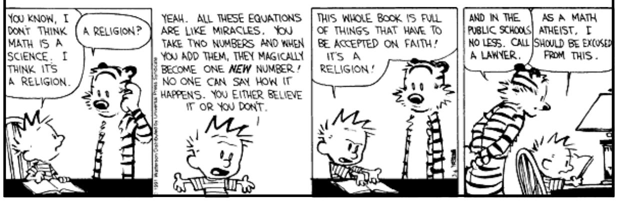
More Lambda Calculus
and
Intro to Type Systems



Plan

- Heavy Class Participation
  - Thus, wake up!
- Lambda Calculus
  - How is it related to real life?
  - Encodings
  - Fixed points
- Type Systems
  - Overview
  - Static, Dyamic
  - Safety, Judgments, Derivations, Soundness

Lambda Review

- λ-calculus is a calculus of functions

$$e := x \mid \lambda x.\ e \mid e_1\ e_2$$

- Several evaluation strategies exist based on β-reduction

$$(\lambda x.e)\ e' \to_\beta [e'/x]\ e$$

- How does this simple calculus relate to real programming languages?

Functional Programming

- The λ-calculus is a prototypical functional language with:
  - no side effects
  - several evaluation strategies
  - lots of functions
  - nothing but functions (pure λ-calculus does not have any other data type)
- How can we program with functions?
- How can we program with only functions?

Programming With Functions

- Functional programming style is a programming style that relies on lots of functions
- A typical functional paradigm is *using functions as arguments or results of other functions*
  - Called "higher-order programming"
- Some "impure" functional languages permit side-effects (e.g., Lisp, Scheme, ML, Python)
  - references (pointers), in-place update, arrays, exceptions
  - Others (and by "others" we mean "Haskell") use monads to model state updates

Variables in Functional Languages

- We can introduce new variables:

$$\text{let } x = e_1 \text{ in } e_2$$

  - x is bound by let
  - x is statically scoped in (a subset of) e_2
- This is pretty much like $(\lambda x.\ e_2)\ e_1$
- In a functional language, variables are never updated
  - they are just *names for expressions or values*
  - e.g., x is a name for the value denoted by e_1 in e_2
- This models the meaning of "let" in math (proofs)

Referential Transparency

- In "pure" functional programs, we can reason equationally, by substitution
  - Called "referential transparency"

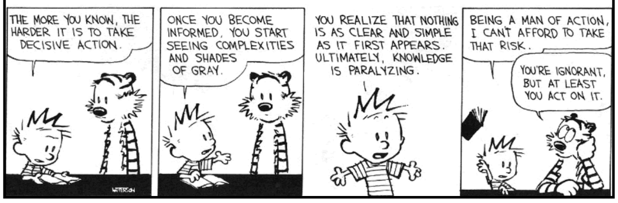
    let x = e_1 in e_2 === $[e_1/x]e_2$
- In an imperative language a "side-effect" in e_1 might invalidate the above equation
- The behavior of a function in a "pure" functional language depends only on the actual arguments
  - Just like a function in math
  - This makes it easier to understand and to reason about functional programs

#7

How Tough Is Lambda?

- How complex (a la CS theory) is it to determine if:

$$e_1 \to_\beta^* e \quad \textit{and} \quad e_2 \to_\beta^* e$$



Expressiveness of λ-Calculus

- The λ-calculus is a minimal system but can express
  - data types (integers, booleans, lists, trees, etc.)
  - branching
  - recursion
- This is enough to encode Turing machines
  - We say the lambda calculus is Turing-complete
- Corollary: $e =_\beta e'$ is undecidable
- Still, how do we encode all these constructs using only functions?
- Idea: *encode the "behavior" of values* and not their structure

#9

Encoding Booleans in λ-Calculus

- What can we *do* with a boolean?
  - we can make a binary choice (= "if" statement)
- A boolean is a function that, given two choices, selects one of them:
  - true $=_{def}$ $\lambda x.\ \lambda y.\ x$
  - false $=_{def}$ $\lambda x.\ \lambda y.\ y$
  - if E_1 then E_2 else E_3 $=_{def}$ $E_1\ E_2\ E_3$
- Example: "if true then u else v" is

  $(\lambda x.\ \lambda y.\ x)\ u\ v \to_\beta (\lambda y.\ u)\ v \to_\beta u$

#10

Encoding Pairs in λ-Calculus

- What can we *do* with a pair?
  - we can access one of its elements (= ".field access")
- A pair is a function that, given a boolean, returns the first or second element

  mkpair x y $=_{def}$ $\lambda b.\ b\ x\ y$

  fst p $=_{def}$ p true

  snd p $=_{def}$ p false
- fst (mkpair x y)

  $\to_\beta$ (mkpair x y) true

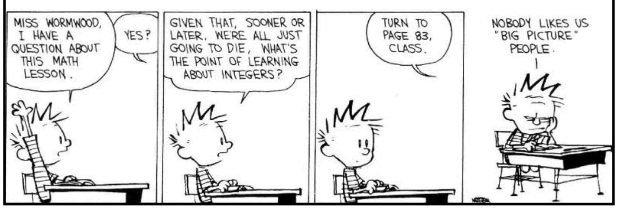
  $\to_\beta$ true x y

  $\to_\beta$ x

#11

Encoding Numbers in λ–Calculus

- What can we *do* with a natural number?
  - What do you, the viewers at home, think?



Encoding Numbers λ-Calculus

- What can we *do* with a natural number?
  - we can iterate a number of times over some function (= "for loop")
- A natural number is a function that given an operation f and a starting value s, applies f a number of times to s:

  0 $=_{def}$ λf. λs. s

  1 $=_{def}$ λf. λs. f s

  2 $=_{def}$ λf. λs. f (f s)

  - Very similar to List.fold_left and friends
- These are numerals in a unary representation
- Called Church numerals

Test Time!

- How would you encode the successor function (succ x = x+1)?
- How would you encode more general addition?
- Recall: 4 $=_{def}$ λf. λs. f f f (f s)

Computing with Natural Numbers

- The successor function

  succ n $=_{def}$ λf. λs. f (n f s)

  or succ n $=_{def}$ λf. λs. n f (f s)
- Addition

  add n_1 n_2 $=_{def}$ n_1 succ n_2
- Multiplication

  mult n_1 n_2 $=_{def}$ n_1 (add n_2) 0
- Testing equality with 0

  iszero n $=_{def}$ n (λb. false) true
- Subtraction
  - Is not instructive, but makes a fun exercise …

Computation Example

- What is the result of the application add 0?

  (λn_1. λn_2. n_1 succ n_2) 0 $\rightarrow_\beta$

  λn_2. 0 succ n_2 =

  λn_2. (λf. λs. s) succ n_2 $\rightarrow_\beta$

  λn_2. n_2 =

  λx. x
- By computing with functions we can express some optimizations
  - But we need to reduce under the lambda
  - Thus this "never" happens in practice

Toward Recursion

- Given a predicate P, encode the function "find" such that "find P n" is the smallest natural number which is larger than n and satisfies P
- Claim: with find we can encode all recursion

  Intuitively, why is this true?

Encoding Recursion

- Given a predicate P encode the function "find" such that "find P n" is the smallest natural number which is larger than n and satisfies P
- find satisfies the equation

  find p n = if p n then n else find p (succ n)
- Define

  F = λf.λp.λn.(p n) n (f p (succ n))
- We need a fixed point of F

  find = F find

  or

  find p n = F find p n

The Fixed-Point Combinator Y

- Let $Y = \lambda F.\ (\lambda y.F(y\ y))\ (\lambda x.\ F(x\ x))$
  - This is called the fixed-point combinator
  - Verify that $Y\ F$ is a fixed point of F

    $Y\ F \to_\beta (\lambda y.F\ (y\ y))\ (\lambda x.\ F\ (x\ x)) \to_\beta F\ (Y\ F)$
  - Thus $Y\ F =_\beta F\ (Y\ F)$
- Given any function in λ-calculus we can compute its fixed-point (w00t! why do we not win here?)
- Thus we can define "find" as the fixed-point of the function F from the previous slide
- Essence of recursion is the self-application "y y"

Expressiveness of Lambda Calculus

- Encodings are fun
  - Yes! Yes they are!
- But programming in pure λ-calculus is painful
- So we will add constants (0, 1, 2, …, true, false, if-then-else, etc.)
- Next we will add *types*

Still Going!

- One minute break
- Stretch!

Types

- A program variable can assume a range of values during the execution of a program
- An upper bound of such a range is called a type of the variable
  - A variable of type "bool" is supposed to assume only boolean values
  - If x has type "bool" then the boolean expression "not(x)" has a sensible meaning during every run of the program

Typed and Untyped Languages

- Untyped languages
  - Do *not* restrict the range of values for a given variable
  - Operations might be applied to inappropriate arguments. The behavior in such cases might be unspecified
  - The pure λ-calculus is an extreme case of an untyped language (however, its behavior is completely specified)
- (Statically) Typed languages
  - Variables are assigned (non-trivial) types
  - A type system keeps track of types
  - Types might or might not appear in the program itself
  - Languages can be explicitly typed or implicitly typed

The Purpose Of Types

- The foremost purpose of types is *to prevent certain types of run-time execution errors*
- Traditional trapped execution errors
  - Cause the computation to stop immediately
  - And are thus well-specified behavior
  - Usually enforced by hardware
  - e.g., Division by zero, floating point op with a NaN
  - e.g., Dereferencing the address 0 (on most systems)
- Untrapped execution errors
  - Behavior is unspecified (depends on the state of the machine = this is very bad!)
  - e.g., accessing past the end of an array
  - e.g., jumping to an address in the data segment

Execution Errors

- A program is deemed safe if it does *not* cause untrapped errors
  - Languages in which all programs are safe are safe languages
- For a given language we can designate a set of forbidden errors
  - A superset of the untrapped errors, usually including some trapped errors as well
    - e.g., null pointer dereference
- Modern Type System Powers:
  - prevent race conditions (e.g., Flanagan TLDI '05)
  - prevent insecure information flow (e.g., Li POPL '05)
  - prevent resource leaks (e.g., Vault, Weimer)
  - help with generic programming, probabilistic languages, …
  - … are often combined with dynamic analyses (e.g., CCured)

#25

Preventing Forbidden Errors - Static Checking

- Forbidden errors can be caught by a combination of static and run-time checking
- Static checking
  - Detects errors early, *before testing*
  - Types provide the necessary static information for static checking
  - e.g., ML, Modula-3, Java
  - Detecting certain errors statically is undecidable in most languages

#26

Preventing Forbidden Errors - Dynamic Checking

- Required when static checking is undecidable
  - e.g., array-bounds checking
- Run-time encodings of types are still used (e.g. Lisp)
- Should be limited since it delays the manifestation of errors
- Can be done in hardware (e.g. null-pointer)

#27

Safe Languages

- There are typed languages that are not safe ("weakly typed languages")
- *All safe languages use types* (statica or dynamic)

	Typed		Untyped
	Static	Dynamic	
Safe	ML, Java, Ada, C#, Haskell, …	Lisp, Scheme, Ruby, Perl, Smalltalk, PHP, Python, …	λ-calculus
Unsafe	C, C++, Pascal, …	?	Assembly

- We focus on statically typed languages

#28

Why Typed Languages?

- Development
  - *Type checking catches early many mistakes*
  - Reduced debugging time
  - Typed signatures are a powerful basis for design
  - Typed signatures enable separate compilation
- Maintenance
  - Types act as checked specifications
  - Types can enforce abstraction
- Execution
  - Static checking reduces the need for dynamic checking
  - Safe languages are easier to analyze statically
    - the compiler can generate better code

#29

Homework

- Read Cardelli article
  - Spread it over the break …
- Read great works of literature
- Homework 5 Due Today
  - Don't ruin your Spring Break by having it hanging over you …
- No Class Next Week (Spring Break!)
  - Next Lecture: Tue Mar 14

#30