Proof Techniques for Operational Semantics

Wei Hu Memorial Lecture

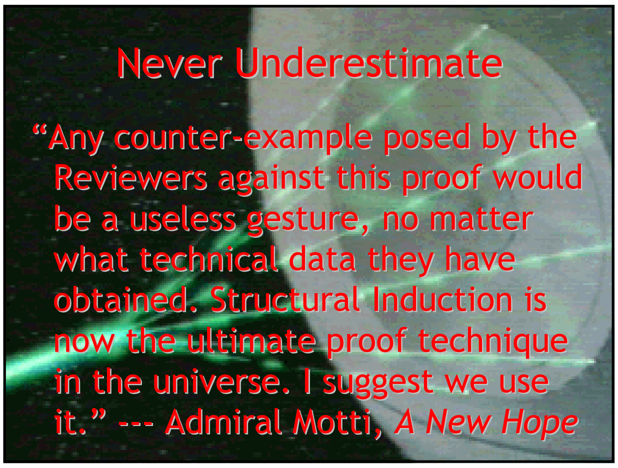
- I will give a completely optional bonus survey lecture: “A Recent History of PL in Context”
  - It will discuss what has been hot in various PL subareas in the last 20 years
  - This may help you get ideas for your class project or locate things that will help your real research
  - Put a tally mark on the sheet if you’d like to attend that day – I’ll pick a most popular day
- Likely Topics:
  - Bug-Finding, Software Model Checking, Automated Deduction, Proof-Carrying Code, PL/Security, Alias Analysis, Constraint-Based Analysis, Run-Time Code Generation

Today’s Cunning Plan

- Why Bother?
- Mathematical Induction
- Well-Founded Induction
- Structural Induction
  - “Induction On The Structure Of The Derivation”

Why Bother?

- I am loathe to teach you anything that I think is a waste of your time.
- Thus I must convince you that inductive opsem proof techniques are useful.
  - Recall class goals: understand PL research techniques and apply them to your research
- This should also highlight where you might use such techniques in your own research.



Classic Example (Schema)

- “A well-typed program cannot go wrong.”
  - Robin Milner
- When you design a new type system, you must show that it is safe (= that the type system is sound with respect to the operational semantics).
- A Syntactic Approach to Type Soundness. Andrew K. Wright, Matthias Felleisen, 1992.
  - Type preservation: “if you have a well-typed program and apply an opsem rule, the result is well-typed.”
  - Progress: “a well-typed program will never get stuck in a state with no applicable opsem rules”
- Done for real languages: SML/NJ, SPARK ADA, Java
  - Plus basically every toy PL research language ever.

Classic Examples

- CCured Project (Berkeley)
  - A program that is instrumented with CCured run-time checks (= "adheres to the CCured type system") will not segfault (= "the x86 opsem rules will never get stuck").
- Vault Language (Microsoft Research)
  - A well-typed Vault program does not leak any tracked resources and invokes tracked APIs correctly (e.g., handles IRQL correctly in asynchronous Windows device drivers, cf. Capability Calculus)
- RC – Reference-Counted Regions For C (Intel Research)
  - A well-typed RC program gains the speed and convenience of region-based memory management but need never worry about freeing a region too early (run-time checks).
- Typed Assembly Language (Cornell)
  - Reasonable C programs (e.g., device drivers) can be translated to TALx86. Well-typed TALx86 programs are type- and memory-safe.
- Secure Information Flow (Many, e.g,. Volpano et al. '96)
  - Lattice model of secure flow analysis is phrased as a type system, so type soundness = noninterference.

Recent Examples

- "The proof proceeds by rule induction over the target term producing translation rules."
  - Chakravarty et al. '05
- "Type preservation can be proved by standard induction on the derivation of the evaluation relation."
  - Hosoya et al. '05
- "Proof: By induction on the derivation of $N \Downarrow W$."
  - Sumi and Pierce '05
- Method: chose four POPL 2005 papers at random, the three above mentioned structural induction.

Induction

- Most important technique for studying the formal semantics of prog languages
  - If you want to perform or understand PL research, you must grok this!
- Mathematical Induction (simple)
- Well-Founded Induction (general)
- Structural Induction (widely used in PL)

Mathematical Induction

- Goal: prove $\forall n \in \mathbb{N}.\ P(n)$
- Base Case: prove P(0)
- Inductive Step:
  - Prove $\forall\ n>0.\ p(n) \Rightarrow p(n+1)$
  - "Pick arbitrary n, assume p(n), prove p(n+1)"

Why Does It Work?

- There are no infinite descending chains of natural numbers
- For any n, P(n) can be obtained by starting from the base case and applying n instances of the inductive step

Well-Founded Induction

- A relation $\prec \subseteq A \times A$ is well-founded if there are no infinite descending chains in A
  - Example: $<_1 = \{ (x, x+1) \mid x \in \mathbb{N} \}$
    - the predecessor relation
  - Example: $< \ = \{ (x, y) \mid x, y \in \mathbb{N}$ and $x < y \}$
- Well-founded induction:
  - To prove $\forall x \in A.\ P(x)$ it is enough to prove $\forall x \in A.\ [\forall y \prec x \Rightarrow P(y)] \Rightarrow P(x)$
- If $\prec$ is $<_1$ then we obtain mathematical induction as a special case

Structural Induction

- Recall $e ::= n \mid e_1 + e_2 \mid e_1 * e_2 \mid x$
- Define $\prec \subseteq$ Aexp * Aexp such that

  $e_1 \prec e_1 + e_2 \qquad e_2 \prec e_1 + e_2$

  $e_1 \prec e_1 * e_2 \qquad e_2 \prec e_1 * e_2$

  - no other elements of Aexp * Aexp are related by $\prec$
- To prove $\forall e \in$ Aexp. P(e)
  1. $\vdash \forall n \in \mathbb{Z}.\ P(n)$
  2. $\vdash \forall x \in L.\ P(x)$
  3. $\vdash \forall e_1, e_2 \in \text{Aexp}.\ P(e_1) \wedge P(e_2) \Rightarrow P(e_1 + e_2)$
  4. $\vdash \forall e_1, e_2 \in \text{Aexp}.\ P(e_1) \wedge P(e_2) \Rightarrow P(e_1 * e_2)$

Notes on Structural Induction

- Called structural induction because the proof is guided by the structure of the expression
- One proof case per form of expression
  - Atomic expressions (with no subexpressions) are all base cases
  - Composite expressions are the inductive case
- This is the most useful form of induction in PL study

Example of Induction on Structure of Expressions

- Let
  - L(e) be the # of literals and variable occurrences in e
  - O(e) be the # of operators in e
- Prove that $\forall e \in$ Aexp. L(e) = O(e) + 1
- Proof: by induction on the structure of e
  - Case e = n. L(e) = 1 and O(e) = 0
  - Case e = x. L(e) = 1 and O(e) = 0
  - Case $e = e_1 + e_2$.
    - $L(e) = L(e_1) + L(e_2)$ and $O(e) = O(e_1) + O(e_2) + 1$
    - By induction hypothesis $L(e_1) = O(e_1) + 1$ and $L(e_2) = O(e_2) + 1$
    - Thus $L(e) = O(e_1) + O(e_2) + 2 = O(e) + 1$
  - Case $e = e_1 * e_2$. Same as the case for +

Other Proofs by Structural Induction on Expressions

- Most proofs for Aexp sublanguage of IMP
- Small-step and natural semantics obtain equivalent results:

  $\forall e \in \text{Exp}.\ \forall n \in \mathbb{N}.\ e \rightarrow^* n \Leftrightarrow e \Downarrow n$

- Structural induction on expressions works here because all of the semantics are syntax directed

Stating The Obvious (With a Sense of Discovery)

- You are given a concrete state σ.
- You have $\vdash \langle x + 1, \sigma\rangle \Downarrow 5$
- You also have $\vdash \langle x + 1, \sigma\rangle \Downarrow 88$
- Is this possible?

Why That Is Not Possible

- Prove that IMP is deterministic

  $\forall e \in \text{Aexp}.\ \forall \sigma \in \Sigma.\ \forall n, n' \in \mathbb{N}.\ \langle e, \sigma\rangle \Downarrow n \wedge \langle e, \sigma\rangle \Downarrow n' \Rightarrow n = n'$

  $\forall b \in \text{Bexp}.\ \forall \sigma \in \Sigma.\ \forall t, t' \in \mathbb{B}.\ \langle b, \sigma\rangle \Downarrow t \wedge \langle b, \sigma\rangle \Downarrow t' \Rightarrow t = t'$

  $\forall c \in \text{Comm}.\ \forall \sigma, \sigma', \sigma'' \in \Sigma.\ \langle c, \sigma\rangle \Downarrow \sigma' \wedge \langle c, \sigma\rangle \Downarrow \sigma'' \Rightarrow \sigma' = \sigma''$
- No immediate way to use mathematical induction
- For commands we cannot use induction on the structure of the command
  - `while`'s evaluation does **not** depend only on the evaluation of its strict subexpressions

$$\frac{\langle b, \sigma\rangle \Downarrow \text{true} \quad \langle c, \sigma\rangle \Downarrow \sigma' \quad \langle \text{while } b \text{ do } c, \sigma'\rangle \Downarrow \sigma''}{\langle \text{while } b \text{ do } c, \sigma\rangle \Downarrow \sigma''}$$

Recall Opsem

- Operational semantics assigns meanings to programs by listing rules of inference that allow you to prove judgments by making derivations.
- A derivation is a tree-structured object made up of valid instances of inference rules.

Induction on the Structure of Derivations

- Key idea: The hypothesis does not just assume a $c \in \text{Comm}$ but the **existence of a derivation of $\langle c, \sigma \rangle \Downarrow \sigma'$**
- Derivation trees are also defined inductively, just like expression trees
- A derivation is built of subderivations:

$$\dfrac{\dfrac{\langle x, \sigma_{i+1}\rangle \Downarrow 5 - i \quad 5 - i \le 5}{\langle x \le 5, \sigma_{i+1}\rangle \Downarrow \text{true}} \qquad \dfrac{\dfrac{\langle x + 1, \sigma_{i+1}\rangle \Downarrow 6 - i}{\langle x{:=}x{+}1, \sigma_{i+1}\rangle \Downarrow \sigma_i} \quad \langle W, \sigma_i\rangle \Downarrow \sigma_0}{\langle x{:=}x{+}1;\ W, \sigma_{i+1}\rangle \Downarrow \sigma_0}}{\langle \text{while } x \le 5 \text{ do } x := x + 1, \sigma_{i+1}\rangle \Downarrow \sigma_0}$$

- Adapt the structural induction principle to work on the structure of derivations

Induction on Derivations

- To prove that for all derivations D of a judgment, property P holds

1. For each derivation rule of the form

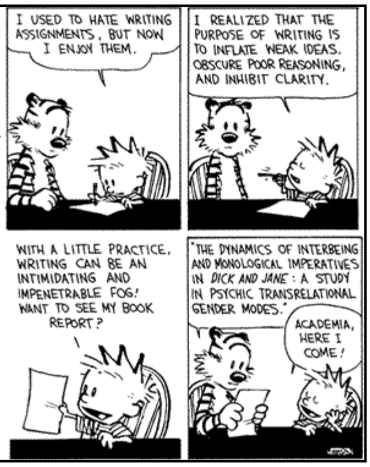
$$\frac{H_1 \ldots H_n}{C}$$

2. Assume that P holds for derivations of H_i ($i = 1, .., n$)
3. Prove the the property holds for the derivation obtained from the derivations of H_i using the given rule

New Notation

- Write D :: Judgment to mean "D is the derivation that proves Judgment"
- Example:

D :: $\langle x{+}1, \sigma \rangle \Downarrow 2$



Induction on Derivations (2)

- Prove that evaluation of commands is deterministic:
  $\langle c, \sigma \rangle \Downarrow \sigma' \Rightarrow \forall \sigma'' \in \Sigma.\ \langle c, \sigma \rangle \Downarrow \sigma'' \Rightarrow \sigma' = \sigma''$
- Pick arbitrary c, σ, σ' and D :: $\langle c, \sigma \rangle \Downarrow \sigma'$
- To prove: $\forall \sigma'' \in \Sigma.\ \langle c, \sigma \rangle \Downarrow \sigma'' \Rightarrow \sigma' = \sigma''$
- Proof: by induction on the structure of the derivation D
- Case: last rule used in D was the one for skip

$$D :: \frac{}{\langle \text{skip}, \sigma \rangle \Downarrow \sigma}$$

  - This means that c = skip, and $\sigma' = \sigma$
  - By inversion $\langle c, \sigma \rangle \Downarrow \sigma''$ uses the rule for skip
  - Thus $\sigma'' = \sigma$
  - This is a base case in the induction

Induction on Derivations (3)

- Case: the last rule used in D was the one for sequencing

$$D :: \frac{D_1 :: \langle c_1, \sigma \rangle \Downarrow \sigma_1 \qquad D_2 :: \langle c_2, \sigma_1 \rangle \Downarrow \sigma'}{\langle c_1; c_2, \sigma \rangle \Downarrow \sigma'}$$

- Pick arbitrary σ'' such that D'' :: $\langle c_1; c_2, \sigma \rangle \Downarrow \sigma''$.
  - by inversion D'' uses the rule for sequencing
  - and has subderivations $D''_1 :: \langle c_1, \sigma \rangle \Downarrow \sigma''_1$ and $D''_2 :: \langle c_2, \sigma''_1 \rangle \Downarrow \sigma''$
- By induction hypothesis on D_1 (with D''_1): $\sigma_1 = \sigma''_1$
  - Now $D''_2 :: \langle c_2, \sigma_1 \rangle \Downarrow \sigma''$
- By induction hypothesis on D_2 (with D''_2): $\sigma'' = \sigma'$
- This is a simple inductive case

Induction on Derivations (4)

- Case: the last rule used in D was `while true`

$$D :: \frac{D_1 :: \langle b, \sigma\rangle \Downarrow true \quad D_2 :: \langle c, \sigma\rangle \Downarrow \sigma_1 \quad D_3 :: \langle \text{while } b \text{ do } c, \sigma_1\rangle \Downarrow \sigma'}{\langle \text{while } b \text{ do } c, \sigma\rangle \Downarrow \sigma'}$$

- Pick arbitrary σ'' such that D''::<while b do c, σ> ⇓ σ''
  - by inversion and determinism of boolean expressions, D'' also uses the rule for `while true`
  - and has subderivations D''_2 :: <c, σ> ⇓ σ''_1 and D''_3 :: <W, σ''_1> ⇓ σ''
- By induction hypothesis on D_2 (with D''_2): $\sigma_1 = \sigma''_1$
  - Now D''_3 :: <while b do c, σ_1> ⇓ σ''
- By induction hypothesis on D_3 (with D''_3): σ'' = σ'

What Do You, The Viewers At Home, Think?

- Let's do `if true` together!
- Case: the last rule in D was `if true`

$$D :: \frac{D_1 :: \langle b, \sigma\rangle \Downarrow true \qquad D_2 :: \langle c1, \sigma\rangle \Downarrow \sigma_1}{\langle \text{if } b \text{ do } c1 \text{ else } c2, \sigma\rangle \Downarrow \sigma_1}$$

- Try to do this on a piece of paper. In a few minutes I'll have some lucky winners come on down.

Induction on Derivations (5)

- Case: the last rule in D was `if true`

$$D :: \frac{D_1 :: \langle b, \sigma\rangle \Downarrow true \qquad D_2 :: \langle c1, \sigma\rangle \Downarrow \sigma'}{\langle \text{if } b \text{ do } c1 \text{ else } c2, \sigma\rangle \Downarrow \sigma'}$$

- Pick arbitrary σ'' such that D'' :: <if b do c1 else c2, σ> ⇓ σ''
  - By inversion and determinism, D'' also uses `if true`
  - And has subderivations D''_1 :: <b, σ> ⇓ true and D''_2 :: <c1, σ> ⇓ σ''
- By induction hypothesis on D_2 (with D''_2): σ' = σ''

Induction on Derivations Summary

- If you must prove ∀x ∈ A. P(x) ⇒ Q(x)
  - with A inductively defined and P(x) rule-defined
  - we pick arbitrary x ∈ A and D :: P(x)
  - we could do induction on both facts
    - x ∈ A leads to induction on the structure of x
    - D :: P(x) leads to induction on the structure of D
  - Generally, the induction on the structure of the derivation is more powerful and a safer bet
- Sometimes there are many choices for induction
  - choosing the right one is a trial-and-error process
  - a bit of practice can help a lot

Equivalence

- Two expressions (commands) are equivalent if they yield the same result from all states

$$e_1 \approx e_2 \text{ iff } \forall \sigma \in \Sigma.\ \forall n \in \mathbb{N}.\ \langle e_1, \sigma\rangle \Downarrow n \text{ iff } \langle e_2, \sigma\rangle \Downarrow n$$

and for commands

$$c_1 \approx c_2 \text{ iff } \forall \sigma, \sigma' \in \Sigma.\ \langle c_1, \sigma\rangle \Downarrow \sigma' \text{ iff } \langle c_2, \sigma\rangle \Downarrow \sigma'$$

Notes on Equivalence

- Equivalence is like logical validity
  - It must hold in all states (= all valuations)
  - 2 ≈ 1 + 1 is like "2 = 1 + 1 is valid"
  - 2 ≈ 1 + x might or might not hold.
    - So, 2 is not equivalent to 1 + x
- Equivalence (for IMP) is undecidable
  - If it were decidable we could solve the halting problem for IMP. How?
- Equivalence justifies code transformations
  - compiler optimizations
  - code instrumentation
  - abstract modeling
- Semantics is the basis for proving equivalence

Equivalence Examples

- skip; c $\approx$ c
- while b do c $\approx$
  if b then c; while b do c else skip
- If $e_1 \approx e_2$ then x := $e_1 \approx$ x := e_2
- while true do skip $\approx$ while true do x := x + 1
- If c is

  while x $\neq$ y do

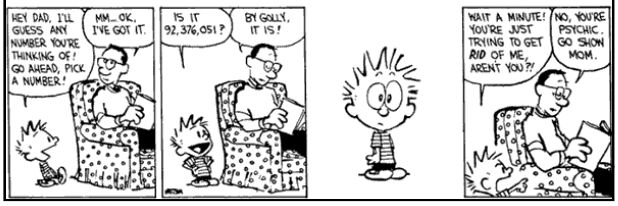
  if x $\geq$ y then x := x - y else y := y - x

  then

  (x := 221; y := 527; c) $\approx$ (x := 17; y := 17)

Potential Equivalence

- (x := e_1; x := e_2) $\approx$ x := e_2
- Is this a valid equivalence?



Not An Equivalence

- (x := e_1; x := e_2) $\not\approx$ x := e_2
- Iie. Chigau yo. Dame desu!
- Not a valid equivalence for all e_1, e_2.
- Consider:
  - (x := x+1; x := x+2) $\not\approx$ x := x+2
- But for n_1, n_2 it's fine:
  - (x := n_1; x := n_2) $\approx$ x := n_2

Proving An Equivalence

- Prove that "skip; c $\approx$ c" for all c
- Assume that D :: $\langle$skip; c, $\sigma\rangle \Downarrow \sigma'$
- By inversion (twice) we have that

$$D :: \frac{\overline{\langle \text{skip}, \sigma\rangle \Downarrow \sigma} \qquad D_1 :: \langle c, \sigma\rangle \Downarrow \sigma'}{\langle \text{skip}; c, \sigma\rangle \Downarrow \sigma'}$$

- Thus, we have $D_1 :: \langle c,\sigma\rangle \Downarrow \sigma'$
- The other direction is similar

Proving An Inequivalence

- Prove that x := y $\not\approx$ x := z when y $\neq$ z
- It suffices to exhibit a σ in which the two commands yield different results
- Let σ(y) = 0 and σ(z) = 1
- Then

  $\langle$x := y, $\sigma\rangle \Downarrow \sigma$[x := 0]

  $\langle$x := z, $\sigma\rangle \Downarrow \sigma$[x := 1]

Summary of Operational Semantics

- Precise specification of dynamic semantics
  - order of evaluation (or that it doesn't matter)
  - error conditions (sometimes implicitly, by rule applicability; "no applicable rule" = "get stuck")
- Simple and abstract (vs. implementations)
  - no low-level details such as stack and memory management, data layout, etc.
- Often not compositional (see while)
- Basis for many proofs about a language
  - Especially when combined with type systems!
- Basis for much reasoning about programs
- Point of reference for other semantics

Homework

- Homework 1 Due Today
- Homework 2 Due Thursday
  - No more homework overlaps.
- Read Winskel Chapter 5
  - Pay careful attention.
- Read Winskel Chapter 8
  - Summarize.