Introduction to Denotational Semantics

Class Likes/Dislikes Survey

- “would change [the bijection question] to be one that still tested students' recollection of set theory but that didn't take as much time”
- “I liked the bijection proof in the homework. I thought it ended up being pretty neat.”
- “my guess is the student would benefit more from a rephrasing or alternate explanation”
- “I don't need to hear the things explained in another way”

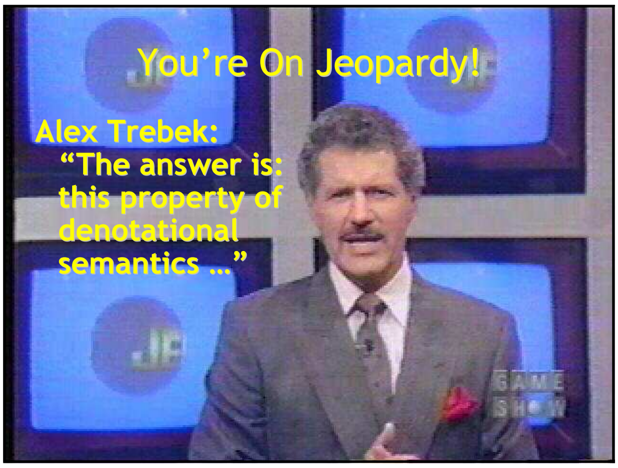
Dueling Semantics

- Operational semantics is
  - simple
  - of many flavors (natural, small-step, more or less abstract)
  - not compositional
  - commonly used in the real (modern research) world
- Denotational semantics is
  - mathematical (the meaning of a syntactic expression is a mathematical object)
  - compositional
- Denotational semantics is also called: fixed-point semantics, mathematical semantics, Scott-Strachey semantics

Typical Student Reaction To Denotation Semantics

Denotational Semantics Learning Goals

- DS is compositional
- When should I use DS?
- In DS, meaning is a “math object”
- DS uses $\perp$ (“bottom”) to mean non-termination
- DS uses fixed points and domains to handle `while`
  - This is the tricky bit



DS In The Real World

- ADA was formally specified with it
- Handy when you want to study non-trivial models of computation
  - e.g., "actor event diagram scenarios", process calculi
- Nice when you want to compare a program in Language 1 to a program in Language 2

Deno-Challenge

- You may skip the homework assignment of your choice if you can find a post-1995 paper in a first- or second-tier PL conference that uses denotational semantics.

Foreshadowing

- Denotational semantics assigns meanings to programs
- The meaning will be a mathematical object
  - A number $a \in \mathbb{Z}$
  - A boolean $b \in \{true, false\}$
  - A function $c : \Sigma \to (\Sigma \cup \{non\text{-}terminating\})$
- The meaning will be determined compositionally
  - Denotation of a command is based on the denotations of its immediate sub-commands (= syntax-directed)

New Notation

- 'Cause, why not?
  - $[\![\ ]\!]$ = "means" or "denotes"
- Example:
  - $[\![foo]\!]$ = "denotation of foo"
  - $[\![3 < 5]\!]$ = true
  - $[\![3 + 5]\!]$ = 8
- Sometimes we write $A[\![\cdot]\!]$ for arith, $B[\![\cdot]\!]$ for boolean, $C[\![\cdot]\!]$ for command

Rough Idea of Denotational Semantics

- The meaning of an arithmetic expression e in state σ is a number n
- So, we try to define $A[\![e]\!]$ as a function that maps the current state to an integer:

$$A[\![\cdot]\!] : Aexp \to (\Sigma \to \mathbb{Z})$$

- The meaning of boolean expressions is defined in a similar way

$$B[\![\cdot]\!] : Bexp \to (\Sigma \to \{true, false\})$$

- All of these denotational function are total
  - Defined for all syntactic elements
  - For other languages it might be convenient to define the semantics only for well-typed elements

Denotational Semantics of Arithmetic Expressions

- We inductively define a function

$$A[\![\cdot]\!] : Aexp \to (\Sigma \to \mathbb{Z})$$

$A[\![n]\!]\ \sigma$ = the integer denoted by literal n

$A[\![x]\!]\ \sigma = \sigma(x)$

$A[\![e_1+e_2]\!]\ \sigma = A[\![e_1]\!]\sigma + A[\![e_2]\!]\sigma$

$A[\![e_1-e_2]\!]\ \sigma = A[\![e_1]\!]\sigma - A[\![e_2]\!]\sigma$

$A[\![e_1*e_2]\!]\ \sigma = A[\![e_1]\!]\sigma * A[\![e_2]\!]\sigma$

- This is a total function (= defined for all expressions)

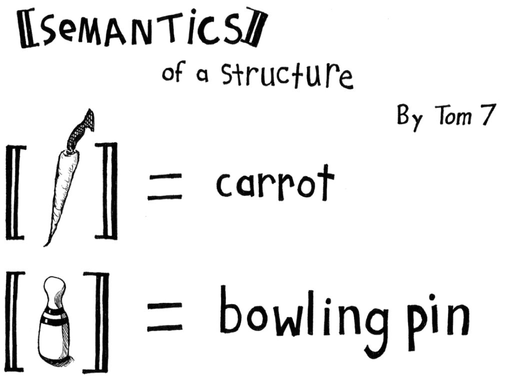
Denotational Semantics of Boolean Expressions

- We inductively define a function

$$B[\![\cdot]\!] : \text{Bexp} \rightarrow (\Sigma \rightarrow \{\textbf{true}, \textbf{false}\})$$

$$
\begin{aligned}
B[\![\text{true}]\!]\sigma &= \textbf{true} \\
B[\![\text{false}]\!]\sigma &= \textbf{false} \\
B[\![b_1 \wedge b_2]\!]\sigma &= B[\![b_1]\!]\,\sigma \wedge B[\![b_2]\!]\,\sigma \\
B[\![e_1 = e_2]\!]\sigma &= \text{if } A[\![e_1]\!]\,\sigma = A[\![e_2]\!]\,\sigma \\
&\qquad \text{then } \textbf{true} \text{ else } \textbf{false}
\end{aligned}
$$

Seems Easy So Far



Denotational Semantics for Commands

- Running a command c starting from a state σ yields another state σ'
- So, we try to define $C[\![c]\!]$ as a function that maps σ to σ'

$$C[\![\cdot]\!] : \text{Comm} \rightarrow (\Sigma \rightarrow \Sigma)$$

- Will this work? Bueller?

$\perp$ = Non-Termination

- We introduce the special element $\perp$ to denote a special resulting state that stands for non-termination
- For any set X, we write $X_\perp$ to denote $X \cup \{\perp\}$

Convention:

whenever $f \in X \rightarrow X_\perp$ we extend f to $X_\perp \rightarrow X_\perp$ so that $f(\perp) = \perp$

- This is called strictness

Denotational Semantics of Commands

- We try:

$$C[\![\cdot]\!] : \text{Comm} \rightarrow (\Sigma \rightarrow \Sigma_\perp)$$

$$
\begin{aligned}
C[\![\text{skip}]\!]\,\sigma &= \sigma \\
C[\![x := e]\!]\,\sigma &= \sigma[x := A[\![e]\!]\,\sigma] \\
C[\![c_1; c_2]\!]\,\sigma &= C[\![c_2]\!]\,(C[\![c_1]\!]\,\sigma) \\
C[\![\text{if } b \text{ then } c_1 \text{ else } c_2]\!]\,\sigma &= \\
&\text{if } B[\![b]\!]\sigma \text{ then } C[\![c_1]\!]\sigma \text{ else } C[\![c_2]\!]\sigma \\
C[\![\text{while } b \text{ do } c]\!]\,\sigma &= \,?
\end{aligned}
$$

Examples

- $C[\![x:=2;\ x:=1]\!]\ \sigma =$
  $\sigma[x := 1]$
- $C[\![\text{if true then } x:=2;\ x:=1 \text{ else } \ldots]\!]\ \sigma =$
  $\sigma[x := 1]$
- The semantics does not care about intermediate states
- We haven't used $\perp$ yet

Denotational Semantics of WHILE

- Notation: W = C⟦while b do c⟧
- Idea: rely on the equivalence (from last time)
  while b do c ≈ if b then c; while b do c else skip
- Try
  W(σ) = if B⟦b⟧σ then W(C⟦c⟧σ) else σ
- This is called the unwinding equation
- It is not a good denotation of W because:
  - It defines W in terms of itself
  - It is not evident that such a W exists
  - It does not describe W uniquely
  - It is not compositional

More on WHILE

- The unwinding equation does not specify W uniquely
- Take C⟦while true do skip⟧
- The unwinding equation reduces to W(σ) = W(σ), which is satisfied by every function!
- Take C⟦while x ≠ 0 do x := x - 2⟧
- The following solution satisfies equation (for any σ')

$$W(\sigma) = \begin{cases} \sigma[x := 0] & \text{if } \sigma(x) = 2k \wedge \sigma(x) \geq 0 \\ \sigma' & \text{otherwise} \end{cases}$$

Denotational Game Plan

- Since WHILE is recursive
  - always have something like: W(σ) = F(W(σ))
- Admits many possible values for W(σ)
- We will order them
  - With respect to non-termination
- And then find the least fixed point
- LFP W(σ)=F(W(σ)) == meaning of "while"

WHILE Semantics

- Define W_k: $\Sigma \to \Sigma_\perp$ (for $k \in \mathbb{N}$) such that

$$W_k(\sigma) = \begin{cases} \sigma' & \text{if "while b do c" in state } \sigma \text{ terminates in fewer than } k \text{ iterations in state } \sigma' \\ \perp & \text{otherwise} \end{cases}$$

- We can define the W_k functions as follows:

$$W_0(\sigma) = \perp$$

$$W_k(\sigma) = \begin{cases} W_{k-1}(C[\![c]\!]\sigma) & \text{if } B[\![b]\!]\sigma \text{ for } k \geq 1 \\ \sigma & \text{otherwise} \end{cases}$$

WHILE Semantics

- How do we get W from W_k?

$$W(\sigma) = \begin{cases} \sigma' & \text{if } \exists k. W_k(\sigma) = \sigma' \neq \perp \\ \perp & \text{otherwise} \end{cases}$$

- This is a valid compositional definition of W
  - Depends only on C⟦c⟧ and B⟦b⟧
- Try the examples again:
  - For C⟦while true do skip⟧
    $W_k(\sigma) = \perp$ for all k, thus $W(\sigma) = \perp$
  - For C⟦while x ≠ 0 do x := x - 2⟧

$$W(\sigma) = \begin{cases} \sigma[x:=0] & \text{if } \sigma(x) = 2k \wedge \sigma(x) \geq 0 \\ \perp & \text{otherwise} \end{cases}$$

More on WHILE

- The solution is not quite satisfactory because
  - It has an operational flavor
  - It does not generalize easily to more complicated semantics (e.g., higher-order functions)
- However, precisely due to the operational flavor this solution is easy to prove sound w.r.t operational semantics

That Wasn't Good Enough!?

Simple Domain Theory

- Consider programs in an eager, deterministic language with one variable called "x"
  - All these restrictions are just to simplify the examples
- A state σ is just the value of x
  - Thus we can use $\mathbb{Z}$ instead of Σ
- The semantics of a command give the value of final x as a function of input x

$$C[\![c]\!] : \mathbb{Z} \to \mathbb{Z}_\perp$$

Examples - Revisited

- Take C⟦while true do skip⟧
  - Unwinding equation reduces to W(x) = W(x)
  - Any function satisfies the unwinding equation
  - Desired solution is W(x) = $\perp$
- Take C⟦while x $\neq$ 0 do x := x - 2⟧
  - Unwinding equation:
    W(x) = if x $\neq$ 0 then W(x - 2) else x
  - Solutions (for all values n, m $\in \mathbb{Z}_\perp$):
    W(x) = if x $\geq$ 0 then
    if x even then 0 else n
    else m
  - Desired solution: W(x) = if x $\geq$ 0 $\wedge$ x even then 0 else $\perp$

An Ordering of Solutions

- The desired solution is the one in which all the arbitrariness is replaced with non-termination
  - The arbitrary values in a solution are not uniquely determined by the semantics of the code
- We introduce an ordering of semantic functions
- Let f, g $\in \mathbb{Z} \to \mathbb{Z}_\perp$
- Define f $\sqsubseteq$ g as
  $\forall x \in \mathbb{Z}.\ f(x) = \perp$ or $f(x) = g(x)$
  - A "smaller" function terminates at most as often, and when it terminates it produces the same result

Alternative Views of Function Ordering

- A semantic function $f \in \mathbb{Z} \to \mathbb{Z}_\perp$ can be written as $S_f \subseteq \mathbb{Z} \times \mathbb{Z}$ as follows:

$$S_f = \{ (x, y) \mid x \in \mathbb{Z},\ f(x) = y \neq \perp \}$$

  - A list of the "terminating" values for the function
- If f $\sqsubseteq$ g then
  - $S_f \subseteq S_g$ (and viceversa)
  - We say that g refines f
  - We say that f approximates g
  - We say that g provides more information than f

The "Best" Solution

- Consider again C⟦while x $\neq$ 0 do x := x - 2⟧
  - Unwinding equation:
    W(x) = if x $\neq$ 0 then W(x - 2) else x
- Not all solutions are comparable:
  W(x) = if x $\geq$ 0 then if x even then 0 else 1 else 2
  W(x) = if x $\geq$ 0 then if x even then 0 else $\perp$ else 3
  W(x) = if x $\geq$ 0 then if x even then 0 else $\perp$ else $\perp$
  (last one is least and best)
- Is there always a least solution?
- How do we find it?
- If only we had a general framework for answering these questions …

Fixed-Point Equations

- Consider the general unwinding equation for while
  while b do c ≡ if b then c; while b do c else skip
- We define a context C (command with a hole)
  C = if b then c; • else skip
  while b do c ≡ C[while b do c]
  - The grammar for C does not contain "while b do c"
- We can find such a (recursive) context for any looping construct
  - Consider: fact n = if n = 0 then 1 else n * fact (n - 1)
  - C = λn. if n = 0 then 1 else n * • (n - 1)
  - fact = C [fact]

Fixed-Point Equations

- The meaning of a context is a semantic functional F : $(\mathbb{Z} \to \mathbb{Z}_\perp) \to (\mathbb{Z} \to \mathbb{Z}_\perp)$ such that
  F ⟦C[w]⟧ = F ⟦w⟧
- For "while": C = if b then c; • else skip
  F w x = if ⟦b⟧ x then w (⟦c⟧ x) else x
  - F depends only on ⟦c⟧ and ⟦b⟧
- We can rewrite the unwinding equation for while
  - W(x) = if ⟦b⟧ x then W(⟦c⟧ x) else x
  - or, W x = F W x for all x,
  - or, W = F W (by function equality)

Fixed-Point Equations

- The meaning of "while" is a solution for W = F W
- Such a W is called a fixed point of F
- We want the least fixed point
  - We need a general way to find least fixed points
- Whether such a least fixed point exists depends on the properties of function F
  - Counterexample: F w x = if w x = $\perp$ then 0 else $\perp$
  - Assume W is a fixed point
  - F W x = W x = if W x = $\perp$ then 0 else $\perp$
  - Pick an x, then if W x = $\perp$ then W x = 0 else W x = $\perp$
  - Contradiction. This F has no fixed point!

Can We Solve This?

- Good news: the functions F that *correspond to contexts in our language* have least fixed points!
- The only way F w x uses w is by invoking it
- If any such invocation diverges, then F w x diverges!
- It turns out: F is monotonic, continuous
  - Not shown here!

The Fixed-Point Theorem

- If F is a semantic functional corresponding to a context in our language
  - F is monotonic and continuous (we assert)
  - For any fixed-point G of F and $k \in \mathbb{N}$
    $F^k(\lambda x.\perp) \sqsubseteq G$
  - The least of all fixed points is
    $\sqcup_k F^k(\lambda x.\perp)$
- Proof (not detailed in the lecture):
  1. By mathematical induction on k.
     Base: $F^0(\lambda x.\perp) = \lambda x.\perp \sqsubseteq G$
     Inductive: $F^{k+1}(\lambda x.\perp) = F(F^k(\lambda x.\perp)) \sqsubseteq F(G) = G$
  2. Suffices to show that $\sqcup_k F^k(\lambda x.\perp)$ is a fixed-point
     $F(\sqcup_k F^k(\lambda x.\perp)) = \sqcup_k F^{k+1}(\lambda x.\perp) = \sqcup_k F^k(\lambda x.\perp)$

WHILE Semantics

- We can use the fixed-point theorem to write the denotational semantics of while:
  ⟦while b do c⟧ = $\sqcup_k F^k(\lambda x.\perp)$
  where F f x = if ⟦b⟧ x then f (⟦c⟧ x) else x
- Example: ⟦while true do skip⟧ = $\lambda x.\perp$
- Example: ⟦while x ≠ 0 then x := x - 1⟧
  - F $(\lambda x.\perp)$ x = if x = 0 then x else $\perp$
  - F^2 $(\lambda x.\perp)$ x = if x = 0 then x else if x - 1 = 0 then x - 1 else $\perp$
    = if $1 \geq x \geq 0$ then 0 else $\perp$
  - F^3 $(\lambda x.\perp)$ x = if $2 \geq x \geq 0$ then 0 else $\perp$
  - LFP_F = if $x \geq 0$ then 0 else $\perp$
- Not easy to find the closed form for general LFPs!

Discussion

- We can write the denotational semantics but we cannot always compute it.
  - Otherwise, we could decide the halting problem
  - H is halting for input 0 iff $[\![H]\!]\ 0 \neq \perp$
- We have derived this for programs with one variable
  - Generalize to multiple variables, even to variables ranging over richer data types, even higher-order functions: domain theory

Can You Remember?

Recall: Learning Goals

- DS is compositional
- When should I use DS?
- In DS, meaning is a "math object"
- DS uses $\perp$ ("bottom") to mean non-termination
- DS uses fixed points and domains to handle `while`
  - This is the tricky bit

Homework

- Homework 2 Due Today
- Homework 3 Out Today
  - Not as long as it looks – separated out every exercise sub-part for clarity.
  - Your denotational answers must be compositional (e.g., $W_k(\sigma)$ or LFP)
- Read Winskel Chapter 6
- Read Hoare article
- Read Floyd article