = \text{prove}(H, G[a/x])$

(a is "fresh")

prove(H, L) = ???

Theorem Prover for Literals

- We have reduced the problem to

$$\mathbf{prove(H,L)}$$

- But H is a conjunction of literals $L_1 \wedge \ldots \wedge L_k$
- Thus we really have to prove that

$$L_1 \wedge \ldots \wedge L_k \Rightarrow L$$

- Equivalently, that $L_1 \wedge \ldots \wedge L_k \wedge \neg L$ is <u>unsatisfiable</u>
  - For any assignment of values to variables the truth value of the conjunction is false
- Now we can say

$$\mathbf{prove(H,L) = Unsat(H \wedge \neg L)}$$

Theory Terminology

- A theory consists of a set of functions and predicate symbols (*syntax*) and definitions for the meanings of those symbols (*semantics*)
- Examples:
  - **0, 1, -1, 2, -3, …, +, -, =, <** (usual meanings; "theory of integers with arithmetic" or "Presburger arithmetic")
  - **=, $\leq$** (axioms of transitivity, anti-symmetry, and $\forall x.\ \forall y.\ x \leq y \lor y \leq x$; "theory of total orders")
  - **sel, upd** (McCarthy's "theory of lists")

Decision Procedures for Theories

- The Decision Problem
  - Decide whether a formula in a theory with first-order logic is true
- Example:
  - Decide "$\forall x.\ x>0 \Rightarrow (\exists y.\ x=y+1)$" in $\{\mathbb{N}, +, =, >\}$
- A theory is decidable when there is an algorithm that solves the decision problem
  - This algorithm is the decision procedure for that theory

Satisfiability Procedures

- The Satisfiability Problem
  - Decide whether a *conjunction of literals* in the theory is satisfiable
  - Factors out the first-order logic part
  - The decision problem can be reduced to the satisfiability problem
    - Parameters for $\forall$, skolem functions for $\exists$, negate and convert to DNF (sorry; I won't explain this here)
- "Easiest" Theory = Propositional Logic = SAT
  - A decision procedure for it is a "SAT solver"

Theory of Equality

- Theory of equality with *uninterpreted functions*
- Symbols: =, ≠, **f**, **g**, …
- Axiomatically defined (A,B,C ∈ Expressions):

$$\frac{}{A=A} \qquad \frac{B=A}{A=B} \qquad \frac{A=B \quad B=C}{A=C} \qquad \frac{A=B}{f(A)=f(B)}$$

- Example satisfiability problem:

$$g(g(g(x)))=x \wedge g(g(g(g(g(x)))))=x \wedge g(x)\neq x$$

More Satisfying Examples

- Theory of Linear Arithmetic
  - Symbols: $\geq$, =, +, -, **integers**
  - Example: $y > 2x + 1,\ x > 1,\ y < 0$ is unsat
  - Satisfiability problem is in P (loosely, no multiplication means no tricky encodings)
- Theory of Lists
  - Symbols: **cons, head, tail, nil**

$$\frac{}{\text{head}(\text{cons}(A,B)) = A} \qquad \frac{}{\text{tail}(\text{cons}(A,B) = B}$$

  - Theorem: $\text{head}(x) = \text{head}(y) \wedge \text{tail}(x) = \text{tail}(y) \Rightarrow x = y$

Mixed Theories

- Often we have facts involving *symbols from multiple theories*
  - E's symbols =, $\neq$, f, g, … (uninterp function equality)
  - R's symbols =, $\neq$, +, -, $\leq$, 0, 1, … (linear arithmetic)
  - Running Example (and Fact):

    $\models x \leq y \ \wedge\ y + z \leq x \ \wedge\ 0 \leq z \ \Rightarrow\ f(f(x) - f(y)) = f(z)$
  - To prove this, we must decide:

    Unsat$(x \leq y,\ y + z \leq x,\ 0 \leq z,\ f(f(x) - f(y)) \neq f(z))$
- We may have a sat procedure for each theory
  - E's sat procedure by Ackermann in 1924
  - R's proc by Fourier
- The sat proc for their combination is much harder
  - Only in 1979 did we get E+R

Satisfiability of Mixed Theories

$\text{Unsat}(x \leq y,\ y + z \leq x,\ 0 \leq z,\ f(f(x) - f(y)) \neq f(z))$

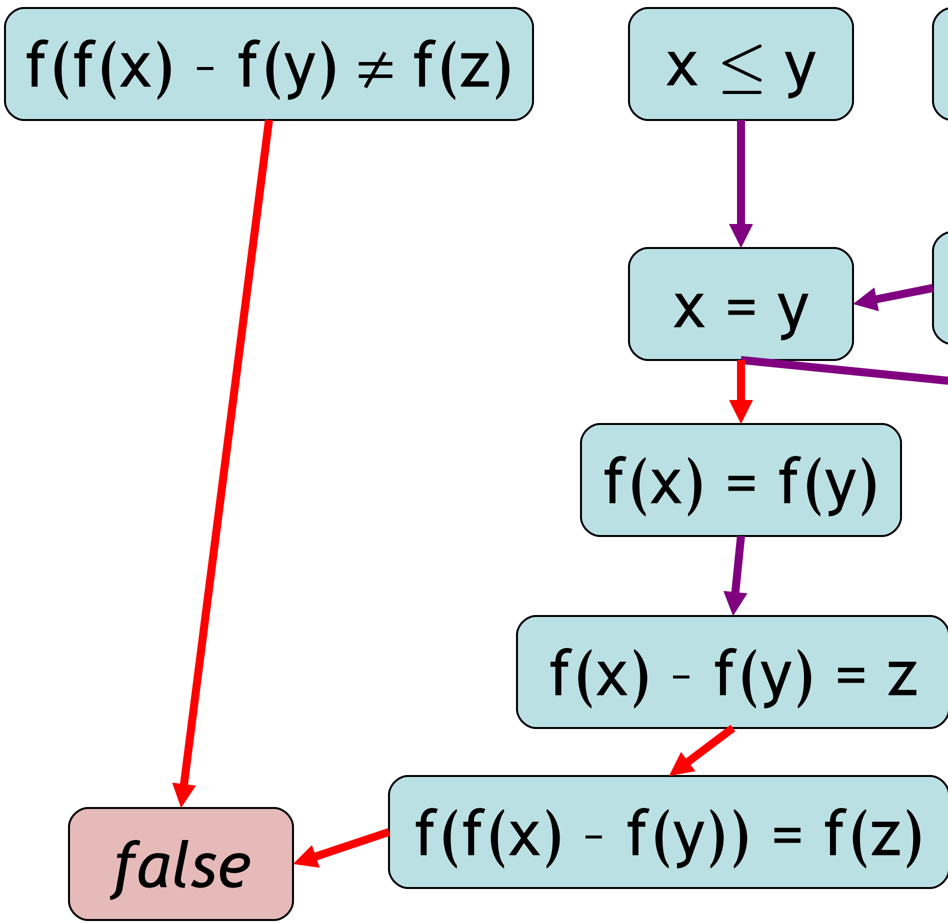
- Can we just separate out the terms in Theory 1 from the terms in Theory 2 and see if they are separately safisfiable?
  - No, unsound, equi-sat $\neq$ equivalent.
- The problem is that the two satisfying assignments may be incompatible
- Idea (Nelson and Oppen): Each sat proc announces all equalities between variables that it discovers

Handling Multiple Theories

- We'll use cooperating decision procedures
- Each sat proc works on the literals it understands
- Sat procs share information (equalities)

"THEN, AS YOU CAN SEE, WE GIVE THEM SOME MULTIPLE CHOICE TESTS."

Consider Equality and Arith

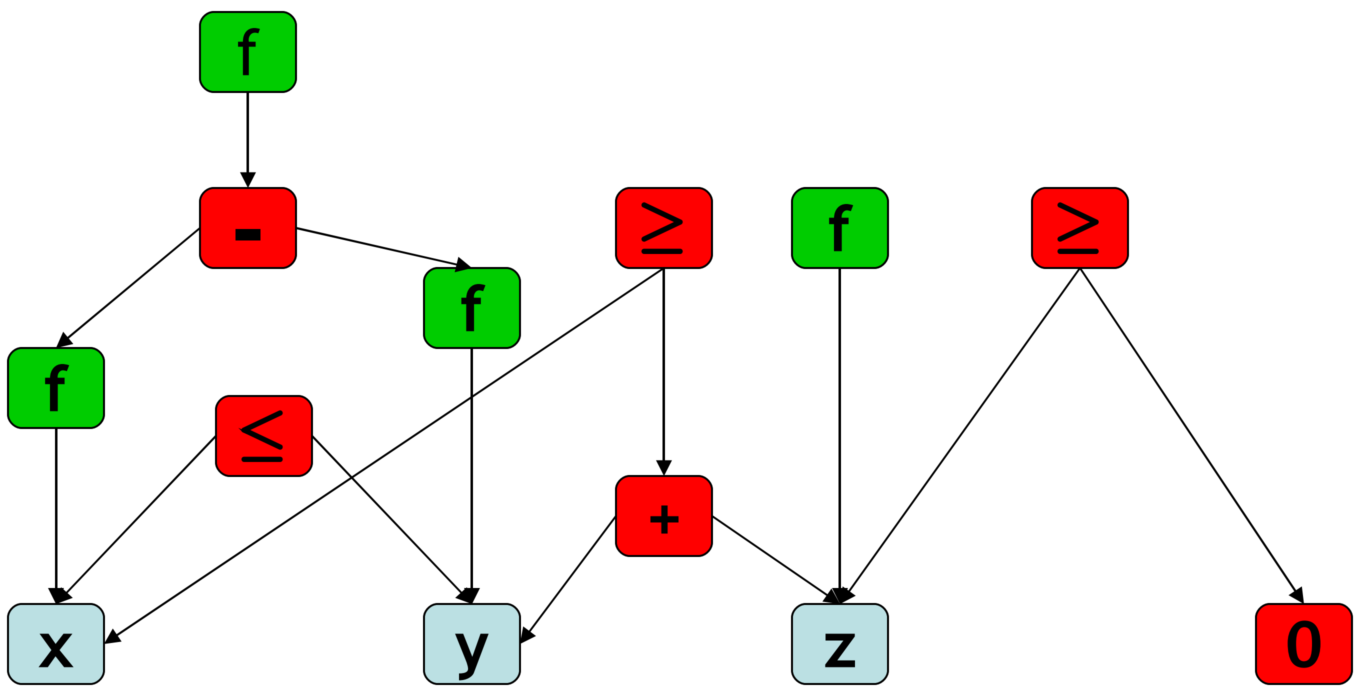


- *How can we do this in our prover?*

Nelson-Oppen: The E-DAG

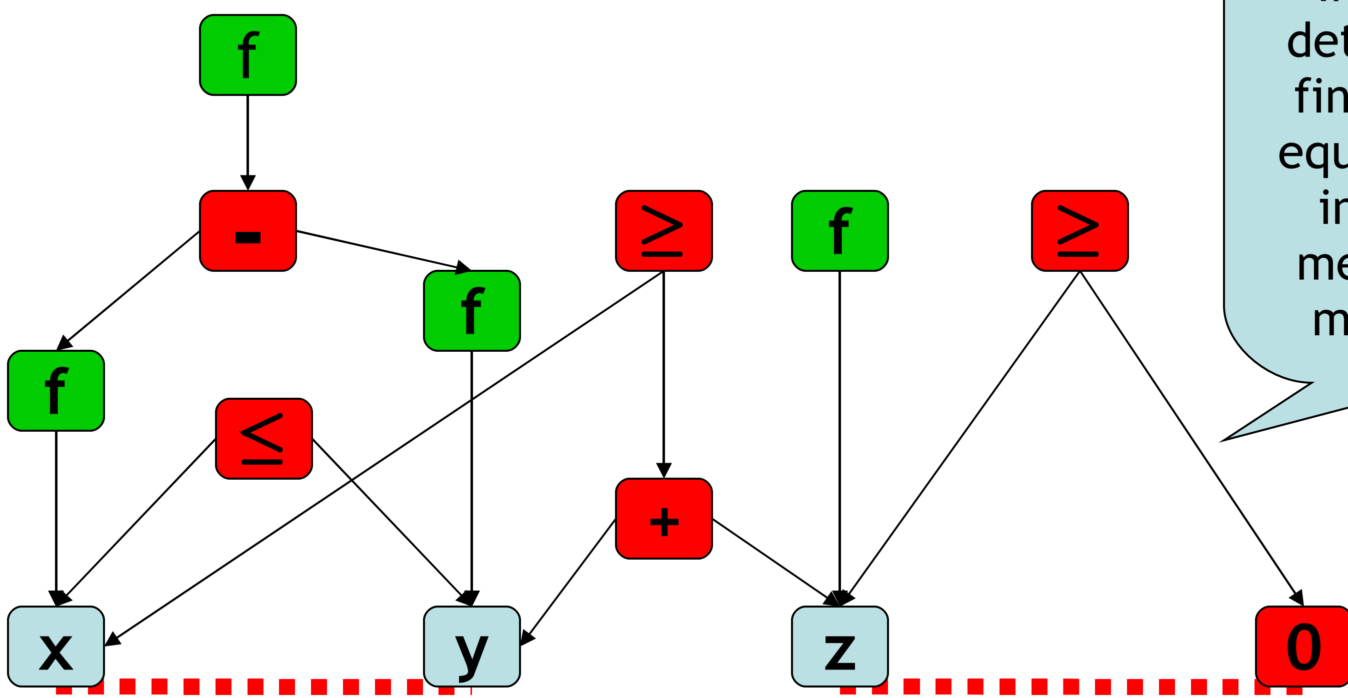
- Represent all terms in one <u>Equivalence DAG</u>
  - Node names act as variables shared between theories!

$$f(f(x) - f(y)) \neq f(z) \wedge y \geq x \wedge x \geq y + z \wedge z \geq 0$$



Nelson-Oppen: Processing

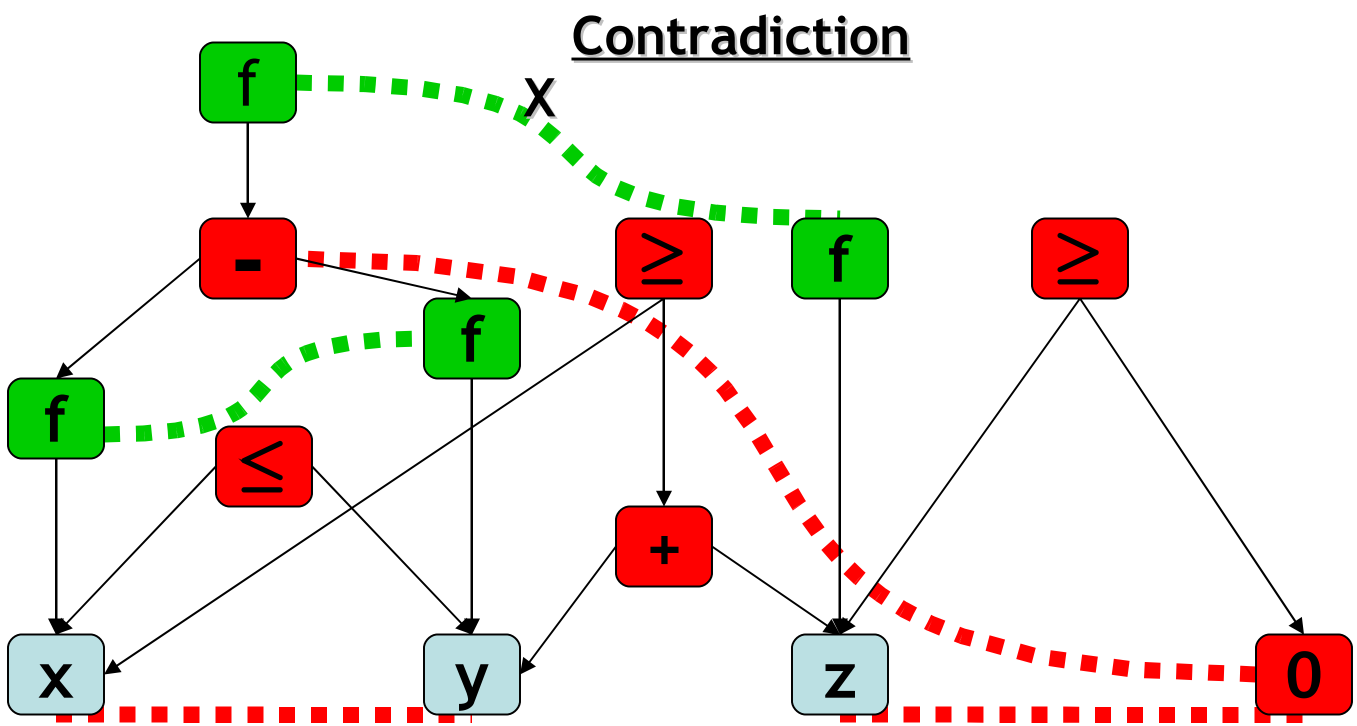
- Run each sat proc
  - Report all contradictions (as usual)
  - <u>Report all equalities</u> between nodes (key idea)



Implementation details: Use union-find to track node equivalence classes in E-DAG. When merging A=B, also merge f(A)=f(B).

Nelson-Oppen: Processing

- Broadcast all discovered equalities
  - Rerun sat procedures
  - <u>Until no more equalities or a contradiction</u>



Does It Work?

- If a contradiction is found, then unsat
  - This is sound if sat procs are sound
  - Because only sound equalities are ever found
- If there are no more equalities, then sat
  - Is this complete? Have they shared enough info?
  - Are the two satisfying assignments compatible?
  - Yes!
  - *(Countable theories with infinite models admit isomorphic models, convex theories have necessary interpretations, etc.)*

SAT-Based Theorem Provers

- Recall separation of concerns:
  - #1 Prover handles connectives ($\forall$, $\wedge$, $\Rightarrow$)
  - #2 Sat procs handle literals (+, $\leq$, 0, head)
- Idea: reduce proof obligation into propositional logic, feed to SAT solver (CVC)
  - To Prove: $3*x=9 \Rightarrow (x = 7 \wedge x \leq 4)$
  - Becomes Prove: $A \Rightarrow (B \wedge C)$
  - Becomes Unsat: $A \wedge \neg(B \wedge C)$
  - Becomes Unsat: $A \wedge (\neg B \vee \neg C)$

SAT-Based Theorem Proving

- To Prove: $3*x=9 \Rightarrow (x = 7 \wedge x \leq 4)$
  - Becomes Unsat: $A \wedge (\neg B \vee \neg C)$
  - SAT Solver Returns: A=1, C=0
  - Ask sat proc: unsat($3*x=9$, $\neg\ x \leq 4$) = true
  - Add constraint: $\neg(A \wedge \neg C)$
  - Becomes Unsat: $A \wedge (\neg B \vee \neg C) \wedge \neg(A \wedge \neg C)$
  - SAT Solver Returns: A=1, B=0, C=1
  - Ask sat proc: unsat($3*x=9$, $\neg\ x=7$, $x \leq 4$) = false
    - (x=3 is a satisfying assignment)
  - We're done! (original to-prove goal is false)
  - If SAT Solver returns "no satisfying assignment" then original to-prove goal is true

Proofs

“Checking proofs ain’t like dustin’ crops, boy!”

Proof Generation

- We want our theorem prover to **emit proofs**
  - No need to trust the prover
  - Can find bugs in the prover
  - Can be used for proof-carrying code
  - Can be used to extract invariants
  - Can be used to extract models (e.g., in SLAM)
- Implements the soundness argument
  - On every run, a soundness proof is constructed

Proof Representation

- Proofs are trees
  - Leaves are hypotheses/axioms
  - Internal nodes are inference rules
- Axiom: “true introduction”
  - Constant: truei : pf
  - pf is the type of proofs
- Inference: “conjunction introduction”
  - Constant: andi : pf $\rightarrow$ pf $\rightarrow$ pf
- Inference: “conjunction elimination”
  - Constant: andel : pf $\rightarrow$ Pf
- Problem:
  - “andel truei : pf” but does not represent a valid proof
  - Need a more powerful *type system that checks content*

$$\frac{}{\vdash \mathrm{true}}\ \textit{truei}$$

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\ \textit{andi}$$

$$\frac{\vdash A \wedge B}{\vdash A}\ \textit{andel}$$

Dependent Types

- Make pf a family of types indexed by formulas
  - f : Type (type of encodings of formulas)
  - e : Type (type of encodings of expressions)
  - pf : f $\rightarrow$ Type (the type of proofs indexed by formulas: it is a proof ***that f is true***)
- Examples:
  - true : f
  - and : f $\rightarrow$ f $\rightarrow$ f
  - truei : pf true
  - andi : pf A $\rightarrow$ pf B $\rightarrow$ pf (and A B)
  - **andi : ΠA:f. ΠB:f. pf A $\rightarrow$ pf B $\rightarrow$ pf (and A B)**

Proof Checking

- Validate proof trees by type-checking them
- Given a proof tree X claiming to prove A ∧ B
- Must check X : pf (and A B)
- We use "expression tree equality", so
  - andel (andi "1+2=3" "x=y") does ***not*** have type pf (3=3)
  - This is already a proof system! If the proof-supplier wants to use the fact that 1+2=3 ⇔ 3=3, she can include a proof of it somewhere!
- Thus Type Checking = Proof Checking
  - And it's quite easily *decidable*! □

Parametric Judgment (Time?)

- Universal Introduction Rule of Inference

$$\frac{\vdash [a/x]A \quad \text{(a is fresh)}}{\vdash \forall x.\ A}$$

- We represent bound variables in the logic using bound variables in the meta-logic
  - all : $(e \rightarrow f) \rightarrow f$
  - Example: $\forall x.\ x=x$ represented as (all (λx. eq x x))
  - Note: $\forall y.\ y=y$ has an α-equivalent representation
  - Substitution is done by β-reduction in meta-logic
    - **[E/x](x=x) is** **(λx. eq x x) E**

Parametric ∀ Proof Rules (Time?)

$$\frac{\vdash [a/x]A \quad \text{(a is fresh)}}{\vdash \forall x.\ A}$$

- Universal Introduction
  - alli: $\Pi A{:}(e \to f).\ (\Pi a{:}e.\ \text{pf}\ (A\ a)) \to \text{pf}\ (\text{all}\ A)$

$$\frac{\vdash \forall x.\ A}{\vdash [E/x]A}$$

- Universal Elimination
  - alle: $\Pi A{:}(e \to f).\ \Pi E{:}e.\ \text{pf}\ (\text{all}\ A) \to \text{pf}\ (A\ E)$

Parametric $\exists$ Proof Rules (Time?)

$$\frac{\vdash [E/x]A}{\vdash \exists x.\ A}$$

- Existential Introduction
  - existi: ΠA:(e $\to$ f). ΠE:e. pf (A E) $\to$ pf (exists A)

$$\frac{\vdash \exists x.\ A \quad \begin{array}{c} \vdash [a/x]A \\ \ldots \\ \vdash B \end{array}}{\vdash B}$$

- Existential Elimination
  - existe: ΠA:(e $\to$ f). ΠB:f.

    pf (exists A) $\to$ (Πa:e. pf (A a) $\to$ pf B) $\to$ pf B

Homework

- Project
  - Need help? Stop by my office or send email.