Dependant Type Systems
(saying what you are)

Data Abstraction
(hiding what you are)

Review

- We studied a variety of type systems
- We repeatedly made the type system more expressive to enable the type checker to catch more errors
- But we have steered clear of undecidable systems
  - Thus there must still be many errors that are not caught
- Now we explore more complex type systems that bring type checking closer to program verification

Dependent Types

- Say that we have the functions
  zero : nat → vector (creates vector of requested length)
  dotprod : vector → vector → real (dot product)
- The types do not prevent using dotprod on vectors of different length
  - If they did, we could catch more bugs!
- Idea: Make "vector" a type family annotated by a natural number
  "vector n" is the type of vectors of length n
  dotprod: vector n → vector n → real (where is n bound?)
  zero : nat → vector ?

Need a way to refer to the *value* of the first argument in the type!

Dependent Type Notation

- How to write the type of zero : nat → vector ?
- Given two sets A and B verify the isomorphism
  $A \to B \simeq \Pi_{x \in A} B$
  - The latter is the cartesian product of B with itself as many times as there are elements in A
  - Also written as Πx:A.B (x plays no role so far!)
  - But now we can make B depend on x!
- Definition: Πx:A.B is the type of functions with argument in A and with the result type B (possibly depending on the value of the argument x in A)
  - We write "zero : Πx:nat. vector x"
  - Special case when $x \notin B$ we abbreviate as A → B
  - We play "fast and loose" with the binding of Π

Dependent Typing Rules

$$\frac{\Gamma, x : \tau_2 \vdash e : \tau}{\Gamma \vdash \lambda x : \tau_2 . e : \Pi x : \tau_2 . \tau} \qquad \frac{\Gamma \vdash e_1 : \Pi x : \tau_2 . \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\ e_2 : [e_2/x]\tau}$$

- Note that expressions are now part of types
- Have types like "vector 5" and "vector (2 + 3)"
- We need type equivalence

$$\frac{\Gamma \vdash e : \tau \quad \Gamma \vdash \tau \equiv \tau'}{\Gamma \vdash e : \tau'}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2}{\Gamma \vdash \mathtt{vector}\ e_1 \equiv \mathtt{vector}\ e_2}$$

Dependent Types and Program Specifications

- Types act as specifications
- With dependent types we can specify *any property!*
- For example, define the following types:
  "eq e" - the type of values equal to "e".
  Also named "sng e" (the singleton type)
  "ge e" - the type of values larger or equal to "e"
  "lt e" - the type of values smaller than "e"
  "and τ_1 τ_2" - the type of values having both type τ_1 and τ_2
- Need appropriate typing rules for the new types
- The precondition for vector-accessing (cf. HW5)
  - read: Πn:nat.vector n → (and (ge 0) (lt n)) → int
- The type checker must do program verification

Dependent Type Commentary

- Type checking with Π types can be *as hard as full program verification*
- Type equivalence can be undecidable
  - If types are dependent on expressions drawn from a powerful language ("powerful" = "arithmetic")
  - Then even type checking will be undecidable
- Dependent types play an important role in the formalization of logics
  - Started with Per Martin-Lof
  - Proof checking via type checking
  - Proof-carrying code uses a dependent type checker to check proofs
  - There are program specification tools based on Π types

Dependent Sum Types

- We want to pack a vector with its length
  - e = (n, v) where "v : vector n"
  - The type of an element of a pair depends on the *value* of another element
  - This is another form of dependency
  - The type of e is "nat $\times$ vector ?"
- Given two sets A and B verify the isomorphism

  $$A \times B \simeq \Sigma_{x \in A} B$$

  - The latter is the *disjoint* union of B with itself as many times as there are elements in A
  - Also written as Σx:A.B   (x here plays no role)
  - But now we can make B depend on x!

Dependent Sum Types

- Definition: Σx:A.B is the type of pairs with first element of type A and second element of type B *(possibly depending on the **value** of first element x)*
  - Now we can write e : Σx:nat. vector x
- Functions that compute the length of a vector
  - vlength : Πn:nat.vector n $\to$ nat
    - (the result is not constrained)
  - slength : Πn:nat.vector n $\to$ sng n
    - "sng n" is a dependent type that contains only n
    - called the singleton type (recall from 3 slides ago …)
- What if the vector is packed with its length?
  - pvlength : Σn:nat.vector n $\to$ nat
  - pslength : Σn:nat.vector n $\to$ sng n

Dependent Sum Types Static Semantics

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : [e_1/x]\tau_2}{\Gamma \vdash (e_1, e_2) : \Sigma x : \tau_1.\tau_2}$$

$$\frac{\Gamma \vdash e : \Sigma x : \tau_1.\tau_2}{\Gamma \vdash \mathtt{snd}\ e : [\mathtt{fst}\ e/x]\tau_2}$$

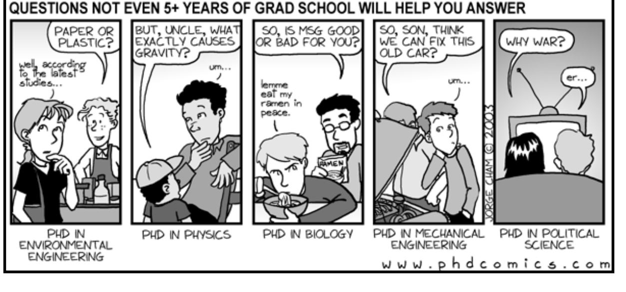
- Note how this rule reduces to the usual rules for tuples when there is no dependency
- The evaluation rules are unchanged

Weimeric Commentary

- Dependant types seem obscure: why care?
- Grand Unified Theory
  - Type Checking = Verification (= Model Checking = Proof Checking = Abstract Interpretation …)
- CCured Project
  - Rumor has it this project was successful
  - The whole thing is dependant sum types
    - SEQ = (pointer + lower bound + upper bound)
    - FSEQ = (pointer + upper bound)
    - WILD = (pointer + lower bound + upper bound + rtti)

Types for Data Abstraction

What's inside the implementation?
We don't know!



Data Abstraction

- Ability to hide (abstract) concrete implementation details
- Modularity builds on data abstraction
- Improves program structure and minimizes dependencies
- One of the most influential developments of the 1970's
- Key element for much of the success of object orientation in the 1980's

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Example of Abstraction

- Cartesian points (gotta love it!)
- Introduce the "abstype" language construct:

abstype point implements
mk : real × real → point
xc : point → real
yc : point → real
is
< point = real × real,
mk = λx. x,
xc = fst,
yc = snd >

- Shows a concrete implementation
- Allows the rest of the program to access the implementation *through an abstract interface*
- Only the interface need to be publicized
- Allows separate compilation

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Data Abstraction

- It is useful to separate the creation of the abstract type and its use (newsflash …)
- Extend the syntax (t = imp, σ = interface):
  Terms ::= … | < t = τ, e : σ > | open e_a as t, x : σ in e_b
  Types ::= … | ∃t. σ
- The expression <t=τ, e : σ> takes the concrete implementation e and "packs it" as a value of an abstract type
  - Alternative notation: "pack e as ∃t. σ with t = τ"
  - "existential types" - used to model the stack, etc.
- The "open" expression allows e_b to access the abstract type expression e_a using the name x, the unknown type of the concrete implementation "t" and the interface σ

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Example with Abstraction

- C = {mk = λx.x, xc = fst, yc = snd } is a *concrete implementation* of points as real × real
- We want to hide the type of the representation σ is the following type:
  { mk : real × real → point,
  xc : point → real, yc : point → real}
- Note that C : [real×real/point]σ
- A = <point=real×real, C : σ> is an expression of the abstract type ∃point.σ
- We want clients to access only the second component of A and just use the abstract name "point" for the first component:
  open A as point, P : σ in … P.xc(P.mk(1.0, 2.0)) …

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Typing Rules for Existential Types

- We add the following typing rules:

$$\frac{\Gamma \vdash [\tau/t]e : [\tau/t]\sigma}{\Gamma \vdash \langle t = \tau, e : \sigma \rangle : \exists t.\sigma}$$

$$\frac{\Gamma \vdash e_a : \exists t.\sigma \qquad \Gamma, t, p : \sigma \vdash e_b : \tau}{\Gamma \vdash \texttt{open}\ e_a\ \texttt{as}\ t, p : \sigma\ \texttt{in}\ e_b : \tau} \qquad t \notin FV(\Gamma \cup \tau)$$

- The restriction in the rule for "open" ensures that t does not escape its scope

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Evaluation Rules for Abstract Types

- We add a new form of value
  v ::= … | <t=τ, v : σ>
  - This is just like v but with some type decorations that make it have an existential type

$$\frac{e_a \Downarrow \langle t = \tau, v : \sigma \rangle \qquad [v/x][\tau/t]e_b \Downarrow v'}{\texttt{open}\ e_a\ \texttt{as}\ t, x : \sigma\ \texttt{in}\ e_b \Downarrow v'}$$

- At the time e_b is evaluated, abstract-type variables are replaced with concrete values
  - If we ignore the type issues "open e_a as t, x : σ in e_b" is like "let x : σ = e_a in e_b"
  - Difference: e_b *cannot know statically* what is the concrete type of x so it cannot take advantage of it

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Abstract Types as a Specification Mechanism

- Just like polymorphism, existential types are mostly a type checking mechanism
- A function of type $\forall t.\ t\ List \rightarrow int$ does not know *statically* what is the type of the list elements. Therefore no operations are allowed on them
  - But it will have at run-time the actual value of t
  - "There are no type variables at run-time"
- Same goes for existentials
- These type mechanisms are a very powerful (and widely used!) form of static checking
  - Recall Wadler's "Theorems for Free"

Data Abstraction and the Real World

- Example: file descriptors
- Solution 1:
  - Represent file descriptors as "int" and export the interface {open:string→int, read:int→ data}
- An untrusted client of the interface calls "read"
- How can we know that "read" is invoked with a file descriptor that was obtained from "open"? Anyone?

Data Abstraction and the Real World

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- Solution 1:
  - Represent file descriptors as "int" and export the interface {open:string→int, read:int→ data}
- An untrusted client of the interface calls "read"
- How can we know that "read" is invoked with a file descriptor that was obtained from "open"?
  - We must keep track of all integers that represent file descriptors
  - We design the interface such that all such integers are small integers and we can essentially keep a bitmap
  - This becomes expensive with more complex (e.g. pointer-based) representations

Data Abstraction, Static Checking

- Solution 2: Use the same representation but *export an abstraction* of it.
  - ∃fd. File or
  - ∃fd. {open : string → fd, read : fd → data}
  - A possible value:
  - Fd = < fd = int, { open = …, read = …} : File> : ∃fd. File
- Now the *untrusted* client e

  open Fd as fd, x : File in e
- At run-time "e" can see that file descriptors are integers
  - But cannot cast 187 as a file descriptor.
  - Static checking with no run-time costs!
  - Catch: you must be able to type check e!

Modularity

- A module is a program fragment along with *visibility constraints*
- Visibility of functions and data
  - Specify the function interface but hide its implementation
- Visibility of type definitions
  - More complicated because the type might appear in specifications of the visible functions and data
  - Can use data abstraction to handle this
- A module is represented as a type component and an implementation component

  $<t = \tau,\ e : \sigma>$ (where t can occur in e and σ)
  - even though the specification (σ) refers to the implementation type we can still hide the latter

Problems with Existentialists

- Existentialist types
  - Assert that truth is subjectivity
  - Oppose the rational tradition and positivism
  - Are subject to an "absurd" universe
- Problems:
  - "In so far as Existentialism is a philosophical doctrine, it remains an idealistic doctrine: it hypothesizes specific historical conditions of human existence into ontological and metaphysical characteristics. Existentialism thus becomes part of the very ideology which it attacks, and its radicalism is illusory." (Herbert Marcuse, "Sartre's Existentialism", p. 161)

Problems with Existentials

- Existential types
  - Allow representation (type) hiding
  - Allow separate compilation. Need to know only the type of a module to compile its client
  - First-class modules. They can be selected at run-time. (cf. OO interface subtyping)
- Problems:
  - Closed scope. Must open an existential before using it!
  - Poor support for module hierarchies

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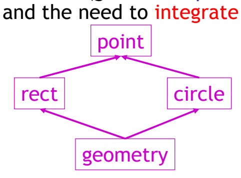
Problems with Existentials (Cont.)

- There is an inherent tension between handling modules in isolation (good for separate compilation, interchangeability) and the need to integrate them



(the arrow means "depends on")

- Solution 1: open "point" at top level
  - Inversion of program structure
  - The most basic construct has the widest scope

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Give Up Abstraction?

- Solution 2: incorporate point in rect and circle

  R = < point = …, <rect = point × point, …> … >

  C = < point = …, <circle = point × real, …> … >
- When we open R and C we get *two distinct notions of point*!
  - And we will *not* be able to combine them
- Another option is to allow the type checker to see the representation type
  - and thus give up representation hiding

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Strong Sums

- New way to open a package

  Terms e ::= ... | Ops(e)

  Types τ ::= ... Σt.τ | Typ(e)
  - Use Typ and Ops to decompose the module
  - Operationally, they are just like "fst" and "snd"
  - Σt.τ is the dependent sum type
  - It is like $\exists$t.τ except we can look at the type

$$\frac{\Gamma \vdash e : \Sigma t.\tau}{\Gamma \vdash \mathtt{Ops}(e) : \tau[\mathtt{Typ}(e)/t]}$$

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Modularity with Strong Sums

- Consider the R and C defined as before:

  Pt = <point = real × real, ...> : Σpoint. τ_P

  R = <point = Typ(Pt),

  < rect = point × point, ...> : Σrect. τ_R

  C = <point = Typ(Pt),

  < circle = point × real, ...> : Σcircle. τ_C
- Since we use strong-sums the type checker sees that the two point types are the same

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Modules with Strong Sums

- ML's module system is based on strong sums

Problems:
- Poorer data abstraction
- Expressions appear in types (Typ(e))
  - Types might not be known until at run time
  - Lost separate compilation
  - Trouble if e has side-effects (but we can use a value restriction – e.g., "IntSet.t")
- Second-class modules (because of value restriction)
- We can combine existentials with strong sums
  - Translucent sums: partially visible

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Homework

- Project Status Update
- Project Due Tue Nov 28
  - You have ~35 days (including holidays) to complete it.
  - Need help? Stop by my office or send email.