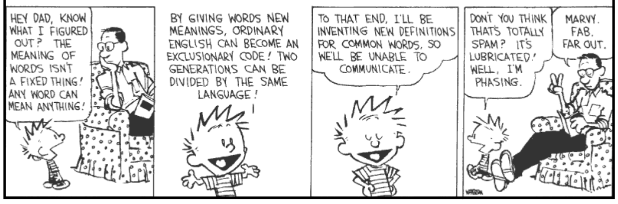
Abstract Interpretation

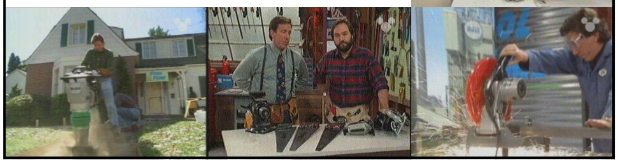
(Galois, Collections, Widening)



Tool Time

- How's Homework 5 going?
- Get started early
- Compilation problems?
  - See FAQ

  (trivia: what tool brand is this?)



More Power!

- You can handle it!

Review

- We introduced abstract interpretation
- An abstraction mapping from concrete to abstract values
  - Has a concretization mapping which forms a Galois connection
- We'll look a bit more at Galois connections
- We'll lift AI from expressions to programs
- … and we'll discuss the mythic "widening"

Why Galois Connections?

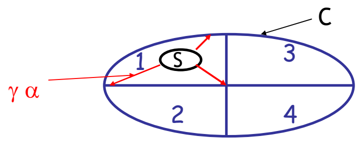
- We have an abstract domain A
  - An abstraction function $\beta : \mathbb{Z} \to A$
  - Induces $\alpha : \mathcal{P}(\mathbb{Z}) \to A$ and $\gamma : A \to \mathcal{P}(\mathbb{Z})$
- We argued that for correctness

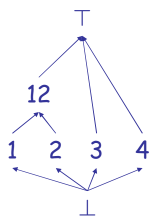
  $$\gamma(a_1 \underline{\text{op}}\ a_2) \supseteq \gamma(a_1) \text{ op } \gamma(a_2)$$

  - We wish for the set on the left to be as small as possible
  - To reduce the loss of information through abstraction
- For each set $S \subseteq C$, define $\alpha(S)$ as follows:
  - Pick smallest S' that includes S and is in the image of γ
  - Define $\alpha(S) = \gamma^{-1}(S')$
  - Then we define: $a_1 \underline{\text{op}}\ a_2 = \alpha(\gamma(a_1) \text{ op } \gamma(a_2))$
- Then α and γ form a Galois connection

Galois Connections

- A Galois connection between complete lattices A and $\mathcal{P}(C)$ is a pair of functions α and γ such that:
  - γ and α are monotonic (with the $\subseteq$ ordering on $\mathcal{P}(C)$)
  - $\alpha(\gamma(a)) = a$ for all $a \in A$
  - $\gamma(\alpha(S)) \supseteq S$ for all $S \in \mathcal{P}(C)$





More on Galois Connections

- All Galois connections are monotonic
- In a Galois connection one function uniquely and absolutely determines the other

Abstract Interpretation for Imperative Programs

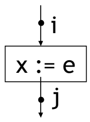
- So far we abstracted the value of expressions
- Now we want to abstract the state at each point in the program
- First we define the concrete semantics that we are abstracting
  - We'll use a collecting semantics

Collecting Semantics

- Recall
  - A state $\sigma \in \Sigma$. Any state σ has type Var $\to \mathbb{Z}$
  - States vary from program point to program point
- We introduce a set of program points: labels
- We want to answer questions like:
  - Is x always positive at label i?
  - Is x always greater or equal to y at label j?
- To answer these questions we'll construct
  $C \in$ Contexts. C has type Labels $\to \mathcal{P}(\Sigma)$
  - For each label i, C(i) = all possible states at label i
  - This is called the collecting semantics of the program
  - This is basically what SLAM (and BLAST, ESP, …) approximate (using BDDs to store $\mathcal{P}(\Sigma)$ efficiently)

Defining the Collecting Semantics

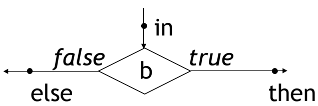
- We first define relations between the collecting semantics at different labels
  - We do it for unstructured CFGs (cf. HW5!)
  - Can do it for IMP with careful notion of program points
- Define a label on each edge in the CFG
- For assignment



$$C_j = \{\sigma[x := n] \mid \sigma \in C_i \wedge [\![e]\!]\sigma = n\}$$

Defining the Collecting Semantics

- For conditionals

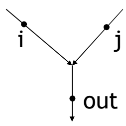


$$C_{else} = \{ \sigma \mid \sigma \in C_{in} \wedge [\![b]\!]\sigma = false\}$$
$$C_{then} = \{ \sigma \mid \sigma \in C_{in} \wedge [\![b]\!]\sigma = true\}$$

- Assumes b has no side effects (as in IMP or HW5)

Defining the Collecting Semantics

- For a join

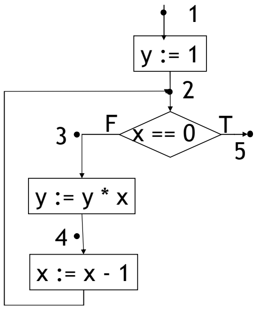


$$C_{out} = C_i \cup C_j$$

- Verify that these relations are monotonic
  - If we increase a C_x all other C_y can only increase

Collecting Semantics: Example

- Assume $x \geq 0$ initially *(explain this?)*



$C_1 = \{\sigma \mid \sigma(x) \geq 0\}$

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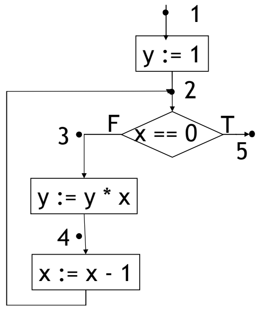

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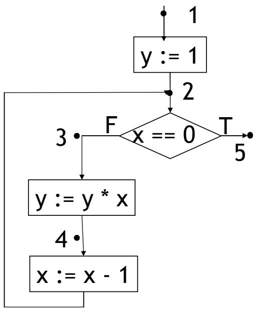

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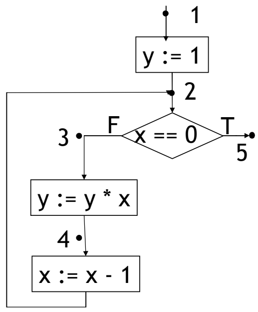

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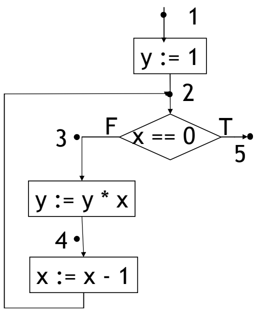

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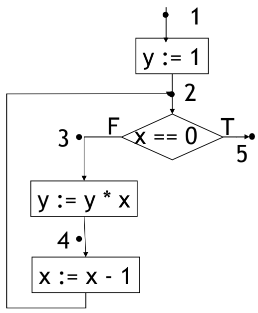

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Why Does This Work?

- We just made a system of recursive equations that are defined largely in terms of themselves
  - e.g., $C_2 = F(C_4)$, $C_4 = G(C_3)$, $C_3 = H(C_2)$
- Why do we have any reason to believe that this will get us what we want?

The Collecting Semantics

- We have an equation with the unknown C
  - The equation is defined by a monotonic and continuous function on domain Labels $\rightarrow \mathcal{P}(\Sigma)$
- We can use the least fixed-point theorem
  - Start with $C^0(L)=\emptyset$ (aka $C^0 = \lambda L.\emptyset$)
  - Apply the relations between C_i and C_j to get C^1_i from C^0_j
  - Stop when all $C^k = C^{k-1}$
  - Problem: we'll go on forever for most programs
  - But we know the fixed point exists

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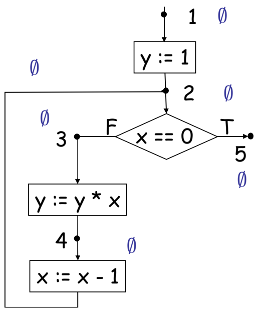
Collecting Semantics: Example

- (assume $x \geq 0$ initially)



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#21

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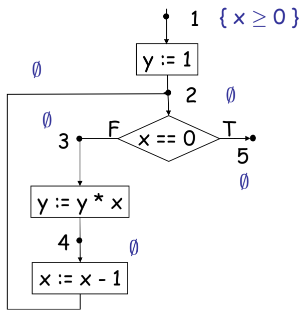

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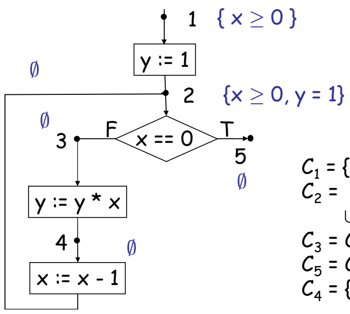

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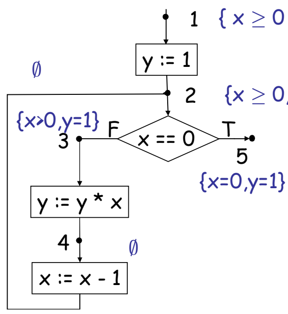

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#24

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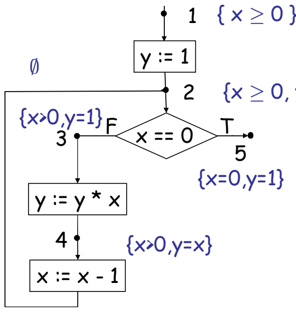

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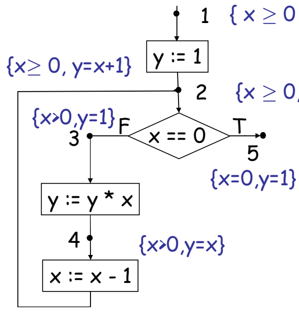

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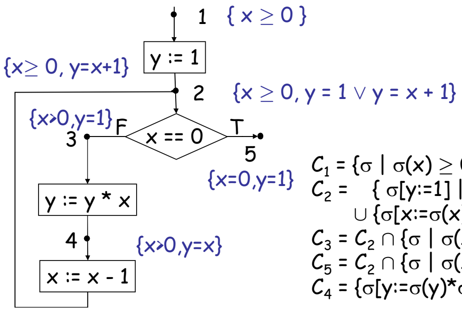

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Abstract Interpretation

- Pick a complete lattice A (abstractions for $\mathcal{P}(\Sigma)$)
  - Along with a monotonic abstraction $\alpha : \mathcal{P}(\Sigma) \to A$
  - Alternatively, pick $\beta : \Sigma \to A$
  - This uniquely defines its Galois connection γ
- Take the relations between C_i and move them to the abstract domain:

$$a : \text{Label} \to A$$

- Assignment

  Concrete: $C_j = \{\sigma[x := n] \mid \sigma \in C_i \land [\![e]\!]\sigma = n\}$

  Abstract: $a_j = \alpha\, \{\sigma[x := n] \mid \sigma \in \gamma(a_i) \land [\![e]\!]\sigma = n\}$

Abstract Interpretation

- Conditional

  Concrete: $C_j = \{ \sigma \mid \sigma \in C_i \land [\![b]\!]\sigma = \text{false}\}$ and

  $C_k = \{ \sigma \mid \sigma \in C_i \land [\![b]\!]\sigma = \text{true}\}$

  Abstract: $a_j = \alpha\, \{ \sigma \mid \sigma \in \gamma(a_i) \land [\![b]\!]\sigma = \text{false}\}$ and

  $a_k = \alpha\, \{ \sigma \mid \sigma \in \gamma(a_i) \land [\![b]\!]\sigma = \text{true}\}$
- Join

  Concrete: $C_k = C_i \cup C_j$

  Abstract: $a_k = \alpha\, (\gamma(a_i) \cup \gamma(a_j)) = \text{lub}\ \{a_i,\ a_j\}$

Least Fixed Points In The Abstract Domain

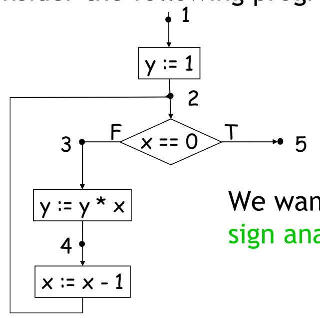
- We have a recursive equation with unknown "a"
  - Defined by a monotonic and continuous function on the domain Labels $\to$ A
- We can use the least fixed-point theorem:
  - Start with $a^0 = \lambda L.\bot$ (aka: $a^0(L) = \bot$)
  - Apply the monotonic function to compute a^{k+1} from a^k
  - Stop when $a^{k+1} = a^k$
- Exactly the same computation as for the collecting semantics
  - What is new?
  - *"There is nothing new under the sun but there are lots of old things we don't know."* - Ambrose Bierce

Least Fixed Points In The Abstract Domain

- We have a hope of termination!
- Classic setup: A has only uninteresting chains (finite number of elements in each chain)
  - A has finite height h (= "finite-height lattice")
- The computation takes O(h × |Labels|²) steps
  - At each step "a" makes progress on at least one label
  - We can only make progress h times
  - And each time we must compute |Labels| elements
- This is a quadratic analysis: good news
  - This is exactly the same as Kildall's 1973 analysis of dataflow's polynomial termination given a finite-height lattice and monotonic transfer functions.

Abstract Interpretation: Example

- Consider the following program, x>0



We want to do the sign analysis on it.

Abstract Domain for Sign Analysis

- Invent the complete sign lattice

  S = { ⊥, -, 0, +, ⊤ }

- Construct the complete lattice

  A = {x, y} → S

  - With the usual point-wise ordering
  - Abstract state gives the sign for x and y
- We start with $a^0 = \lambda L.\lambda v{\in}\{x,y\}.\bot$
  - aka: $a^0(L,v) = \bot$

Let's Do It!

Label		Iterations →									
1	x	+									+
	y	⊤									⊤
2	x	⊥	+			⊤					⊤
	y	⊥	+						⊤		⊤
3	x	⊥		+			⊤				⊤
	y	⊥		+						⊤	⊤
4	x	⊥			+			⊤			⊤
	y	⊥			+			⊤			⊤
5	x	⊥					0				0
	y	⊥					+			⊤	⊤

Notes, Weaknesses, Solutions

- We abstracted the state of each variable independently

  A = {x, y } → {⊥, -, 0, +, ⊤ }

- We lost relationships between variables
  - E.g., at a point x and y may always have the same sign
  - In the previous abstraction we get {x := ⊤, y := ⊤} at label 2 (when in fact y is always positive!)
- We can also abstract the state as a whole

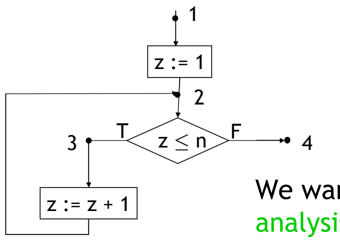
  A = 𝒫({⊥, -, 0, +, ⊤} × {⊥, -, 0, +, ⊤})

Other Abstract Domains

- Range analysis
  - Lattice of ranges: R ={ ⊥, [n..m], (-∞, m], [n, +∞), ⊤ }
  - It is a complete lattice
    - [n..m] ⊔ [n'..m'] = [min(n, n')..max(m,m')]
    - [n..m] ⊓ [n'..m'] = [max(n, n')..min(m, m')]
    - With appropriate care in dealing with ∞
  - $\beta : \mathbb{Z} \to R$ such that $\beta(n) = [n..n]$
  - $\alpha : \mathcal{P}(\mathbb{Z}) \to R$ such that $\alpha(S) = \mathrm{lub}\ \{\beta(n) \mid n \in S\}$ = [min(S)..max(S)]
  - $\gamma : R \to \mathcal{P}(Z)$ such that $\gamma(r) = \{ n \mid n \in r \}$
- This lattice has infinite-height chains
  - So the abstract interpretation might not terminate!

Example of Non-Termination

- Consider this (common) program fragment



We want to do range analysis on it.

Example of Non-Termination

- Consider the sequence of abstract states at point 2
  - [1..1], [1..2], [1..3], …
  - The analysis never terminates
  - Or terminates very late if the loop bound is known statically
- It is time to approximate even more: widening
- We redefine the join (lub) operator of the lattice to ensure that from [1..1] upon union with [2..2] the result is [1..+∞) and not [1..2]
- Now the sequence of states is
  - [1..1], [1, +∞), [1, +∞) Done (no more infinite chains)

Formal Definition of Widening

(Cousot 16.399 "Abstract Interpretation", 2005)

- A widening $\triangledown$: $(P \times P) \rightarrow P$ on a poset $\langle P, \sqsubseteq \rangle$ satisfies:
  - $\forall x, y \in P . \quad x \sqsubseteq (x \triangledown y) \quad \wedge \quad y \sqsubseteq (x \triangledown y)$
  - For all increasing chains $x^0 \sqsubseteq x^1 \sqsubseteq \ldots$ the increasing chain $y^0 =^{def} x^0, \ldots, y^{n+1} =^{def} y^n \triangledown x^{n+1}, \ldots$ is not strictly increasing.
- Two different main uses:
  - Approximate missing lubs. *(Not for us.)*
  - Convergence acceleration. *(This is the real use.)*
    - A widening operator can be used to effectively compute an upper approximation of the least fixpoint of $F \in L \mapsto L$ starting from below when L is computer-representable but does not satisfy the ascending chain condition.

Formal Widening Example

$[1,1] \triangledown [1,2] = [1,+\infty)$

- Range Analysis on z:

L0: z := 1 ;
L1: while z<99 do
L2: z := z+1
L3: done /* z ≥ 99 */
L4:

Original x^i	Widened y^i
$x^{L0}_0 = \bot$	$y^{L0}_0 = \bot$
$x^{L1}_0 = [1,1]$	$y^{L1}_0 = [1,1]$
$x^{L2}_0 = [1,1]$	$y^{L2}_0 = [1,1]$
$x^{L3}_0 = [2,2]$	$y^{L3}_0 = [2,2]$
$x^{L2}_1 = [1,2]$	$y^{L2}_1 = [1,+\infty)$
$x^{L3}_1 = [2,+\infty)$	$y^{L3}_1 = [2,+\infty)$
$x^{L4}_0 = [99,+\infty)$	$y^{L4}_0 = [99,+\infty)$
stable (fewer than 99 iterations!)	

$x^{Li}_j =^{def}$ the jth iterative attempt to compute an abstract value for z at label Li

Recall lub S = [min(S)..max(S)]
lub {[2,+∞),[1,+∞)} = {[1,+∞)}

Other Abstract Domains

- Linear relationships between variables
  - A convex polyhedron is a subset of $\mathbb{Z}^k$ whose elements satisfy a number of inequalities:

$$a_1x_1 + a_2x_2 + \ldots + a_kx_k \geq c_i$$

  - This is a complete lattice; linear programming methods compute lubs
- Linear relationships with at most two variables
  - Convex polyhedra but with ≤ 2 variables per constraint
  - Octagons ($x \pm y \geq c$) have efficient algorithms
- Modulus constraints (e.g. even and odd)

Abstract Chatter

- AI, Dataflow and Software Model Checking
  - The big three (aside from flow-insensitive type systems) for program analyses
- Are in fact quite related:
  - David Schmidt. *Data flow analysis is model checking of abstract interpretation*. POPL '98.
- AI is usually flow-sensitive (per-label answer)
- AI can be path-sensitive (if your abstract domain includes ∨, for example), which is just where model checking uses BDD's
- Metal, SLAM, ESP, … can all be viewed as AI

Abstract Interpretation Conclusions

- AI is a very powerful technique that underlies a large number of program analyses
- AI can also be applied to functional and logic programming languages
- There are a few success stories
  - Strictness analysis for lazy functional languages
  - PolySpace for linear constraints
- In most other cases however AI is still slow
- When the lattices have infinite height and widening heuristics are used the result becomes unpredictable

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Homework

- Project Proposal Due Today
- Read Pierce Article, pages 1-10 only
- Homework 5 Due Later

- (Assuming I finished with a few minutes left) You may stay as we run through some on-the-board examples.

#44

On The Board Questions

- What is the VC for:

$$\text{for } i = e_{low} \text{ to } e_{high} \text{ do } c \text{ done}$$

- This axiomatic rule is unsound. Why?

$$\frac{\vdash \{A \land p\}\ c_{then}\ \{B_{then}\} \qquad \vdash \{A \land \neg p\}\ c_{else}\ \{B_{else}\}}{\vdash \{A\}\ \text{if } p \text{ then } c_{then} \text{ else } c_{else}\ \{B_{then} \lor B_{else}\}}$$

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