Introduction to Denotational Semantics (1/2)

Gone in Sixty Seconds

- **Denotation semantics** is a formal way of assigning meanings to programs. In it, the meaning of a program is a **mathematical** object.
- Denotation semantics is **compositional**: the meaning of an expression depends on the meanings of subexpressions.
- Denotational semantics uses $\perp$ ("bottom") to mean **non-termination**.
- DS uses **fixed points** and **domains** to handle `while`.

Induction on Derivations Summary

- If you must prove $\forall x \in A.\ P(x) \Rightarrow Q(x)$
  - A is some structure (e.g., AST), P(x) is some property
  - we pick arbitrary $x \in A$ and $D :: P(x)$
  - we could do induction on both facts
    - $x \in A$ leads to induction on the structure of x
    - $D :: P(x)$ leads to induction on the structure of D
  - Generally, the induction on the structure of the derivation is more powerful and a safer bet
- Sometimes there are many choices for induction
  - choosing the right one is a trial-and-error process
  - a bit of practice can help a lot

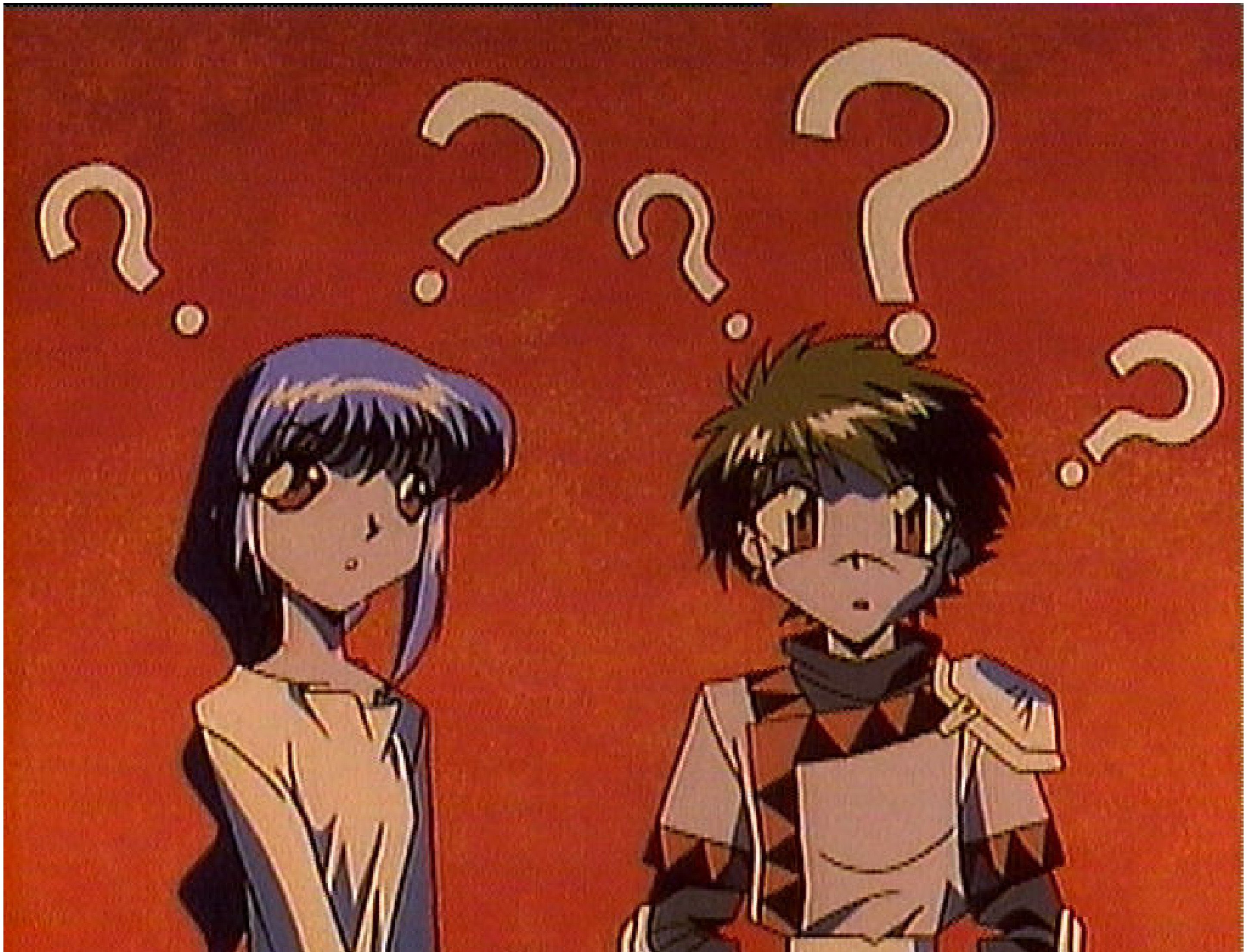
Summary of Operational Semantics

- Precise specification of dynamic semantics
  - order of evaluation (or that it doesn't matter)
  - error conditions (sometimes implicitly, by rule applicability; "no applicable rule" = "get stuck")
- Simple and abstract (vs. implementations)
  - no low-level details such as stack and memory management, data layout, etc.
- Often not compositional (see while)
- Basis for many proofs about a language
  - Especially when combined with type systems!
- Basis for much reasoning about programs
- Point of reference for other semantics

Dueling Semantics

- Operational semantics is
  - simple
  - of many flavors (natural, small-step, more or less abstract)
  - not compositional
  - commonly used in the real (modern research) world
- Denotational semantics is
  - mathematical (the meaning of a syntactic expression is a mathematical object)
  - compositional
- Denotational semantics is also called: fixed-point semantics, mathematical semantics, Scott-Strachey semantics

Typical Student Reaction To Denotation Semantics



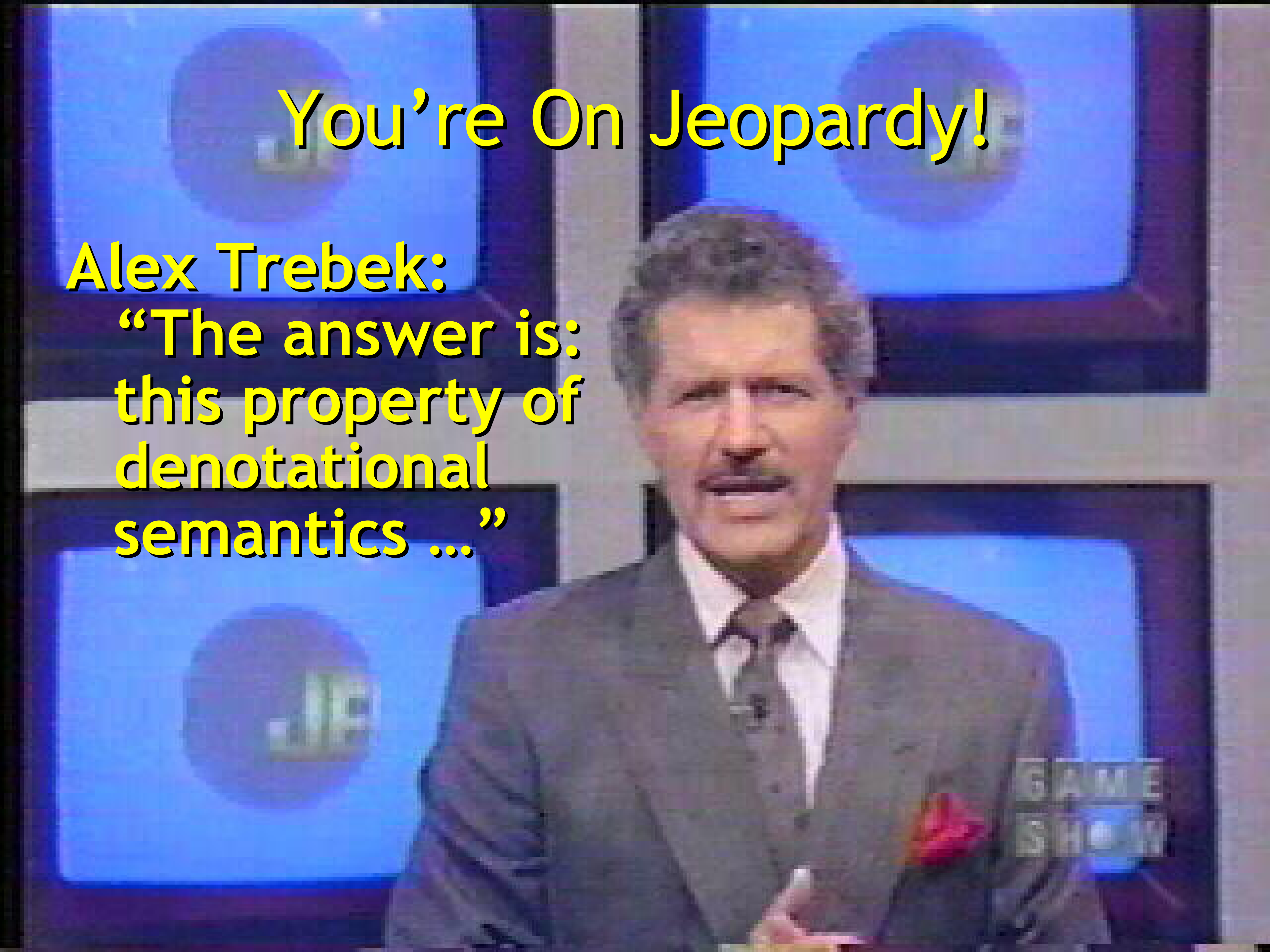
Denotational Semantics Learning Goals

- DS is **compositional** (!)
- When should I use DS?
- In DS, meaning is a "math object"
- DS uses $\perp$ ("bottom") to mean non-termination
- DS uses fixed points and domains to handle `while`
  - This is the tricky bit

You're On Jeopardy!

Alex Trebek:

"The answer is: this property of denotational semantics …"



DS In The Real World

- ADA was formally specified with it
- Handy when you want to study non-trivial models of computation
  - e.g., "actor event diagram scenarios", process calculi
- Nice when you want to compare a program in Language 1 to a program in Language 2

Deno-Challenge

- You may skip homework assignment 3 or 4 if you can find two (2) post-2000 papers in first- or second-tier PL conferences that use denotational semantics *and* you write me a two paragraph summary of each paper.

Foreshadowing

- Denotational semantics assigns meanings to programs
- The meaning will be a mathematical object
  - A number $\quad a \in \mathbb{Z}$
  - A boolean $\quad b \in \{true, false\}$
  - A function $\quad c : \Sigma \to (\Sigma \cup \{non\text{-}terminating\})$
- The meaning will be determined compositionally
  - Denotation of a command is based on the denotations of its immediate sub-commands (= more than merely syntax-directed)

New Notation

- ‘Cause, why not?

  ⟦ ⟧ = “means” or “denotes”

- Example:

  ⟦foo⟧ = “denotation of foo”

  ⟦3 < 5⟧ = true

  ⟦3 + 5⟧ = 8

- Sometimes we write A⟦·⟧ for arith, B⟦·⟧ for boolean, C⟦·⟧ for command

Rough Idea of Denotational Semantics

- The meaning of an arithmetic expression e in state σ is a number n
- So, we try to define A⟦e⟧ as a function that maps the current state to an integer:

$$A[\![\cdot]\!] : \text{Aexp} \to (\Sigma \to \mathbb{Z})$$

- The meaning of boolean expressions is defined in a similar way

$$B[\![\cdot]\!] : \text{Bexp} \to (\Sigma \to \{\text{true, false}\})$$

- All of these denotational function are total
  - Defined for all syntactic elements
  - For other languages it might be convenient to define the semantics only for well-typed elements

Denotational Semantics of Arithmetic Expressions

- We inductively define a function

$$A[\![\cdot]\!] : \text{Aexp} \to (\Sigma \to \mathbb{Z})$$

$A[\![n]\!]\ \sigma$ = the integer denoted by literal n

$A[\![x]\!]\ \sigma = \sigma(x)$

$A[\![e_1+e_2]\!]\ \sigma = A[\![e_1]\!]\sigma + A[\![e_2]\!]\sigma$

$A[\![e_1-e_2]\!]\ \sigma = A[\![e_1]\!]\sigma - A[\![e_2]\!]\sigma$

$A[\![e_1*e_2]\!]\ \sigma = A[\![e_1]\!]\sigma * A[\![e_2]\!]\sigma$

- This is a total function (= defined for all expressions)

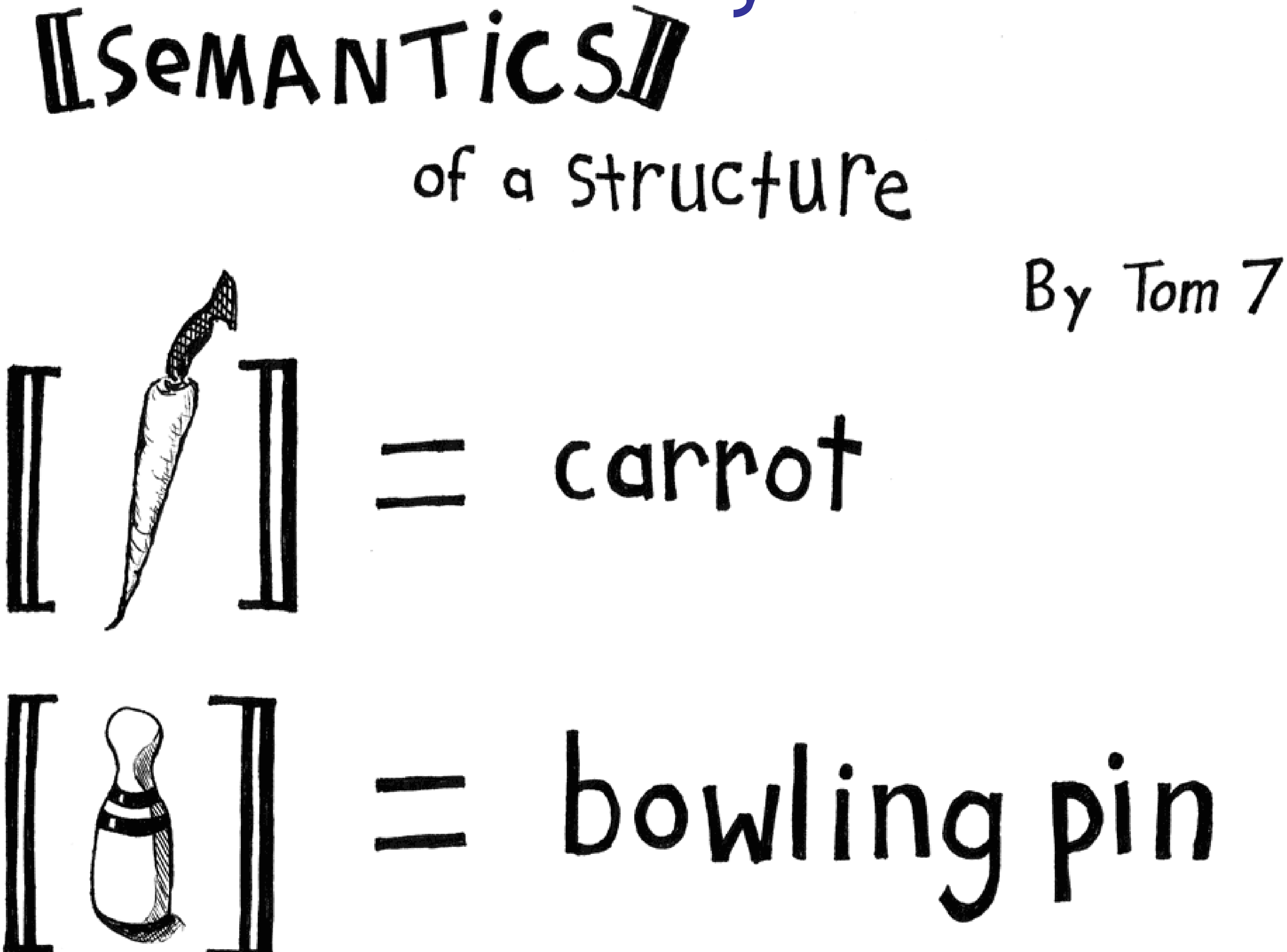
Denotational Semantics of Boolean Expressions

- We inductively define a function

$$B[\![\cdot]\!] : \text{Bexp} \to (\Sigma \to \{\textbf{true}, \textbf{false}\})$$

$$
\begin{aligned}
B[\![\text{true}]\!]\sigma &= \textbf{true} \\
B[\![\text{false}]\!]\sigma &= \textbf{false} \\
B[\![b_1 \wedge b_2]\!]\sigma &= B[\![b_1]\!]\,\sigma \wedge B[\![b_2]\!]\,\sigma \\
B[\![e_1 = e_2]\!]\sigma &= \text{if } A[\![e_1]\!]\,\sigma = A[\![e_2]\!]\,\sigma \\
&\quad\ \text{then } \textbf{true} \text{ else } \textbf{false}
\end{aligned}
$$

Seems Easy So Far



Denotational Semantics for Commands

- Running a command c starting from a state σ yields another state σ'
- So, we try to define C⟦c⟧ as a function that maps σ to σ'

$$C[\![\cdot]\!] : \text{Comm} \to (\Sigma \to \Sigma)$$

- Will this work? Bueller?

$\perp$ = Non-Termination

- We introduce the special element $\perp$ ("bottom") to denote a special resulting state that stands for **non-termination**
- For any set X, we write

  $X_\perp$ to denote $X \cup \{\perp\}$

Convention:

whenever $f \in X \to X_\perp$ we extend f to $X_\perp \to X_\perp$ so that $f(\perp) = \perp$

- This is called strictness

Denotational Semantics of Commands

- We try:

$$C[\![\cdot]\!] : \text{Comm} \to (\Sigma \to \Sigma_\perp)$$

$$C[\![\text{skip}]\!]\ \sigma = \sigma$$

$$C[\![x := e]\!]\ \sigma = \sigma[x := A[\![e]\!]\ \sigma]$$

$$C[\![c_1;\ c_2]\!]\ \sigma = C[\![c_2]\!]\ (C[\![c_1]\!]\ \sigma)$$

$$C[\![\text{if } b \text{ then } c_1 \text{ else } c_2]\!]\ \sigma = \text{if } B[\![b]\!]\sigma \text{ then } C[\![c_1]\!]\sigma \text{ else } C[\![c_2]\!]\sigma$$

$$C[\![\text{while } b \text{ do } c]\!]\ \sigma = ?$$

Examples

- $C[\![x:=2;\ x:=1]\!]\ \sigma =$

  $\sigma[x := 1]$
- $C[\![$if true then $x:=2;\ x:=1$ else …$]\!]\ \sigma =$

  $\sigma[x := 1]$
- The semantics does not care about intermediate states (cf. "big-step")
- We haven't used $\perp$ yet

Q: Theatre (012 / 842)

- Name the author or the 1953 play about McCarthyism that features John Proctor's famous cry of *"More weight!"* .

Q: General (450 / 842)

- Identify the children's dance here parodied in faux-Shakespearean English:
  - *O proud left foot, that ventures quick within*
  - *Then soon upon a backward journey lithe.*
  - *Anon, once more the gesture, then begin:*
  - *Command sinistral pedestal to writhe.*

Q: Games (557 / 842)

- Name the company that manufactures **Barbie** (a $1.9 billion dollar a year industry in 2005 with two dolls being bought every second).

Q: Music (207 / 842)

- In 1995 the Swedish euro-dance group Rednex released a version of this late 1800's American bluegrass tune about an attractive man of unknown provenance.

Denotational Semantics of WHILE

- Notation: $W = C[\![\text{while b do c}]\!]$
- Idea: rely on the equivalence (see end of notes)

  while b do c ≈ if b then c; while b do c else skip
- Try

$$W(\sigma) = \text{if } B[\![b]\!]\sigma \text{ then } W(C[\![c]\!]\sigma) \text{ else } \sigma$$

- This is called the unwinding equation
- It is not a good denotation of W because:
  - It defines W in terms of itself
  - It is not evident that such a W exists
  - It does not describe W uniquely
  - It is not compositional

More on WHILE

- The unwinding equation does not specify W uniquely
- Take C⟦while true do skip⟧
- The unwinding equation reduces to W(σ) = W(σ), which is satisfied by every function!
- Take C⟦while x ≠ 0 do x := x - 2⟧
- The following solution satisfies equation (for any σ')

$$W(\sigma) = \begin{cases} \sigma[x := 0] & \text{if } \sigma(x) = 2k \wedge \sigma(x) \geq 0 \\ \sigma' & \text{otherwise} \end{cases}$$

Denotational Game Plan

- Since WHILE is recursive
  - always have something like: $W(\sigma) = F(W(\sigma))$
- Admits *many* possible values for $W(\sigma)$
- We will *order* them
  - With respect to non-termination = "least"
- And then find the least fixed point
- LFP $W(\sigma)=F(W(\sigma))$ == meaning of "while"

WHILE k-steps Semantics

- Define $W_k: \Sigma \to \Sigma_\perp$ (for $k \in \mathbb{N}$) such that

$$W_k(\sigma) = \begin{cases} \sigma' & \text{if "while b do c" in state } \sigma \text{ terminates in fewer than k iterations in state } \sigma' \\ \perp & \text{otherwise} \end{cases}$$

- We can define the W_k functions as follows:

$$W_0(\sigma) = \perp$$

$$W_k(\sigma) = \begin{cases} W_{k-1}(C[\![c]\!]\sigma) & \text{if } B[\![b]\!]\sigma \text{ for } k \geq 1 \\ \sigma & \text{otherwise} \end{cases}$$

WHILE Semantics

- How do we get W from W_k?

$$W(\sigma) = \begin{cases} \sigma' & \text{if } \exists k. W_k(\sigma) = \sigma' \neq \bot \\ \bot & \text{otherwise} \end{cases}$$

- This is a valid compositional definition of W
  - Depends only on $C[\![c]\!]$ and $B[\![b]\!]$
- Try the examples again:
  - For $C[\![\text{while true do skip}]\!]$

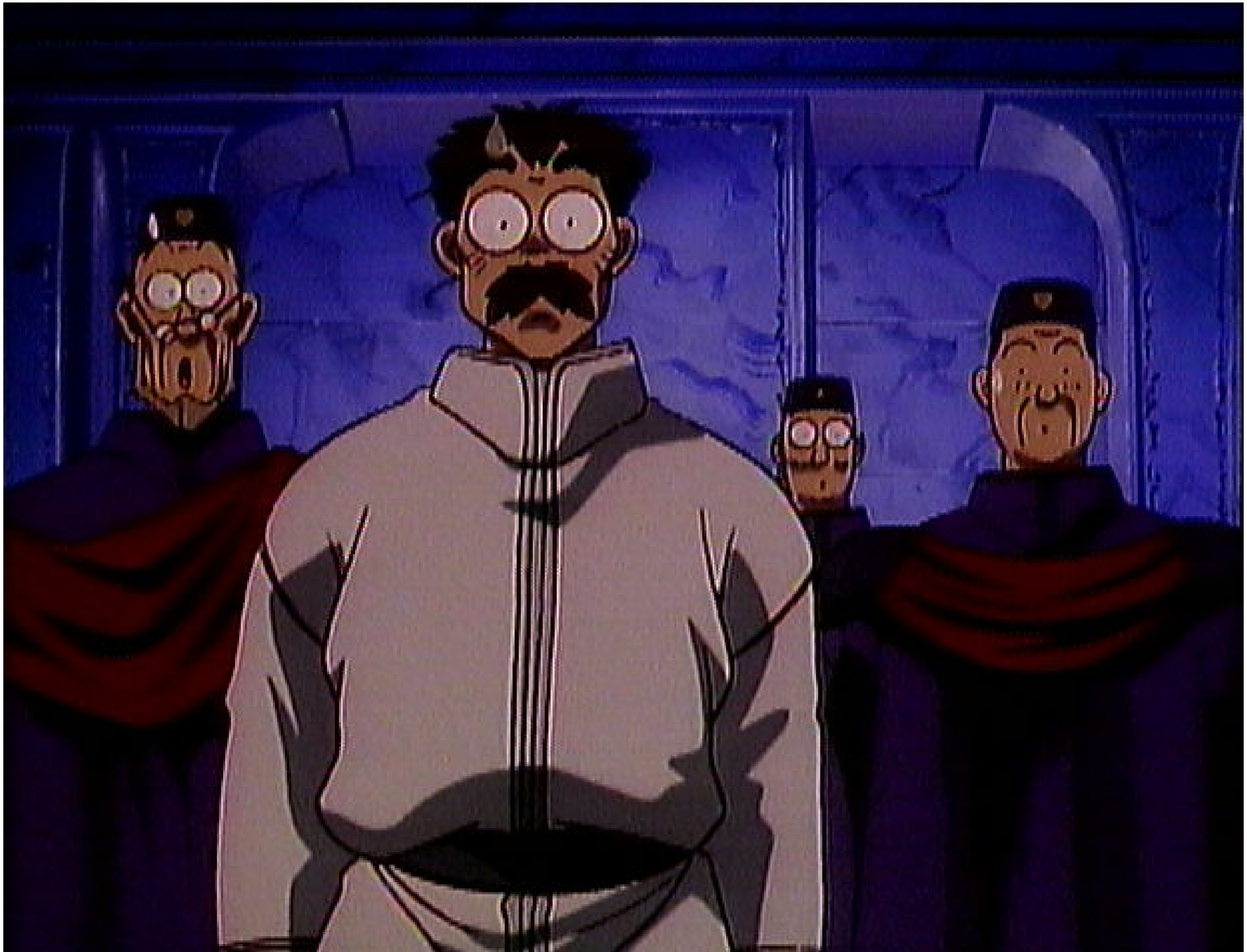
    $W_k(\sigma) = \bot$ for all k, thus $W(\sigma) = \bot$
  - For $C[\![\text{while } x \neq 0 \text{ do } x := x - 2]\!]$

$$W(\sigma) = \begin{cases} \sigma[x:=0] & \text{if } \sigma(x) = 2n \wedge \sigma(x) \geq 0 \\ \bot & \text{otherwise} \end{cases}$$

More on WHILE

- The solution is not quite satisfactory because
  - It has an operational flavor (= “run the loop”)
  - It does not generalize easily to more complicated semantics (e.g., higher-order functions)
- However, precisely due to the operational flavor this solution is easy to prove sound w.r.t operational semantics

That Wasn't Good Enough!?



Simple Domain Theory

- Consider programs in an eager, deterministic language with one variable called "x"
  - All these restrictions are just to simplify the examples
- A state σ is just the value of x
  - Thus we can use $\mathbb{Z}$ instead of Σ
- The semantics of a command give the value of final x as a function of input x

$$C[\![c]\!] : \mathbb{Z} \to \mathbb{Z}_{\perp}$$

Examples - Revisited

- Take C⟦while true do skip⟧
  - Unwinding equation reduces to W(x) = W(x)
  - Any function satisfies the unwinding equation
  - Desired solution is W(x) = $\perp$
- Take C⟦while x $\neq$ 0 do x := x – 2⟧
  - Unwinding equation:

    W(x) = if x $\neq$ 0 then W(x – 2) else x
  - Solutions (for all values n, m $\in \mathbb{Z}_\perp$):

    W(x) = if x $\geq$ 0 then
    if x even then 0 else n
    else m
  - Desired solution: W(x) = if x $\geq$ 0 $\wedge$ x even then 0 else $\perp$

An Ordering of Solutions

- The desired solution is the one in which all the arbitrariness is replaced with non-termination
  - The arbitrary values in a solution are not uniquely determined by the semantics of the code
- We introduce an ordering of semantic functions
- Let $f, g \in \mathbb{Z} \to \mathbb{Z}_{\perp}$
- Define $f \sqsubseteq g$ as

  $$\forall x \in \mathbb{Z}.\ f(x) = \perp \text{ or } f(x) = g(x)$$

  - A "smaller" function terminates *at most as often*, and when it terminates it produces the same result

Alternative Views of Function Ordering

- A semantic function $f \in \mathbb{Z} \to \mathbb{Z}_\perp$ can be written as $S_f \subseteq \mathbb{Z} \times \mathbb{Z}$ as follows:

$$S_f = \{ (x, y) \mid x \in \mathbb{Z}, f(x) = y \neq \perp \}$$

  - set of "terminating" values for the function
- If $f \sqsubseteq g$ then
  - $S_f \subseteq S_g$ (and vice-versa)
  - We say that g <u>refines</u> f
  - We say that f <u>approximates</u> g
  - We say that g provides more information than f

The "Best" Solution

- Consider again $C[\![\text{while } x \neq 0 \text{ do } x := x - 2]\!]$
  - Unwinding equation:

    $W(x) = \text{if } x \neq 0 \text{ then } W(x - 2) \text{ else } x$
- Not all solutions are comparable:

  $W(x) = \text{if } x \geq 0 \text{ then if } x \text{ even then } 0 \text{ else } 1 \text{ else } 2$

  $W(x) = \text{if } x \geq 0 \text{ then if } x \text{ even then } 0 \text{ else } \bot \text{ else } 3$

  $W(x) = \text{if } x \geq 0 \text{ then if } x \text{ even then } 0 \text{ else } \bot \text{ else } \bot$

  (last one is least and best)
- Is there always a least solution?
- How do we find it?
- *If only we had a general framework* for answering these questions …

Fixed-Point Equations

- Consider the general unwinding equation for while

  while b do c ≡ if b then c; while b do c else skip

- We define a context C (command with a hole)

  C = if b then c; • else skip

  while b do c ≡ C[while b do c]

  - The grammar for C does not contain "while b do c"

- We can find such a (recursive) context for any looping construct
  - Consider: fact n = if n = 0 then 1 else n * fact (n - 1)
  - C(n) = if n = 0 then 1 else n * • (n - 1)
  - fact = C [fact]

Fixed-Point Equations

- The meaning of a context is a semantic functional $F : (\mathbb{Z} \to \mathbb{Z}_\perp) \to (\mathbb{Z} \to \mathbb{Z}_\perp)$ such that

$$F\ [\![C[w]]\!] = F\ [\![w]\!]$$

- For "while": C = if b then c; • else skip

$$F\ w\ x = \text{if } [\![b]\!]\ x \text{ then } w\ ([\![c]\!]\ x) \text{ else } x$$

  - F depends only on $[\![c]\!]$ and $[\![b]\!]$
- We can rewrite the unwinding equation for while
  - $W(x) = \text{if } [\![b]\!]\ x \text{ then } W([\![c]\!]\ x) \text{ else } x$
  - or, W x = F W x for all x,
  - or, <u>W = F W</u> (by function equality)

Fixed-Point Equations

- The meaning of "while" is a solution for W = F W
- Such a W is called a fixed point of F
- We want the least fixed point
  - We need a general way to find least fixed points
- Whether such a least fixed point exists depends on the properties of function F
  - Counterexample: F w x = if w x = ⊥ then 0 else ⊥
  - Assume W is a fixed point
  - F W x = W x = if W x = ⊥ then 0 else ⊥
  - Pick an x, then if W x = ⊥ then W x = 0 else W x = ⊥
  - Contradiction. This F has no fixed point!

Can We Solve This?

- Good news: the functions F that *correspond to contexts in our language* have least fixed points!
- The only way F w x uses w is by invoking it
- If any such invocation diverges, then F w x diverges!
- It turns out: F is monotonic, continuous
  - Not shown here!

New Notation: λ

- **$\lambda x.\ e$**
  - an anonymous function with body **e** and argument **x**
- Example: double(x) = x+x

  double = $\lambda x.\ x+x$
- Example: allFalse(x) = false

  allFalse = $\lambda x.$ false
- Example: multiply(x,y) = x*y

  multiply = $\lambda x.\ \lambda y.\ x*y$

The Fixed-Point Theorem

- If F is a semantic function corresponding to a context in our language
  - F is monotonic and continuous (we assert)
  - For any fixed-point G of F and $k \in \mathbb{N}$

    $$F^k(\lambda x.\bot) \sqsubseteq G$$

  - The least of all fixed points is

    $$\sqcup_k F^k(\lambda x.\bot)$$

- Proof (not detailed in the lecture):
  1. By mathematical induction on k.

     Base: $F^0(\lambda x.\bot) = \lambda x.\bot \sqsubseteq G$

     Inductive: $F^{k+1}(\lambda x.\bot) = F(F^k(\lambda x.\bot)) \sqsubseteq F(G) = G$

  - Suffices to show that $\sqcup_k F^k(\lambda x.\bot)$ is a fixed-point

    $$F(\sqcup_k F^k(\lambda x.\bot)) = \sqcup_k F^{k+1}(\lambda x.\bot) = \sqcup_k F^k(\lambda x.\bot)$$

WHILE Semantics

- We can use the fixed-point theorem to write the denotational semantics of while:

  $[\![\text{while b do c}]\!] = \sqcup_k F^k (\lambda x.\bot)$

  where $F\ f\ x = \text{if } [\![b]\!]\ x \text{ then } f\ ([\![c]\!]\ x) \text{ else } x$
- Example: $[\![\text{while true do skip}]\!] = \lambda x.\bot$
- Example: $[\![\text{while } x \neq 0 \text{ then } x := x - 1]\!]$
  - $F\ (\lambda x.\bot)\ x = \text{if } x = 0 \text{ then } x \text{ else } \bot$
  - $F^2\ (\lambda x.\bot)\ x = \text{if } x = 0 \text{ then } x \text{ else if } x\text{-}1 = 0 \text{ then } x\text{-}1 \text{ else } \bot$

    $= \text{if } 1 \geq x \geq 0 \text{ then } 0 \text{ else } \bot$
  - $F^3\ (\lambda x.\bot)\ x = \text{if } 2 \geq x \geq 0 \text{ then } 0 \text{ else } \bot$
  - $LFP_F = \text{if } x \geq 0 \text{ then } 0 \text{ else } \bot$
- Not easy to find the closed form for general LFPs!

Discussion

- We can write the denotational semantics but we cannot always compute it.
  - Otherwise, we could decide the halting problem
  - H is halting for input 0 iff $[\![H]\!]\ 0 \neq \bot$
- We have derived this for programs with one variable
  - Generalize to multiple variables, even to variables ranging over richer data types, even higher-order functions: domain theory

Can You Remember?

You just survived the hardest lectures in 615.
It's all downhill from here.

Recall: Learning Goals

- DS is compositional
- When should I use DS?
- In DS, meaning is a "math object"
- DS uses $\perp$ ("bottom") to mean non-termination
- DS uses fixed points and domains to handle `while`
  - This is the tricky bit

Homework

- Homework 2 Due Thursday
- Homework 3
  - Not as long as it looks – separated out every exercise sub-part for clarity.
  - Your denotational answers must be compositional (e.g., $W_k(\sigma)$ or LFP)
- Read Winskel Chapter 6
- Read Hoare article
- Read Floyd article

Equivalence

- Two expressions (commands) are equivalent if they yield the same result from all states

  $e_1 \approx e_2$ iff

  $\forall \sigma \in \Sigma.\ \forall n \in \mathbb{N}.$

  $\langle e_1, \sigma \rangle \Downarrow n$ iff $\langle e_2, \sigma \rangle \Downarrow n$

  and for commands

  $c_1 \approx c_2$ iff

  $\forall \sigma, \sigma' \in \Sigma.$

  $\langle c_1, \sigma \rangle \Downarrow \sigma'$ iff $\langle c_2, \sigma \rangle \Downarrow \sigma'$

Notes on Equivalence

- Equivalence is like logical validity
  - It must hold in all states (= all valuations)
  - $2 \approx 1 + 1$ is like "$2 = 1 + 1$ is valid"
  - $2 \approx 1 + x$ might or might not hold.
    - So, 2 is not equivalent to $1 + x$
- Equivalence (for IMP) is undecidable
  - If it were decidable we could solve the halting problem for IMP. *How?*
- Equivalence justifies code transformations
  - compiler optimizations
  - code instrumentation
  - abstract modeling
- Semantics is the basis for proving equivalence

Equivalence Examples

- skip; c ≈ c
- while b do c ≈
  if b then c; while b do c else skip
- If $e_1 \approx e_2$ then x := e_1 ≈ x := e_2
- while true do skip ≈ while true do x := x + 1
- If c is

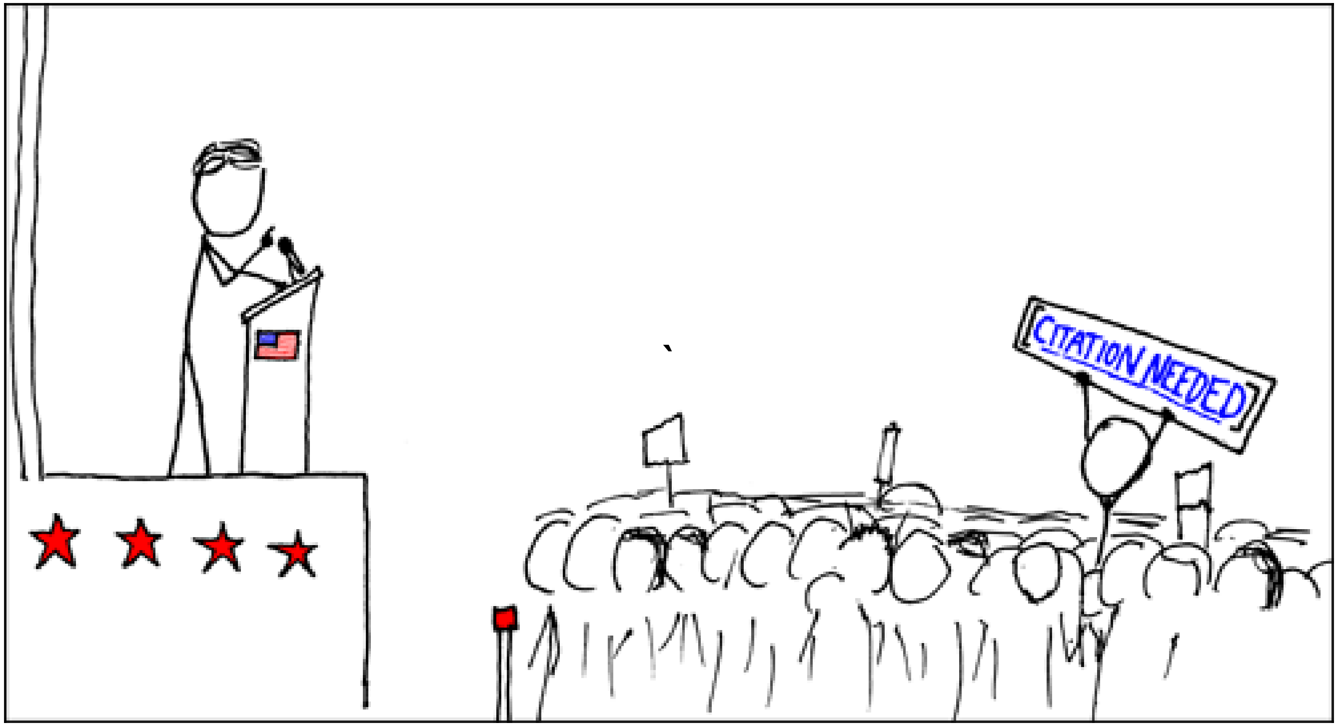
  while x ≠ y do

  if x ≥ y then x := x - y else y := y - x

  then (x := 221; y := 527; c) ≈ (x := 17; y := 17)

Potential Equivalence

- $(x := e_1; x := e_2) \approx x := e_2$
- Is this a valid equivalence?



Not An Equivalence

- $(x := e_1;\ x := e_2) \not\approx x := e_2$
- Iie. Chigau yo. Dame desu!
- Not a valid equivalence for all e_1, e_2.
- Consider:
  - $(x := x+1;\ x := x+2) \not\approx x := x+2$
- But for n_1, n_2 it's fine:
  - $(x := n_1;\ x := n_2) \approx x := n_2$

Proving An Equivalence

- Prove that "skip; c $\approx$ c" for all c
- Assume that D :: <skip; c, σ> $\Downarrow$ σ'
- By inversion (twice) we have that

$$D :: \frac{\dfrac{}{\langle \text{skip}, \sigma\rangle \Downarrow \sigma} \quad D_1 :: \langle c, \sigma\rangle \Downarrow \sigma'}{\langle \text{skip}; c, \sigma\rangle \Downarrow \sigma'}$$

- Thus, we have D_1 :: <c,σ> $\Downarrow$ σ'
- The other direction is similar

Proving An Inequivalence

- Prove that $x := y \not\sim x := z$ when y !=≠z
- It suffices to exhibit a σ in which the two commands yield different results

- Let $\sigma(y) = 0$ and $\sigma(z) = 1$
- Then

$$\langle x := y, \sigma\rangle \Downarrow \sigma[x := 0]$$

$$\langle x := z, \sigma\rangle \Downarrow \sigma[x := 1]$$