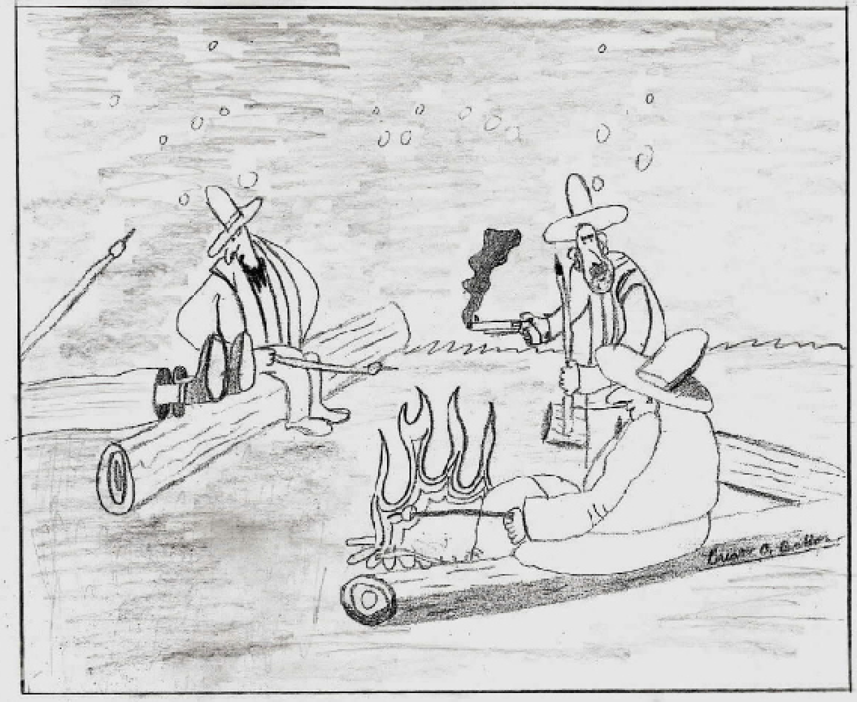
Simply-Typed Lambda Calculus

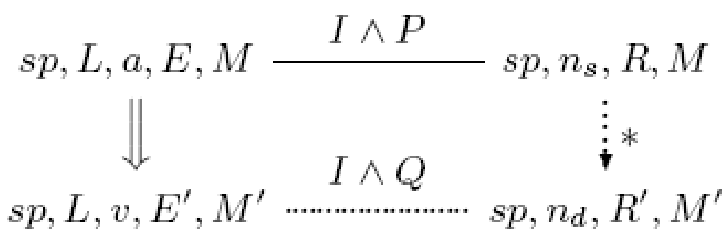


You guys are both my witnesses... He insinuated that ZFC set theory is superior to Type Theory!

The Reading

- Explain the Xavier Leroy article to me …

The correctness of the translation follows from a simulation argument between the executions of the Cminor source and the RTL translation, proved by induction on the Cminor evaluation derivation. In the case of expressions, the simulation property is summarized by the following diagram:

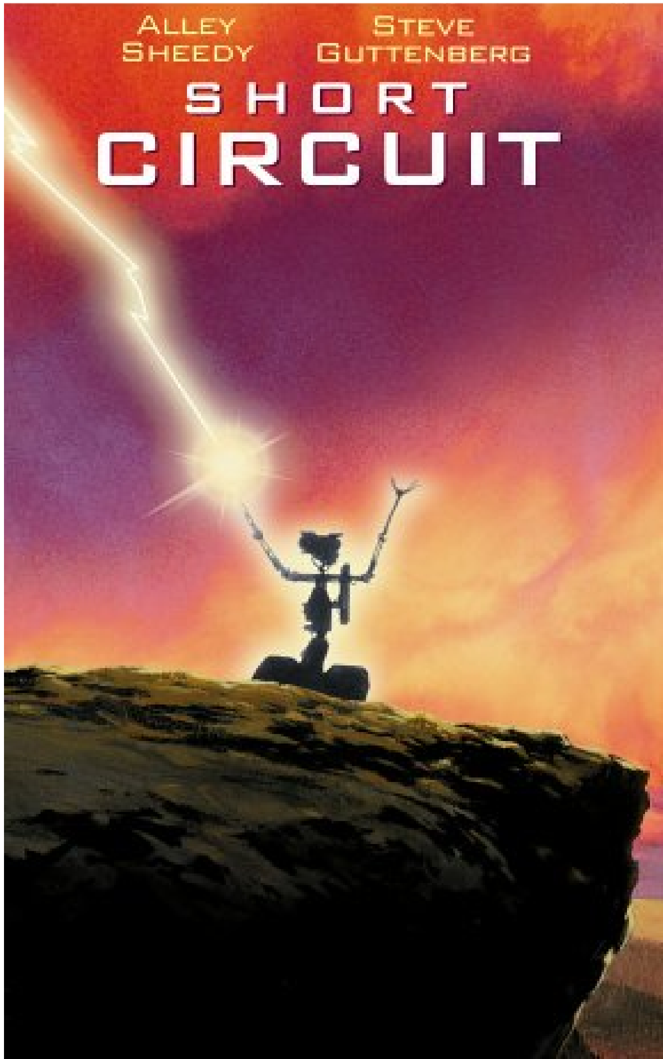


On the choice of semantics We used big-step semantics for the source language, "mixed-step" semantics for the intermediate languages, and small-step semantics for the target language. A consequence of this choice is that our semantic preservation theorems hold only for terminating source programs: they all have premises of the form "if the source program evaluates to result r", which do not hold for non-terminating programs. This is unfortunate for

- How did he do register allocation?

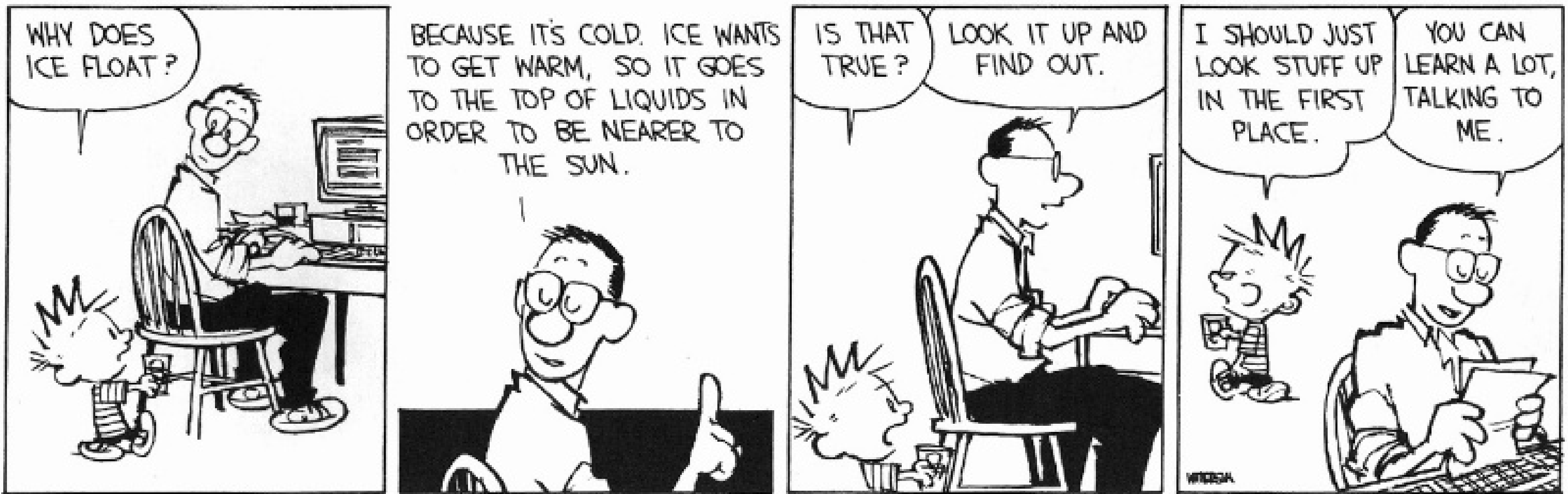
Homework Five Is Alive

- There will be no Number Six



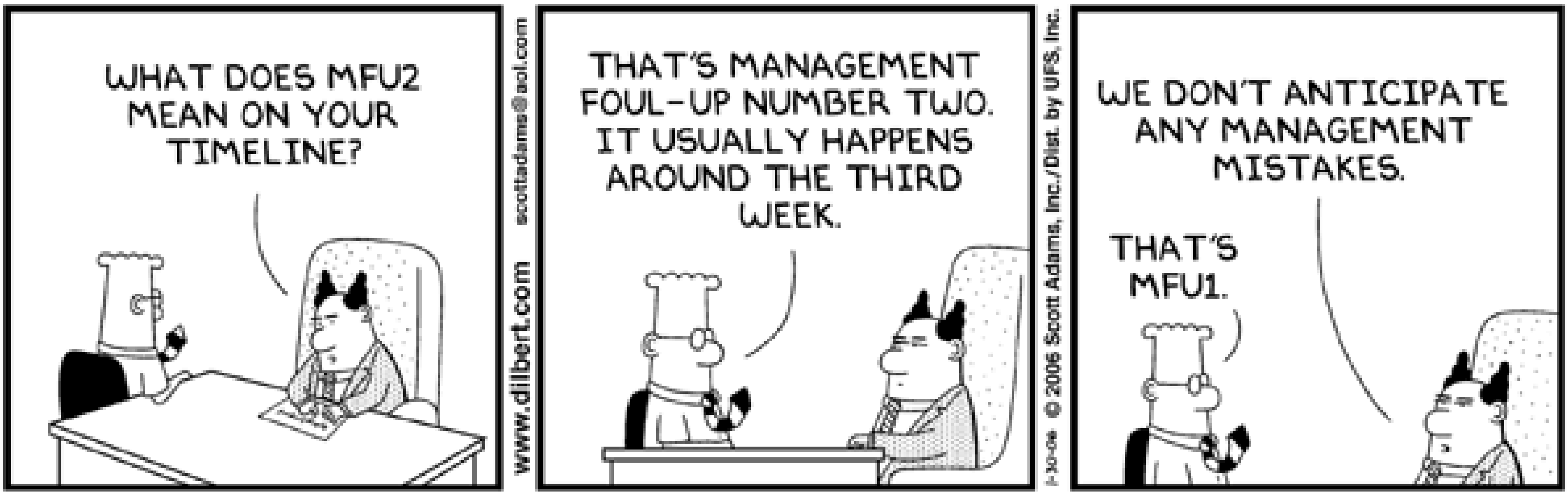
Back to School

- What is operational semantics? When would you use contextual (small-step) semantics?
- What is denotational semantics?
- What is axiomatic semantics? What is a verification condition?



Today's (Short?) Cunning Plan

- Type System Overview
- First-Order Type Systems
- Typing Rules
- Typing Derivations
- Type Safety



Why Typed Languages?

- Development
  - *Type checking catches early many mistakes*
  - Reduced debugging time
  - Typed signatures are a powerful basis for design
  - Typed signatures enable separate compilation
- Maintenance
  - Types act as checked specifications
  - Types can enforce abstraction
- Execution
  - Static checking reduces the need for dynamic checking
  - Safe languages are easier to analyze statically
    - the compiler can generate better code

Why Not Typed Languages?

- Static type checking imposes constraints on the programmer
  - Some valid programs might be rejected
  - But often they can be made well-typed easily
  - Hard to step outside the language (e.g. OO programming in a non-OO language, but cf. Ruby, OCaml, etc.)
- Dynamic safety checks can be costly
  - 50% is a possible cost of bounds-checking in a tight loop
    - In practice, the overall cost is much smaller
  - Memory management must be automatic ⇒ need a garbage collector with the associated run-time costs
  - Some applications are justified in using weakly-typed languages (e.g., by external safety proof)

Safe Languages

- There are typed languages that are not safe ("weakly typed languages")
- *All safe languages use types* (static or dynamic)

	Typed		Untyped
	Static	Dynamic	
Safe	ML, Java, Ada, C#, Haskell, ...	Lisp, Scheme, Ruby, Perl, Smalltalk, PHP, Python, …	λ-calculus
Unsafe	C, C++, Pascal, ...	?	Assembly

- We focus on statically typed languages

Properties of Type Systems

- How do types differ from other program annotations?
  - Types are more precise than comments
  - Types are more easily mechanizable than program specifications
- Expected properties of type systems:
  - Types should be enforceable
  - Types should be checkable algorithmically
  - Typing rules should be transparent
    - Should be easy to see why a program is not well-typed

Why Formal Type Systems?

- Many typed languages have informal descriptions of the type systems (e.g., in language reference manuals)
- A fair amount of careful analysis is required to avoid false claims of type safety
- A formal presentation of a type system is a precise specification of the type checker
  - And allows formal proofs of type safety
- But even informal knowledge of the principles of type systems help

Formalizing a Language

1. Syntax
   - Of expressions (programs)
   - Of types
   - Issues of binding and scoping
- Static semantics (typing rules)
   - Define the typing judgment and its derivation rules
3. Dynamic semantics (e.g., operational)
   - Define the evaluation judgment and its derivation rules
4. Type soundness
   - Relates the static and dynamic semantics
   - State and prove the soundness theorem

Typing Judgments

- Judgment (recall)
  - A statement J about certain formal entities
  - Has a truth value $\models$ J
  - Has a derivation $\vdash$ J (= "a proof")
- A common form of typing judgment:

  $\Gamma \vdash e : \tau$ (e is an expression and τ is a type)
- Γ (Gamma) is a set of type assignments for the free variables of e
  - Defined by the grammar $\Gamma ::= \cdot \mid \Gamma, x : \tau$
  - Type assignments for variables not free in e are not relevant
  - e.g, **x : int, y : int $\vdash$ x + y : int**

Typing rules

- Typing rules are used to derive typing judgments
- Examples:

$$\frac{}{\Gamma \vdash 1 : \texttt{int}}$$

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

$$\frac{\Gamma \vdash e_1 : \texttt{int} \qquad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 + e_2 : \texttt{int}}$$

Typing Derivations

- A typing derivation is a derivation of a typing judgment (big surprise there ...)
- Example:

$$\dfrac{\dfrac{}{x : \texttt{int} \vdash x : \texttt{int}} \qquad \dfrac{\dfrac{}{x : \texttt{int} \vdash x : \texttt{int}} \quad \dfrac{}{x : \texttt{int} \vdash 1 : \texttt{int}}}{x : \texttt{int} \vdash x + 1 : \texttt{int}}}{x : \texttt{int} \vdash x + (x + 1) : \texttt{int}}$$

- We say $\Gamma \vdash e : \tau$ to mean there exists a derivation of this typing judgment (= "we can prove it")
- Type checking: given Γ, e and τ find a derivation
- Type inference: given Γ and e, find τ and a derivation

Proving Type Soundness

- A typing judgment is either true or false
- Define what it means for a value to have a type

$$v \in \| \tau \|$$

(e.g. $5 \in \| \text{int} \|$ and $\text{true} \in \| \text{bool} \|$)

- Define what it means for an expression to have a type

$$e \in | \tau | \quad \text{iff} \quad \forall v. (e \Downarrow v \Rightarrow v \in \| \tau \|)$$

- Prove type soundness

$$\text{If } \cdot \vdash e : \tau \qquad \text{then } e \in | \tau |$$

or equivalently

$$\text{If } \cdot \vdash e : \tau \text{ and } e \Downarrow v \qquad \text{then } v \in \| \tau \|$$

- This implies safe execution (since the result of a unsafe execution is not in $\| \tau \|$ for any τ)

Upcoming Exciting Episodes

- We will give formal description of first-order type systems (no type variables)
  - Function types (simply typed λ-calculus)
  - Simple types (integers and booleans)
  - Structured types (products and sums)
  - Imperative types (references and exceptions)
  - Recursive types (linked lists and trees)
- The type systems of most common languages are first-order
- Then we move to second-order type systems
  - Polymorphism and abstract types

Q: Movies (378 / 842)

- This 1988 animated movie written and directed by Isao Takahata for Studio Ghibli was considered by Roger Ebert to be one of the most powerful anti-war films ever made. It features Seita and his sister Setsuko and their efforts to survive outside of society during the firebombing of Tokyo.

Q: General (468 / 842)

- This country's automobile stickers use the abbreviation **CH** (Confederatio Helvetica). The 1957 Max Miedinger typeface **Helvetica** is also named for this country.

Q: Games (504 / 842)

- This 1985 falling-blocks computer game was invented by Alexey Pajitnov (Алексей Пажитнов) and inspired by pentominoes.

Simply-Typed Lambda Calculus

Notice $:\tau$

- Syntax:

  Terms $e ::= x \mid \lambda x{:}\tau.\ e \mid e_1\ e_2$

  $\mid n \mid e_1 + e_2 \mid \text{iszero } e$

  $\mid \text{true} \mid \text{false} \mid \text{not } e$

  $\mid \text{if } e_1 \text{ then } e_2 \text{ else } e_3$

  Types $\tau ::= \text{int} \mid \text{bool} \mid \tau_1 \to \tau_2$

- $\tau_1 \to \tau_2$ is the function type
- $\to$ associates to the right
- Arguments have typing annotations $:\tau$
- This language is also called F_1

Static Semantics of F_1

- The typing judgment

$$\Gamma \vdash e : \tau$$

- Some (simpler) typing rules:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : \tau \to \tau'}$$

$$\frac{\Gamma \vdash e_1 : \tau_2 \to \tau \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau}$$

More Static Semantics of F_1

$$\frac{}{\Gamma \vdash n : \texttt{int}} \qquad \frac{\Gamma \vdash e_1 : \texttt{int} \quad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 + e_2 : \texttt{int}}$$

Why do we have this mysterious gap? I don't know either!

$$\frac{}{\Gamma \vdash \texttt{true} : \texttt{bool}} \qquad \frac{\Gamma \vdash e : \texttt{bool}}{\Gamma \vdash \texttt{not}\ e : \texttt{bool}}$$

$$\frac{\Gamma \vdash e_1 : \texttt{bool} \quad \Gamma \vdash e_t : \tau \quad \Gamma \vdash e_f : \tau}{\Gamma \vdash \texttt{if}\ e_1\ \texttt{then}\ e_t\ \texttt{else}\ e_f : \tau}$$

Typing Derivation in F_1

- Consider the term

  λx : int. λb : bool. if b then f x else x

  - With the initial typing assignment f : int → Int
  - Where Γ = f : int → int, x : int, b : bool

$$
\dfrac{\dfrac{\dfrac{\Gamma \vdash b : \texttt{bool} \qquad \dfrac{\Gamma \vdash f : \texttt{int} \rightarrow \texttt{int} \qquad \Gamma \vdash x : \texttt{int}}{\Gamma \vdash f\ x : \texttt{int}} \qquad \Gamma \vdash x : \texttt{int}}{f : \texttt{int} \rightarrow \texttt{int}, x : \texttt{int}, b : \texttt{bool} \vdash \texttt{if}\ b\ \texttt{then}\ f\ x\ \texttt{else}\ x : \texttt{int}}}{f : \texttt{int} \rightarrow \texttt{int}, x : \texttt{int} \vdash \lambda b : \texttt{bool}.\ \texttt{if}\ b\ \texttt{then}\ f\ x\ \texttt{else}\ x : \texttt{bool} \rightarrow \texttt{int}}}{f : \texttt{int} \rightarrow \texttt{int} \vdash \underbrace{\lambda x : \texttt{int}.\lambda b : \texttt{bool}.\ \texttt{if}\ b\ \texttt{then}\ f\ x\ \texttt{else}\ x}_{} : \texttt{int} \rightarrow \texttt{bool} \rightarrow \texttt{int}}
$$

Type Checking in F_1

- Type checking is *easy* because
  - Typing rules are syntax directed
  - Typing rules are compositional (what does this mean?)
  - All local variables are annotated with types

- In fact, type inference is *also easy* for F_1
- Without type annotations an expression may have no unique type

$$\cdot \vdash \lambda x.\ x : \text{int} \to \text{int}$$

$$\cdot \vdash \lambda x.\ x : \text{bool} \to \text{bool}$$

Operational Semantics of F_1

- Judgment:

$$e \Downarrow v$$

- Values:

$$v ::= n \mid \text{true} \mid \text{false} \mid \lambda x{:}\tau.\ e$$

- The evaluation rules …
  - Audience participation time: raise your hand and give me an evaluation rule.

Opsem of F_1 (Cont.)

- Call-by-value evaluation rules (sample)

$$\frac{}{\lambda x : \tau . e \Downarrow \lambda x : \tau . e}$$

$$\frac{e_1 \Downarrow \lambda x : \tau . e_1' \quad e_2 \Downarrow v_2 \quad [v_2/x]e_1' \Downarrow v}{e_1 \; e_2 \Downarrow v}$$

$$\frac{}{n \Downarrow n} \qquad \frac{e_1 \Downarrow n_1 \quad e_2 \Downarrow n_2 \quad n = n_1 + n_2}{e_1 + e_2 \Downarrow n}$$

$$\frac{e_1 \Downarrow \texttt{true} \quad e_t \Downarrow v}{\texttt{if } e_1 \texttt{ then } e_t \texttt{ else } e_f \Downarrow v}$$

$$\frac{e_1 \Downarrow \texttt{false} \quad e_f \Downarrow v}{\texttt{if } e_1 \texttt{ then } e_t \texttt{ else } e_f \Downarrow v}$$

Where is the Call-By-Value? How might we change it?

Evaluation is undefined for ill-typed programs !

Type Soundness for F_1

- Theorem: **If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$**
  - Also called, subject reduction theorem, type preservation theorem
- This is one of the most important sorts of theorems in PL
- Whenever you make up a new safe language you are expected to prove this
  - Examples: Vault, TAL, CCured, …
- Proof: next time!

Homework

- Read Wright and Felleisen article
- Work on your projects!
  - Status Update Due Soon
- Work on Homework 5

The reading is **not** optional.