Lambda Calculus

λ

Plan

- Introduce lambda calculus
  - Syntax
  - Substitution
  - Operational Semantics (… with contexts!)
  - Evaluations strategies
  - Equality
- Later:
  - Relationship to programming languages
  - Study of types and type systems

Lambda Background

- Developed in 1930's by Alonzo Church
- Subsequently studied by many people
  - Still studied today!
- Considered the "testbed" for procedural and functional languages
  - Simple
  - Powerful
  - Easy to extend with new features of interest
  - Lambda:PL :: Turning Machine:Complexity
  - Somewhat like a crowbar …

"Whatever the next 700 languages turn out to be, they will surely be variants of lambda calculus."

(Landin '66)

Lambda Syntax

- The λ-calculus has 3 kinds of expressions (terms)

e ::= x		Variables
|	$\lambda x.\ e$	Functions (abstractions)
|	$e_1\ e_2$	Application

- $\lambda x.\ e$ is a one-argument anonymous function with body e
- $e_1\ e_2$ is a function application
- Application associates to the left

$$x\ y\ z \equiv\equiv (x\ y)\ z$$

- Abstraction extends far to the right

$$\lambda x.\ x\ \lambda y.\ x\ y\ z === \lambda x.\ (x\ [\lambda y.\ \{(x\ y)\ z\}])$$

Why Should I Care?

- A language with 3 expressions? Woof!
- Li and Zdancewic. *Downgrading policies and relaxed noninterference*. POPL '05
  - Just one example of a recent PL/security paper

4. LOCAL DOWNGRADING POLICIES

4.1 Label Definition

Definition 4.1.1 (The policy language). *In Figure 1.*

Types	$\tau ::=$	$\mathsf{int} \mid \tau \to \tau$
Constants	$c ::=$	c_i
Operators	$\oplus ::=$	$+, -, =, \ldots$
Terms	$m ::=$	$\lambda x\!:\!\tau.\ m \mid m\ m \mid x \mid c \mid m \oplus m$
Policies	$n ::=$	$\lambda x\!:\!\mathsf{int}.\ m$
Labels	$l ::=$	$\{n_1, \ldots, n_k\} \quad (k \geq 1)$

Figure 1: $\mathbb{L}_{\mathsf{local}}$ Label Syntax

The core of the policy language is a variant of the simply-typed λ-calculus with a base type, binary operators and constants. A **downgrading policy** is a λ-term that specifies how an integer can be downgraded: when this λ-term is applied to the annotated integer, the result becomes public. A

$$\frac{\Gamma \vdash m : \tau}{\Gamma \vdash m \equiv m : \tau} \quad \text{Q-REFL}$$

$$\frac{\Gamma \vdash m_1 \equiv m_2 : \tau}{\Gamma \vdash m_2 \equiv m_1 : \tau} \quad \text{Q-SYMM}$$

$$\frac{\Gamma \vdash m_1 \equiv m_2 : \tau \qquad \Gamma \vdash m_2 \equiv m_3 : \tau}{\Gamma \vdash m_1 \equiv m_3 : \tau} \quad \text{Q-TRANS}$$

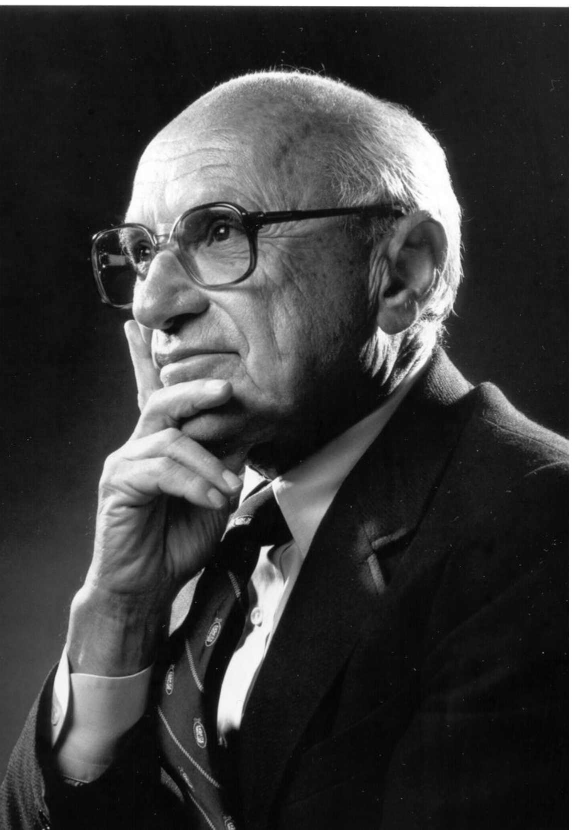
$$\frac{\Gamma, x\!:\!\tau_1 \vdash m_1 \equiv m_2 : \tau_2}{\Gamma \vdash \lambda x\!:\!\tau_1.\ m_1 \equiv \lambda x\!:\!\tau_1.\ m_2 : \tau_1 \to \tau_2} \quad \text{Q-ABS}$$

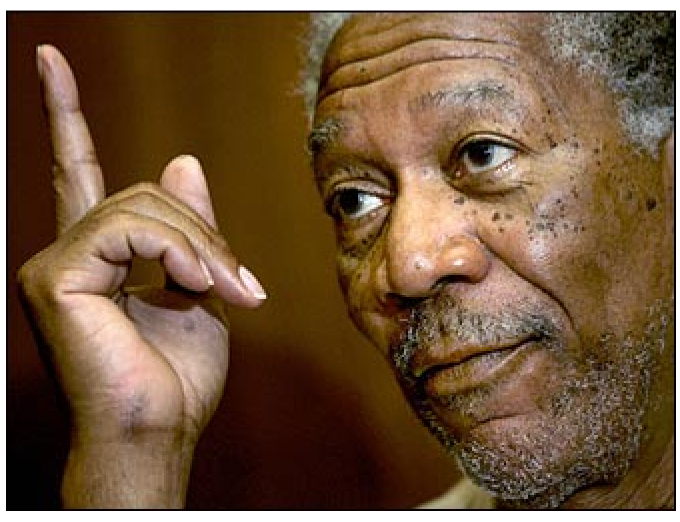
$$\frac{\Gamma \vdash m_1 \equiv m_2 : \tau_1 \to \tau_2 \qquad \Gamma \vdash m_3 \equiv m_4 : \tau_1}{\Gamma \vdash m_1\ m_3 \equiv m_2\ m_4 : \tau_2} \quad \text{Q-APP}$$

$$\frac{\Gamma \vdash m_1 \equiv m_2 : \mathsf{int} \qquad \Gamma \vdash m_3 \equiv m_4 : \mathsf{int}}{\Gamma \vdash m_1 \oplus m_3 \equiv m_2 \oplus m_4 : \mathsf{int}} \quad \text{Q-BINOP}$$

Lambda Celebrity Representative

- Milton Friedman?
- Morgan Freeman?
- C. S. Friedman?





Gordon Freeman

- Best-selling PC FPS to date …

Examples of Lambda Expressions

- The identity function:

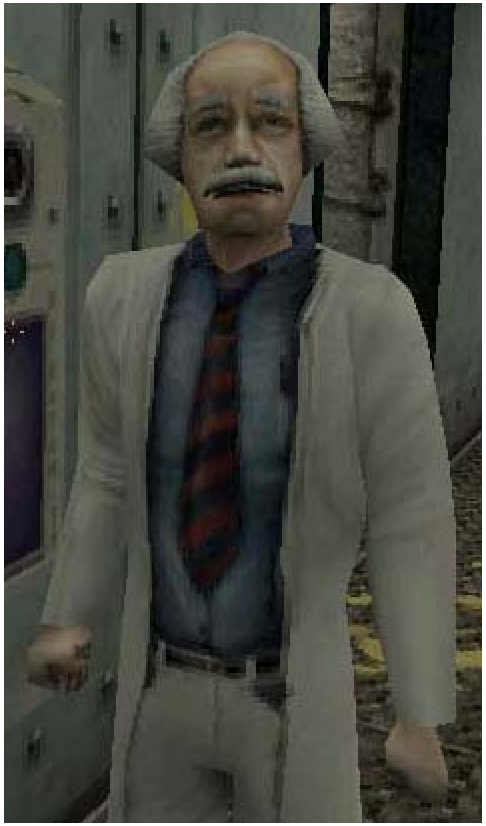
$$I =_{def} \lambda x.\ x$$

- A function that, given an argument y, discards it and yields the identity function:

$$\lambda y.\ (\lambda x.\ x)$$

- A function that, given an function f, invokes it on the identity function:

$$\lambda f.\ f\ (\lambda x.\ x)$$



"There goes our grant money."

Scope of Variables

- As in all languages with variables, it is important to discuss the notion of scope
  - The scope of an identifier is the portion of a program where the identifier is accessible
- An abstraction $\lambda x.\ E$ binds variable x in E
  - x is the newly introduced variable
  - E is the scope of x (unless x is shadowed)
  - We say x is bound in $\lambda x.\ E$
  - Just like formal function arguments are bound in the function body

Free and Bound Variables

- A variable is said to be free in E if it has occurrences that are not bound in E
- We can define the free variables of an expression E recursively as follows:
  - $\text{Free}(x) = \{x\}$
  - $\text{Free}(E_1\ E_2) = \text{Free}(E_1) \cup \text{Free}(E_2)$
  - $\text{Free}(\lambda x.\ E) = \text{Free}(E) - \{x\}$
- Example: $\text{Free}(\lambda x.\ x\ (\lambda y.\ x\ y\ z)) = \{z\}$
- Free variables are (implicitly or explicitly) declared outside the expression

Free Your Mind!

- Just as in any language with statically-nested scoping we have to worry about variable shadowing
  - An occurrence of a variable might refer to different things in different contexts
- Example in IMP with locals:

$$\text{let } x = 5 \text{ in } x + (\text{let } x = \underline{9} \text{ in } \underline{x}) + x$$

- In λ-calculus:

$$\lambda x.\ x\ (\lambda \underline{x}.\ \underline{x})\ x$$

Renaming Bound Variables

- λ-terms that can be obtained from one another by renaming bound variables are considered *identical*
- This is called α-equivalence
- Renaming bound vars is called α-renaming
- Ex: $\lambda x.\ x$ is identical to $\lambda y.\ y$ and to $\lambda z.\ z$
- Intuition:
  - By changing the name of a formal argument and all of its occurrences in the function body, the behavior of the function *does not change*
  - In λ-calculus such functions are considered identical

Make It Easy On Yourself

- Convention: we will always try to rename bound variables so that they are all unique
  - e.g., write $\lambda x.\ x\ (\lambda y.y)\ x$ instead of $\lambda x.\ x\ (\lambda x.x)\ x$
- This makes it easy to see the scope of bindings and also prevents confusion!

Substitution

- The substitution of F for x in E (written [F/x]E)
  - Step 1. Rename bound variables in E and F so they are unique
  - Step 2. Perform the textual substitution of f for X in E
- Called capture-avoiding substitution
- Example: $[y\ (\lambda x.\ x)\ /\ x]\ \lambda y.\ (\lambda x.\ x)\ y\ x$
  - After renaming: $[y\ (\lambda x.\ x)\ /\ x]\ \lambda z.\ (\lambda u.\ u)\ z\ x$
  - After substitution: $\lambda z.\ (\lambda u.\ u)\ z\ (y\ (\lambda x.\ x))$
- If we are not careful with scopes we might get:

$\lambda y.\ (\lambda x.\ x)\ y\ (y\ (\lambda x.\ x))$ $\leftarrow$ wrong!

The De Bruijn Notation

- An alternative syntax that avoids naming of bound variables (and the subsequent confusions)
- The De Bruijn index of a variable *occurrence* is that number of lambda that separate the occurrence from its binding lambda in the abstract syntax tree
- The De Bruijn notation replaces names of occurrences with their De Bruijn indices
- Examples:
  - **λ x. x** — **λ. 0**
  - **λ x. λ x. x** — **λ. λ. 0**
  - **λ x. λ y. y** — **λ. λ. 0**
  - **(λ x. x x) (λ z. z z)** — **(λ. 0 0) (λ. 0 0)**
  - **λ x. (λ x. λ y. x) x** — **λ. (λ. λ. 1) 0**

Identical terms have identical representations!

Combinators

- A λ-term without free variables is closed or a combinator
- Some interesting combinators:

  I = λ x. x

  K = λ x. λ y. x

  S = λ f. λ g. λ x. f x (g x)

  D = λ x. x x

  Y = λ f. (λ x. f (x x)) (λ x. f (x x))

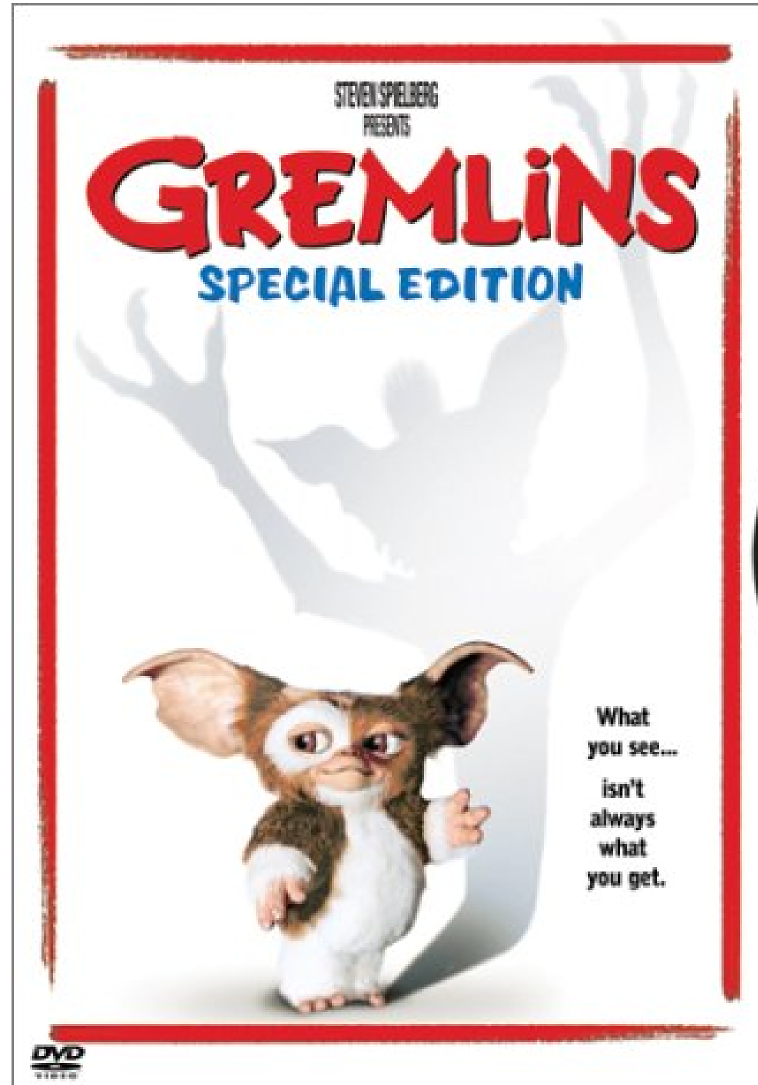
- Theorem: any closed term is equivalent to one written with just S, K and I
  - Example: D $=_\beta$ S I I
  - (we'll discuss this form of equivalence later)

Q: Music (241 / 842)

- Name the singer and his crossover 1982 album that holds (as of 2005) the record of being the best-selling album of all-original material in the US (26 times platinum, 37 weeks as Billboard #1). Much of that success was the result of the singer's use of the MTV music video.

Q: Movies (262 / 842)

- Name two of the three rules given for the pet Gizmo in the 1984 movie **Gremlins.**



Q: Movies (341 / 842)

- This 1993 Mel Brooks parody film features "The Man in Black" as "Kevin Costner" and also stars Patrick Stewart as King Richard. It includes the exchange: *"And why would the people listen to you? / Because, unlike some other Robin Hoods, I can speak with an English accent."*

Q: General (452 / 842)

- Name any 3 of the 22 letters in the Hebrew alphabet.

Q: Games (489 / 842)

- This 1965 **Wham-O** toy is an extremely elastic sphere made of a rubber polymer with a high coefficient of restitution. When dropped from shoulder level onto a hard surface it rebounds to about 90% of its original height.

A: Games (489 / 842)

- **Super Ball**
  - *Trivia: When Lamar Hunt saw his daughter playing with a Super Ball, it inspired him to name the new AFL-NFL World Championship Game the Super Bowl.*

Informal Semantics

- We consider only closed terms
- The evaluation of

$$(\lambda\, x.\ e)\ f$$

  - Binds x to f
  - Evaluates e *with the new binding*
  - Yields the result of this evaluation
- Like a function call, or like "let x = f in e"
- Example:

$(\lambda\, f.\ f\ (f\ e))\ g$ *evaluates to* $g\ (g\ e)$

Operational Semantics

- Many operational semantics for the λ-calculus
- All are based on the equation

  $$(\lambda\ x.\ e)\ f =_\beta [f/x]e$$

  usually read from left to right
- This is called the β-rule and the evaluation step a β-reduction
- The subterm $(\lambda\ x.\ e)\ f$ is a β-redex
- We write $e \to_\beta g$ to say that e β-reduces to g in one step
- We write $e \to_\beta^* g$ to say that e β-reduces to g in 0 or more steps
  - Remind you of the small-step opsem term rewriting?

Examples of Evaluation

- The identity function:

$$(\lambda\ x.\ x)\ E \rightarrow [E\ /\ x]\ x = E$$

- Another example with the identity:

$(\lambda\ f.\ f\ (\lambda\ x.\ x))\ (\lambda\ x.\ x) \rightarrow$

$[\lambda\ x.\ x\ /\ f]\ f\ (\lambda\ x.\ x)) =$

$[\lambda\ x.\ x\ /\ f]\ f\ (\lambda\ y.\ y)) =$

$(\lambda\ x.\ x)\ (\lambda\ y.\ y) \rightarrow$

$[\lambda\ y.\ y\ /\ x]\ x = \lambda\ y.\ y$

- A *non-terminating* evaluation:

$(\lambda\ x.\ xx)\ (\lambda\ y.\ yy) \rightarrow$

$[\lambda\ y.\ yy\ /\ x]\ xx = (\lambda\ y.\ yy)\ (\lambda\ y.\ yy) \rightarrow \ldots$

- Try T T, where T = $\lambda x.\ x\ x\ x$

Evaluation and the Static Scope

- The definition of substitution guarantees that evaluation respets static scoping:

$$(\lambda\ x.\ (\lambda\ y.\ y\ x))\ (y\ (\lambda\ x.\ x)) \to_\beta \lambda\ z.\ z\ (y\ (\lambda\ v.\ v))$$

(y remains free, i.e., defined externally)

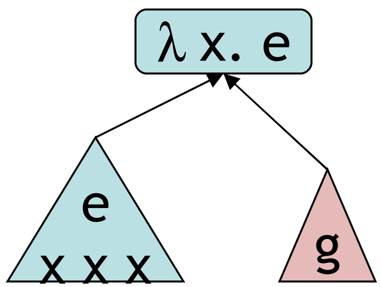
- If we forget to rename the bound y:

$$(\lambda\ x.\ (\lambda\ y.\ y\ x))\ (y\ (\lambda\ x.\ x)) \to_\beta^{*} \lambda\ y.\ y\ (y\ (\lambda\ v.\ v))$$

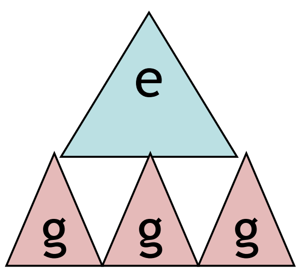
(y was free before but *is bound now*)

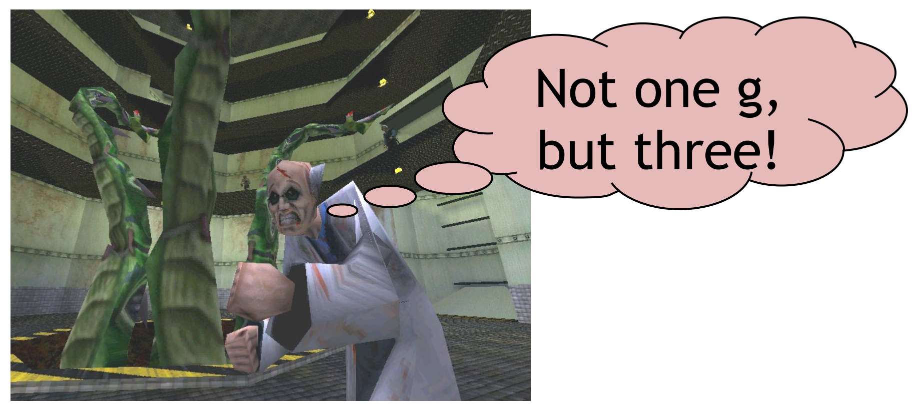
Another View of Reduction

- The application



- Becomes:





(terms can grow substantially through β-reduction!)

Normal Forms

- A term without redexes is in normal form
- A reduction sequence stops at a normal form

- If e is in normal form and $e \to_\beta^* f$ then e is identical to f

- $K = \lambda\ x.\ \lambda\ y.\ x$ is in normal form
- $K\ I$ is *not* in normal form

Nondeterministic Evaluation

- We define a small-step reduction relation

$$\frac{}{(\lambda\ x.\ e)\ f \to [f/x]e}$$

$$\frac{e_1 \to e_2}{e_1\ f \to e_2\ f} \qquad \frac{f_1 \to f_2}{e\ f_1 \to e\ f_2}$$

$$\frac{e \to f}{\lambda\ x.\ e \to \lambda\ x.\ f}$$

- This is a non-deterministic semantics
- Note that we evaluate under λ *(where?)*

Lambda Calculus Contexts

- Define contexts with one hole
- H ::= • | λ x. H | H e | e H
- Write H[e] to denote the filling of the hole in H with the expression e
- Example:

  H = λ x. x •   H[λ y. y] = λ x. x (λ y. y)

- Filling the hole allows variable capture!

  H = λ x. x •   H[x] = λ x. x x

Contextual Opsem

$$\frac{}{(\lambda\ x.\ e)\ f \to [f/x]e} \qquad \frac{e \to f}{H[e] \to H[f]}$$

- Contexts allow concise formulations of congruence rules (application of local reduction rules on subterms)
- Reduction occurs at a β-redex that can be anywhere inside the expression
- The latter rule is called a congruence or structural rule
- The above rules to not specify which redex must be reduced first

The Order of Evaluation

- In a λ-term there could be more than one instance of (λ x. e) f, as in:

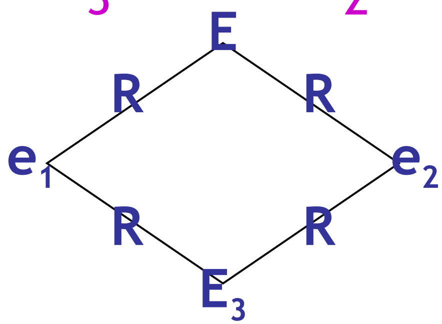
  (λ y. (λ x. x) y) E

  - Could reduce the inner or outer λ
  - Which one should we pick?

$$\begin{tikzcd}[column sep=small] & (\lambda\ y.\ (\lambda\ x.\ x)\ y)\ E \arrow[dl, "\text{inner}"'] \arrow[dr, "\text{outer}"] & \\ (\lambda\ y.\ [y/x]\ x)\ E = (\lambda\ y.\ y)\ E \arrow[dr] & & [E/y]\ (\lambda\ x.\ x)\ y = (\lambda\ x.\ x)\ E \arrow[dl] \\ & E & \end{tikzcd}$$

The Diamond Property

- A relation R has the diamond property if whenever $e\ R\ e_1$ and $e\ R\ e_2$ then there exists e_3 such that $e_1\ R\ e_3$ and $e_2\ R\ e_3$



- $\rightarrow_\beta$ does *not* have the diamond property
- $\rightarrow_\beta{}^*$ has the diamond property
- Also called the confluence property

A Diamond In The Rough

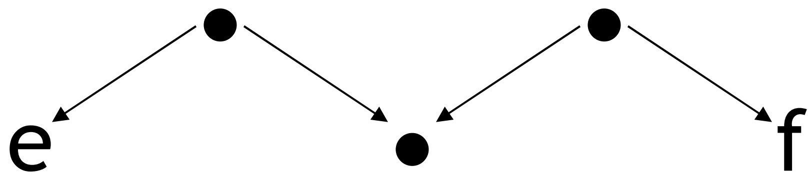
- Languages defined by non-deterministic sets of rules are common
  - Logic programming languages
  - Expert systems
  - Constraint satisfaction systems
    - And thus most pointer analyses …
  - Dataflow systems
  - Makefiles
- It is useful to know whether such systems have the diamond property

(Beta) Equality

- Let $=_\beta$ be the reflexive, transitive and **symmetric** closure of $\rightarrow_\beta$

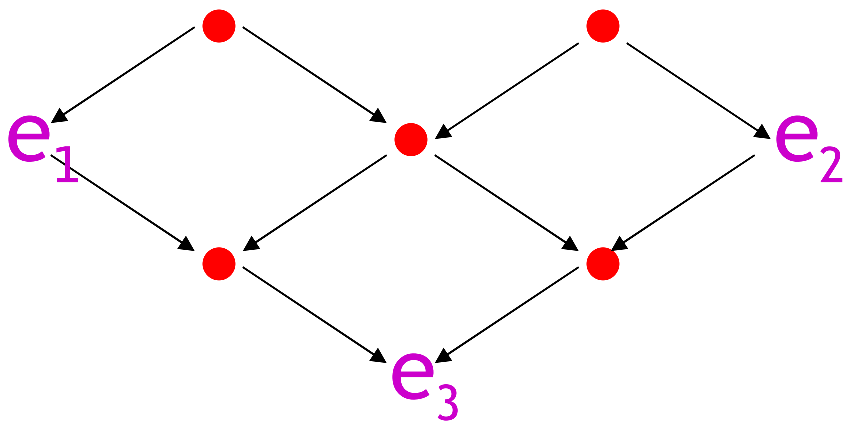
$$\mathbf{=_\beta \text{ is } (\rightarrow_\beta \cup \leftarrow_\beta)^*}$$

- That is, $e =_\beta f$ if e converts to f via a sequence of forward and backward $\rightarrow_\beta$



The Church-Rosser Theorem

- If $e_1 =_\beta e_2$ then there exists e_3 such that $e_1 \to_\beta^* e_3$ and $e_2 \to_\beta^* e_3$



- Proof (informal): apply the diamond property as many times as necessary

Corollaries

- If $e_1 =_\beta e_2$ and e_1 and e_2 are normal forms then e_1 is identical to e_2
  - From C-R we have $\exists e_3.\ e_1 \rightarrow_\beta^* e_3$ and $e_2 \rightarrow_\beta^* e_3$
  - Since e_1 and e_2 are normal forms they are identical to e_3
- If $e \rightarrow_\beta^* e_1$ and $e \rightarrow_\beta^* e_2$ and e_1 and e_2 are normal forms then e_1 is identical to e_2
  - "All terms have a unique normal form."

Evaluation Strategies

- Church-Rosser theorem says that independent of the reduction strategy we will find $\leq$1 normal form
- But some reduction strategies might find 0
- $(\lambda\ x.\ z)\ \underline{((\underline{\lambda}\ y.\ y\ y)\ (\underline{\lambda}\ y.\ y\ y))} \to$

  $(\lambda\ x.\ z)\ \underline{((\underline{\lambda}\ y.\ y\ y)\ (\underline{\lambda}\ y.\ y\ y))} \to \ldots$
- $\underline{(\underline{\lambda}\ x.\ z)\ ((\underline{\lambda}\ y.\ y\ y)\ (\underline{\lambda}\ y.\ y\ y))} \to z$
- There are three traditional strategies
  - normal order (never used, always works)
  - call-by-name (rarely used, cf. TeX)
  - call-by-value (amazingly popular)

Civilization: Call By Value

- Normal Order
  - Evaluates the left-most redex not contained in another redex
  - If there is a normal form, this finds it
  - Not used in practice: requires partially evaluating function pointers and looking “inside” functions
- Call-By-Name (“lazy”)
  - Don’t reduce under λ, don’t evaluate a function argument (until you need to)
  - Does not always evaluate to a normal form
- **Call-By-Value (“eager” or “strict”)**
  - Don’t reduce under λ, *do* evaluate a function’s argument right away
  - Finds normal forms less often than the other two

Endgame

- This time: λ syntax, semantics, reductions, equality, …
- Next time: encodings, real prorams, type systems, and all the fun stuff!

Wisely done, Mr. Freeman. I will see you up ahead.

Homework

- Read Leroy article, think about axiomatic
- Homework 5 Due Later

Tricksy On The Board Answer

- Is this rule unsound?

$$\frac{\vdash \{A \wedge p\}\ c_{then}\ \{B_{then}\} \qquad \vdash \{A \wedge \neg p\}\ c_{else}\ \{B_{else}\}}{\vdash \{A\}\ \text{if } p \text{ then } c_{then} \text{ else } c_{else}\ \{B_{then} \vee B_{else}\}}$$

- Nope: it's our basic rule plus 2x consequence

$$\frac{\vdash \{A \wedge p\}\ c_1\ \{B\} \qquad \vdash \{A \wedge \neg\ p\}\ c_2\ \{B\}}{\vdash \{A\}\ \text{if } p \text{ then } c_1 \text{ else } c_2\ \{B\}}$$

$$\frac{\vdash A' \Rightarrow A \qquad \vdash \{A\}\ c\ \{B\} \qquad \vdash B \Rightarrow B'}{\vdash \{A'\}\ c\ \{B'\}}$$

- Note that $B_{then} \Rightarrow B_{then} \vee B_{else}$