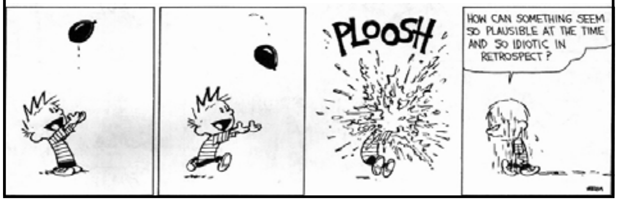
Monomorphic Type Systems



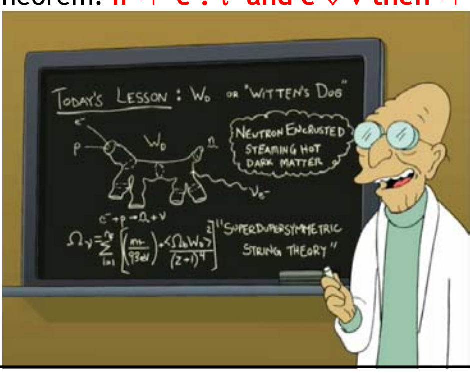
Type Soundness for F_1

What does this mean?

- Theorem: **If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$**
  - Also called, subject reduction theorem, type preservation theorem
- This is one of the most important sorts of theorems in PL
- Whenever you make up a new safe language you are expected to prove this
  - Examples: Vault, TAL, CCured, …

How Might We Prove It?

- Theorem: **If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$**



Proof Approaches To Type Safety

- Theorem: If $\cdot \vdash e : \tau$ and $e \Downarrow v$ then $\cdot \vdash v : \tau$
- Try to prove by induction on e
  - Won't work because $[v_2/x]e'_1$ in the evaluation of $e_1\ e_2$
  - Same problem with induction on $\cdot \vdash e : \tau$
- Try to prove by induction on τ
  - Won't work because e_1 has a "bigger" type than $e_1\ e_2$
- Try to prove by induction on $e \Downarrow v$
  - To address the issue of $[v_2/x]e'_1$
  - This is it!

Type Soundness Proof

- Consider the *function application* case

$$\mathcal{E} :: \frac{e_1 \Downarrow \lambda x : \tau_2 . e'_1 \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e'_1 \Downarrow v}{e_1\ e_2 \Downarrow v}$$

and by inversion on the derivation of $e_1\ e_2 : \tau$

$$\mathcal{D} :: \frac{\cdot \vdash e_1 : \tau_2 \longrightarrow \tau \qquad \cdot \vdash e_2 : \tau_2}{\cdot \vdash e_1\ e_2 : \tau}$$

- From IH on $e_1 \Downarrow$ … we have $\cdot, x : \tau_2 \vdash e_1' : \tau$
- From IH on $e_2 \Downarrow$ … we have $\cdot \vdash v_2 : \tau_2$
- Need to infer that $\cdot \vdash [v_2/x]e_1' : \tau$ and use the IH
  - We need a substitution lemma (by induction on e_1')

Significance of Type Soundness

- The theorem says that the result of an evaluation has the same type as the initial expression
- The theorem does not say that
  - The evaluation *never gets stuck* (e.g., trying to apply a non-function, to add non-integers, etc.), nor that
  - The evaluation *terminates*
- Even though both of the above facts are true of F_1
- We need a small-step semantics to prove that the execution never gets stuck
- I Assert: the execution always terminates in F_1
  - When does the base lambda calculus ever not terminate?

Small-Step Contextual Semantics for F_1

- We define redexes

  $r ::= n_1 + n_2 \mid \text{if } b \text{ then } e_1 \text{ else } e_2 \mid (\lambda x{:}\tau.e_1)\ v_2$
- and contexts

  $H ::= H_1 + e_2 \mid n_1 + H_2 \mid \text{if } H \text{ then } e_1 \text{ else } e_2 \mid H_1\ e_2 \mid (\lambda x{:}\tau.\ e_1)\ H_2 \mid \bullet$
- and local reduction rules

  $n_1 + n_2 \rightarrow n_1 \text{ plus } n_2$

  $\text{if true then } e_1 \text{ else } e_2 \rightarrow e_1$

  $\text{if false then } e_1 \text{ else } e_2 \rightarrow e_2$

  $(\lambda x{:}\tau.\ e_1)\ v_2 \rightarrow [v_2/x]e_1$
- and one global reduction rule

  $H[r] \rightarrow H[e]$ iff $r \rightarrow e$

Decomposition Lemmas for F_1

1. If $\cdot \vdash e : \tau$ and e is not a (final) value then there exist (unique) H and r such that e = H[r]
   - any well typed expression can be decomposed
   - any well-typed non-value can make progress
2. Furthermore, there exists τ' such that $\cdot \vdash r : \tau'$
   - the redex is closed and well typed
3. Furthermore, there exists e' such that $r \rightarrow e'$ and $\cdot \vdash e' : \tau'$
   - local reduction is type preserving
4. Furthermore, for any e', $\cdot \vdash e' : \tau'$ implies $\cdot \vdash H[e'] : \tau$
   - the expression preserves its type if we replace the redex with an expression of same type

Type Safety of F_1

- Type preservation theorem
  - If $\cdot \vdash e : \tau$ and $e \rightarrow e'$ then $\cdot \vdash e' : \tau$
  - Follows from the decomposition lemma
- Progress theorem
  - If $\cdot \vdash e : \tau$ and e is not a value then there exists e' such that e can make progress: $e \rightarrow e'$
- Progress theorem says that execution can make progress *on a well typed expression*
- From type preservation we know the execution of well *typed expressions never gets stuck*
  - This is a (very!) common way to *state and prove type safety* of a language

What's Next?

- We've got the basic simply-typed monomorphic lambda calculus
- Now let's make it more complicated …
- By adding features!

Product Types: Static Semantics

- Extend the syntax with (binary) tuples

  $e ::= \ldots \mid (e_1, e_2) \mid \text{fst } e \mid \text{snd } e$

  $\tau ::= \ldots \mid \tau_1 \times \tau_2$
  - This language is sometimes called $F_1^{\times}$
- Same typing judgment $\Gamma \vdash e : \tau$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{fst } e : \tau_1} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \texttt{snd } e : \tau_2}$$

Dynamic Semantics and Soundness

- New form of values: $v ::= \ldots \mid (v_1, v_2)$
- New (big step) evaluation rules:

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2}{(e_1, e_2) \Downarrow (v_1, v_2)}$$

$$\frac{e \Downarrow (v_1, v_2)}{\texttt{fst } e \Downarrow v_1} \qquad \frac{e \Downarrow (v_1, v_2)}{\texttt{snd } e \Downarrow v_2}$$

- New contexts: $H ::= \ldots \mid (H_1, e_2) \mid (v_1, H_2) \mid \text{fst } H \mid \text{snd } H$
- New redexes:

  $\text{fst } (v_1, v_2) \rightarrow v_1$

  $\text{snd } (v_1, v_2) \rightarrow v_2$
- Type soundness holds just as before

General PL Feature Plan

- The general plan for language feature design
- You invent a new feature (tuples)
- You add it to the lambda calculus
- You invent typing rules and opsem rules
- You extend the basic proof of type safety
- You declare moral victory, and milling throngs of cheering admirers wait to carry you on their shoulders to be knighted by the Queen, etc.

Records

- Records are like tuples with labels (w00t!)
- New form of expressions
  - e ::= … | $\{L_1 = e_1, \ldots, L_n = e_n\}$ | e.L
- New form of values
  - v ::= $\{L_1 = v_1, \ldots, L_n = v_n\}$
- New form of types
  - τ ::= … | $\{L_1 : \tau_1, \ldots, L_n : \tau_n\}$
- … follows the model of $F_1^{\times}$
  - typing rules
  - derivation rules
  - type soundness

Sum Types

- We need disjoint union types of the form:
  - either an int or a float
  - either 0 or a pointer
  - either a (binary tree node with two children) or a (leaf)
- New expressions and types
  - e ::= … | injl e | injr e |
    case e of injl x → e_1 | injr y → e_2
  - τ ::= … | $\tau_1 + \tau_2$
  - A value of type $\tau_1 + \tau_2$ is *either* a τ_1 *or* a τ_2
  - Like union in C or Pascal, but safe
    - distinguishing between components is *under compiler control*
  - case is a *binding operator* (like "let"): x is bound in e_1 and y is bound in e_2 (like OCaml's "match … with")

Examples with Sum Types

- Consider the type unit with a single element called * or ()
- The type integer option defined as "unit + int"
  - Useful for optional arguments or return values
    - No argument: injl * (OCaml's "None")
    - Argument is 5: injr 5 (OCaml's "Some(5)")
  - To use the argument you must test the kind of argument
  - case arg of injl x ⇒ "no_arg_case" | injr y ⇒ "…y…"
  - injl and injr are tags and case is tag checking
- bool is the union type "unit + unit"
  - true is injl *
  - false is injr *
  - if e then e_1 else e_2 is case e of injl x ⇒ e_1 | injr y ⇒ e_2

Static Semantics of Sum Types

- New typing rules

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathbf{injl}\ e : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathbf{injr}\ e : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \quad \Gamma, x : \tau_1 \vdash e_l : \tau \quad \Gamma, y : \tau_2 \vdash e_r : \tau}{\Gamma \vdash \mathbf{case}\ e_1\ \mathbf{of}\ \mathbf{injl}\ x \Rightarrow e_l \mid \mathbf{injr}\ y \Rightarrow e_r : \tau}$$

- Types are *not unique* anymore
  - injl 1 : int + bool
  - injl 1 : int + (int → int)
  - this complicates type checking, but it is still doable

Dynamic Semantics of Sum Types

- New values v ::= … | injl v | injr v
- New evaluation rules

$$\frac{e \Downarrow v}{\mathbf{injl}\ e \Downarrow \mathbf{injl}\ v} \qquad \frac{e \Downarrow v}{\mathbf{injr}\ e \Downarrow \mathbf{injr}\ v}$$

$$\frac{e \Downarrow \mathbf{injl}\ v \quad [v/x]e_l \Downarrow v'}{\mathbf{case}\ e\ \mathbf{of}\ \mathbf{injl}\ x \Rightarrow e_l \mid \mathbf{injr}\ y \Rightarrow e_r \Downarrow v'}$$

$$\frac{e \Downarrow \mathbf{injr}\ v \quad [v/y]e_r \Downarrow v'}{\mathbf{case}\ e\ \mathbf{of}\ \mathbf{injl}\ x \Rightarrow e_l \mid \mathbf{injr}\ y \Rightarrow e_r \Downarrow v'}$$

Type Soundness for F_1^+

- Type soundness *still holds*
- No way to use a $\tau_1 + \tau_2$ inappropriately
- The key is that the only way to use a $\tau_1 + \tau_2$ is with case, which ensures that you are not using a τ_1 as a τ_2
- In C or Pascal checking the tag is the responsibility of the programmer!
  - Unsafe (yes, even Pascal!)

Types for Imperative Features

- So far: types for pure functional languages
- Now: types for imperative features
- Such types are used to characterize non-local effects
  - assignments
  - exceptions
  - typestate
- Contextual semantics is useful here
  - Just when you thought it was safe to forget it …

Reference Types

- Such types are used for mutable memory cells
- Syntax (as in ML)

Why do I need : τ ?

$e ::= \ldots \mid \text{ref } e : \tau \mid e_1 := e_2 \mid \, ! \, e$

$\tau ::= \ldots \mid \tau \text{ ref}$

  - ref e : τ - evaluates e, allocates a new memory cell, stores the value of e in it and returns the address of the memory cell
    - like malloc + initialization in C, or new in C++ and Java
  - $e_1 := e_2$, evaluates e_1 to a memory cell and updates its value with the value of e_2
  - ! e - evaluates e to a memory cell and returns its contents

Global Effects, Reference Cells

- A reference cell can escape the static scope where it was created

  (λf:int $\to$ int ref. !(f 5)) (λx:int. ref x : int)
- The value stored in a reference cell *must be visible from the entire program*
- The "result" of an expression must now include the changes to the heap that it makes (cf. IMP's opsem)
- To model reference cells we must extend the evaluation model

Modeling References

- A heap is a mapping from addresses to values

$$h ::= \cdot \mid h, a \leftarrow v : \tau$$

  - $a \in$ Addresses (Addresses $\neq \mathbb{Z}$?)
  - We tag the heap cells with their types
  - Types are useful only for static semantics. They are not needed for the evaluation $\Rightarrow$ are not a part of the implementation
- We call a program an expression with a heap

  p ::= heap h in e
  - The initial program is "heap $\cdot$ in e"
  - Heap addresses act as bound variables in the expression
  - This is a trick that allows easy *reuse of properties of local variables for heap addresses*
    - e.g., we can rename the address and its occurrences at will

Static Semantics of References

- Typing rules for expressions:

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash (\mathbf{ref}\ e : \tau) : \tau\ \mathbf{ref}} \qquad \frac{\Gamma \vdash e : \tau\ \mathbf{ref}}{\Gamma \vdash !e : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau\ \mathbf{ref} \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 := e_2 : \mathbf{unit}}$$

- and for programs

$$\frac{\Gamma \vdash v_i : \tau_i \ (i = 1 \,..\, n) \qquad \Gamma \vdash e : \tau}{\vdash \mathbf{heap}\ h\ \mathbf{in}\ e : \tau}$$

where $\Gamma = a_1 : \tau_1\ \mathbf{ref}, \ldots, a_n : \tau_n\ \mathbf{ref}$
and $h = a_1 \leftarrow v_1 : \tau_1, \ldots, a_n \leftarrow v_n : \tau_n$

Contextual Semantics for References

- Addresses are values: v ::= ... | a
- New contexts: H ::= ref H | H_1 := e_2 | a_1 := H_2 | ! H
- No new *local* reduction rules
- But some new *global* reduction rules
  - heap h in H[ref v : τ] $\rightarrow$ heap h, a $\leftarrow$ v : τ in H[a]
    - where a is fresh (this models allocation - the heap is extended)
  - heap h in H[! a] $\rightarrow$ heap h in H[v]
    - where a $\leftarrow$ v : $\tau \in$ h (heap lookup - can we get stuck?)
  - heap h in H[a := v] $\rightarrow$ heap h[a $\leftarrow$ v] in H[*]
    - where h[a $\leftarrow$ v] means a heap like h except that the part "a $\leftarrow$ v_1 : τ" in h is replaced by "a $\leftarrow$ v : τ" (memory update)
- Global rules are used to propagate the effects of a write to the entire program (eval order matters!)

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Example with References

- Consider these (the redex is underlined)
  - heap · in (λf:int $\rightarrow$ int ref. !(f 5)) (λx:int. ref x : int)
  - heap · in !((λx:int. ref x : int) 5)
  - heap · in !(ref 5 : int)
  - heap a = 5 : int in !a
  - heap a = 5 : int in 5
- The resulting program has a useless memory cell
- An equivalent result would be

  heap · in 5
- This is a simple way to model garbage collection

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Homework

- Read Wright and Felleisen article
  - … that you didn't read on Tuesday.
  - Or that optional Goodenough one …
- Soon: Bonus Lecture #2 Scheduling
- Work on your projects!
  - Status Update Due Tue Oct 24

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