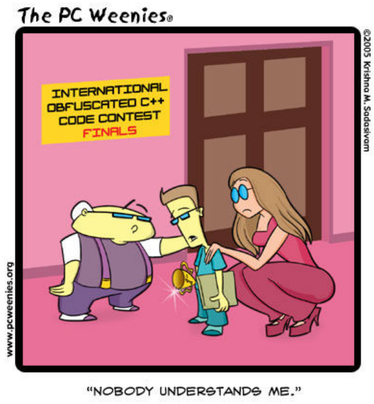


"The Real Deal" Axiomatic Semantics

#1

Soundness of Axiomatic Semantics

- Formal statement of soundness:
  If $\vdash \{ A \}\ c\ \{ B \}$ then $\models \{ A \}\ c\ \{ B \}$

  or, equivalently

  For all σ, if $\sigma \models A$
  and Op :: $<c, \sigma> \Downarrow \sigma'$
  and Pr :: $\vdash \{ A \}\ c\ \{ B \}$
  then $\sigma' \models B$
- "Op" = "Opsem Derivation"
- "Pr" = "Axiomatic Proof"

#2

Simultaneous Induction

- Consider two structures Op and Pr
  - Assume that $x < y$ iff x is a substructure of y
- Define the ordering
  $(o, p) \prec (o', p')$ iff
  $o < o'$ or $o = o'$ and $p < p'$
  - Called lexicographic (dictionary) ordering
- This $\prec$ is a well founded order and leads to simultaneous induction
- If $o < o'$ then p can actually be larger than p'!
- It can even be unrelated to p'!

#3

Soundness of the While Rule

(Indiana Proof and the Slide of Doom)

- Case: last rule used in Pr : $\vdash \{A\}\ c\ \{B\}$ was the while rule:

$$\frac{Pr_1 :: \vdash \{A \wedge b\}\ c\ \{A\}}{\vdash \{A\}\ \text{while } b \text{ do } c\ \{A \wedge \neg b\}}$$

- Two possible rules for the root of Op (*by inversion*)
  - We'll only do the complicated case:

$$\frac{Op_1 :: <b, \sigma> \Downarrow \text{true} \qquad Op_2 :: <c,\sigma> \Downarrow \sigma' \qquad Op_3 :: <\text{while } b \text{ do } c, \sigma' > \Downarrow \sigma''}{<\text{while } b \text{ do } c, \sigma > \Downarrow \sigma''}$$

Assume that $\sigma \models A$
To show that $\sigma'' \models A \wedge \neg b$

- By soundness of booleans and Op_1 we get $\sigma \models b$
  - Hence $\sigma \models A \wedge b$
- By IH on Pr_1 and Op_2 we get $\sigma' \models A$
- By IH on Pr and Op_3 we get $\sigma'' \models A \wedge \neg b$, q.e.d.
  - This is the tricky bit!

#4

Soundness of the While Rule

- Note that in the last use of IH the derivation Pr *did not decrease*
- But Op_3 was a sub-derivation of Op
- See Winskel, Chapter 6.5, for a soundness proof with denotational semantics

#5

Completeness of Axiomatic Semantics

- If $\models \{A\}\ c\ \{B\}$ can we always derive $\vdash \{A\}\ c\ \{B\}$?
- If so, axiomatic semantics is complete
- If not then there are valid properties of programs that we cannot verify with Hoare rules :-(
- Good news: for our language the Hoare triples are complete
- Bad news: only if the underlying logic is complete
  (whenever $\models A$ we also have $\vdash A$)
  - this is called relative completeness

#6

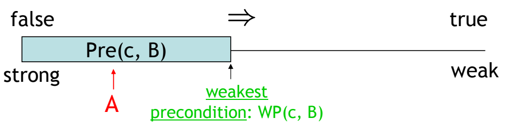
Examples, General Plan

- OK, so:
  $\models \{\ x < 5 \wedge z = 2\ \}\ y := x + 2\ \{\ y < 7\ \}$
- Can we prove it?
  $?\vdash?\ \{\ x < 5 \wedge z = 2\ \}\ y := x + 2\ \{\ y < 7\ \}$
- Well, we *could* easily prove:
  $\vdash \{\ x{+}2 < 7\ \}\ y := x + 2\ \{\ y < 7\ \}$
- And we know …
  $\vdash x < 5 \wedge z = 2 \Rightarrow x{+}2 < 7$
- Shouldn't those two proofs be enough?

Proof Idea

- Dijkstra's idea: To verify that $\{\ A\ \}\ c\ \{\ B\ \}$
  a) Find out all predicates A' such that $\models \{\ A'\ \}\ c\ \{\ B\ \}$
  - call this set Pre(c, B) (Pre = "pre-conditions")

  b) Verify for one $A' \in$ Pre(c, B) that $A \Rightarrow A'$
- Assertions can be ordered:



- Thus: compute WP(c, B) and prove $A \Rightarrow$ WP(c, B)

Proof Idea (Cont.)

- Completeness of axiomatic semantics:
  If $\models \{\ A\ \}\ c\ \{\ B\ \}$ then $\vdash \{\ A\ \}\ c\ \{\ B\ \}$
- Assuming that we can compute wp(c, B) with the following properties:
  1. wp is a precondition (according to the Hoare rules)
     $\vdash \{\ wp(c, B)\ \}\ c\ \{\ B\ \}$
  2. wp is *(truly)* the weakest precondition
     If $\models \{\ A\ \}\ c\ \{\ B\ \}$ then $\models A \Rightarrow wp(c, B)$

$$\frac{\vdash A \Rightarrow wp(c, B) \qquad \vdash \{wp(c, B)\}\ c\ \{B\}}{\vdash \{A\}\ c\ \{B\}}$$

- We also need that whenever $\models A$ then $\vdash A$!

Weakest Preconditions

- Define wp(c, B) inductively on c, following the Hoare rules:
- $wp(c_1;\ c_2, B) = wp(c_1, wp(c_2, B))$

$$\frac{\{A\}\ c_1\ \{C\} \qquad \{C\}\ c_2\ \{B\}}{\{\ A\ \}\ c_1;\ c_2\ \{B\}}$$

- $wp(x := e, B) = [e/x]B$

$$\frac{}{\{\ [e/x]B\ \}\ x := E\ \{B\}}$$

$$\frac{\{A_1\}\ c_1\ \{B\} \qquad \{A_2\}\ c_2\ \{B\}}{\{\ E \Rightarrow A_1 \wedge \neg\, E \Rightarrow A_2\}\ \text{if } E \text{ then } c_1 \text{ else } c_2\ \{B\}}$$

- $wp(\text{if } E \text{ then } c_1 \text{ else } c_2, B) = E \Rightarrow wp(c_1, B) \wedge \neg E \Rightarrow wp(c_2, B)$

Weakest Preconditions for Loops

- We start from the unwinding equivalence
  while b do c =
  if b then c; while b do c else skip
- Let w = while b do c and W = wp(w, B)
- We have that
  $W = b \Rightarrow wp(c, W) \quad \wedge \quad \neg\, b \Rightarrow B$
- But this is a recursive equation!
  - We know how to solve these using domain theory
- But we need a domain for assertions

A Partial Order for Assertions

- Which assertion contains the least information?
  - "true" – does not say anything about the state
- What is an appropriate information ordering ?
  $A \sqsubseteq A'$ iff $\models A' \Rightarrow A$
- Is this partial order complete?
  - Take a chain $A_1 \sqsubseteq A_2 \sqsubseteq \ldots$
  - Let $\wedge A_i$ be the infinite conjunction of A_i
    $\sigma \models \wedge A_i$ iff for all i we have that $\sigma \models A_i$
  - I assert that $\wedge A_i$ is the least upper bound
- Can $\wedge A_i$ be expressed in our language of assertions?
  - In many cases: yes (see Winskel), we'll assume yes for now

Weakest Precondition for WHILE

- Use the fixed-point theorem

$$F(A) = b \Rightarrow wp(c, A) \wedge \neg b \Rightarrow B$$

  - (Where did this come from? Two slides back!)
  - I assert that F is both monotonic and continuous

- The least-fixed point (= the weakest fixed point) is

$$wp(w, B) = \wedge F^i(true)$$

- Notice that unlike for denotational semantics of IMP we are not working on a flat domain!

#13

Weakest Preconditions (Cont.)

- Define a family of wp's
  - wp_k(while e do c, B) = weakest precondition on which the loop terminates in B <u>if</u> it terminates in k or fewer iterations

  $wp_0 = \neg E \Rightarrow B$

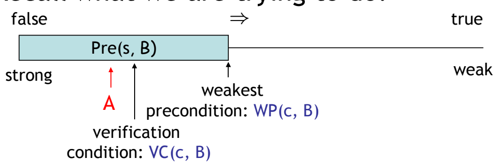
  $wp_1 = E \Rightarrow wp(c, wp_0) \wedge \neg E \Rightarrow B$

  …
- wp(while e do c, B) = $\wedge_{k \geq 0} wp_k$ = lub $\{wp_k \mid k \geq 0\}$
- See Necula document on the web page for the proof of completeness with weakest preconditions
- Weakest preconditions are
  - Impossible to compute (in general)
  - Can we find something easier to compute yet sufficient?

#14

Not Quite Weakest Preconditions

- Recall what we are trying to do:



- Construct a <u>verification condition</u>: VC(c, B)
  - Our loops will be annotated with loop invariants!
  - VC is guaranteed to be stronger than WP
  - But still weaker than A: $A \Rightarrow VC(c, B) \Rightarrow WP(c, B)$

#15

Groundwork

- Factor out the hard work
  - Loop invariants
  - Function specifications (pre- and post-conditions)
- Assume programs are annotated with such specs
  - Good software engineering practice anyway
  - Requiring annotations = Kiss of Death?
- New form of while that includes a <u>loop invariant</u>:

$$while_{Inv}\ b\ do\ c$$

  - Invariant formula Inv must hold every time before b is evaluated
- A process for computing VC(annotated_command, post_condition) is called <u>VCGen</u>

#16

Verification Condition Generation

- Mostly follows the definition of the wp function:

VC(skip, B) = B

$VC(c_1; c_2, B) = VC(c_1, VC(c_2, B))$

$VC(\text{if } b \text{ then } c_1 \text{ else } c_2, B) =$

$b \Rightarrow VC(c_1, B) \wedge \neg b \Rightarrow VC(c_2, B)$

VC(x := e, B) = [e/x] B

VC(let x = e in c, B) = [e/x] VC(c, B)

$VC(while_{Inv}\ b\ do\ c, B)$ = ?

#17

VCGen for WHILE

$VC(while_{Inv}\ e\ do\ c, B) =$

$$\underbrace{Inv}_{\text{Inv holds on entry}} \wedge (\forall x_1 \ldots x_n.\ Inv \Rightarrow \underbrace{(e \Rightarrow VC(c, Inv))}_{\text{Inv is preserved in an arbitrary iteration}} \wedge \underbrace{\neg e \Rightarrow B)}_{\text{B holds when the loop terminates in an arbitrary iteration}})$$

- Inv is the loop invariant (provided externally)
- $x_1, \ldots, x_n$ are all the variables modified in c
- The $\forall$ is similar to the $\forall$ in mathematical induction:

$$P(0) \wedge \forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$$

#18

Example VCGen Problem

- Let's compute the VC of this program with respect to post-condition x ≠ 0

```
x = 0;
y = 2;
while_{x+y=2} y > 0 do
    y := y - 1;
    x := x + 1
```

First, what do we expect? What pre-condition do we need to ensure x≠0 after this?

Example of VC

- By the sequencing rule, first we do the while loop (call it w):
  - $\text{while}_{x+y=2}$ y > 0 do
    - y := y - 1;
    - x := x + 1
- VCGen(w, x ≠ 0) = x+y=2 ∧
  - ∀x,y. x+y=2 ⇒ (y>0 ⇒ VC(c, x+y=2) ∧ y≤0 ⇒ x ≠ 0)
- VCGen(y:=y-1 ; x:=x+1, x+y=2) =
  - (x+1) + (y-1) = 2
- w Result: x+y=2 ∧
  - ∀x,y. x+y=2 ⇒ (y>0 ⇒ (x+1)+(y-1)=2 ∧ y≤0 ⇒ x ≠ 0)

Preserve loop invariant

Ensure post on exit

Example of VC (2)

- VC(w, x ≠ 0) = x+y=2 ∧
  - ∀x,y. x+y=2 ⇒
    - (y>0 ⇒ (x+1)+(y-1)=2 ∧ y≤0 ⇒ x ≠ 0)
- VC(x := 0; y := 2 ; w, x ≠ 0) = 0+2=2 ∧
  - ∀x,y. x+y=2 ⇒
    - (y>0 ⇒ (x+1)+(y-1)=2 ∧ y≤0 ⇒ x ≠ 0)
- So now we ask an automated theorem prover to prove it.

Thoreau, Thoreau, Thoreau

```
$ ./Simplify
> (AND (EQ (+ 0 2) 2)
  (FORALL ( x y ) (IMPLIES (EQ (+ x y) 2)
     (AND (IMPLIES (> y 0)
                   (EQ (+ (+ x 1)(- y 1)) 2))
          (IMPLIES (<= y 0) (NEQ x 0))))))
1: Valid.
```

- Huzzah!
- Simplify is a non-trivial five megabytes

Can We Mess Up VCGen?

- The invariant is from the user (= the adversary, the untrusted code base)
- Let's use a loop invariant that is too weak, like "true".
- VC = true ∧ ∀x,y. true ⇒
  - (y>0 ⇒ true ∧ y≤0 ⇒ x ≠ 0)
- Let's use a loop invariant that is false, like "x ≠ 0".
- VC = 0 ≠ 0 ∧ ∀x,y. x ≠ 0 ⇒
  - (y>0 ⇒ x+1 ≠ 0 ∧ y≤0 ⇒ x ≠ 0)

Emerson, Emerson, Emerson

```
$ ./Simplify
> (AND TRUE
  (FORALL ( x y ) (IMPLIES TRUE
    (AND (IMPLIES (> y 0) TRUE)
         (IMPLIES (<= y 0) (NEQ x 0))))))
Counterexample: context:
    (AND
      (EQ x 0)
      (<= y 0)
    )
1: Invalid.
```

- OK, so we won't be fooled.

Soundness of VCGen

- Simple form

  $\models$ { VC(c,B) } c { B }

- Or equivalently that

  $\models$ VC(c, B) $\Rightarrow$ wp(c, B)

- Proof is by induction on the structure of c
  - Try it!
- Soundness holds for any choice of invariant!
- Next: properties and extensions of VCs

#25

VC and Invariants

- Consider the Hoare triple:

  $\{x \leq 0\}$ $\text{while}_{I(x)}$ $x \leq 5$ do $x := x + 1$ $\{x = 6\}$

- The VC for this is:

  $x \leq 0 \Rightarrow I(x) \;\wedge\; \forall x.\ (I(x) \Rightarrow (x > 5 \Rightarrow x = 6\ \wedge$
  $x \leq 5 \Rightarrow I(x+1)\ ))$

- Requirements on the invariant:
  - Holds on entry $\forall x.\ x \leq 0 \Rightarrow I(x)$
  - Preserved by the body $\forall x.\ I(x) \wedge x \leq 5 \Rightarrow I(x+1)$
  - Useful $\forall x.\ I(x) \wedge x > 5 \Rightarrow x = 6$
- Check that $I(x) = x \leq 6$ satisfies all constraints

#26

Forward VCGen

- Traditionally the VC is computed backwards
  - That's how we've been doing it in class
  - It works well for structured code
- But it can also be computed forward
  - Works even for un-structured languages (e.g., assembly language)
  - Uses symbolic execution, a technique that has broad applications in program analysis
    - e.g., the PREfix tool (Intrinsa, Microsoft) does this

#27

Forward VC Gen Intuition

- Consider the sequence of assignments

  $x_1 := e_1;\ x_2 := e_2$

- The $VC(c, B) = [e_1/x_1]([e_2/x_2]B)$

  $= [e_1/x_1,\ e_2[e_1/x_1]/x_2]\ B$

- We can compute the substitution in a forward way using symbolic execution (aka symbolic evaluation)
  - Keep a symbolic state that maps variables to expressions
  - Initially, $\Sigma_0 = \{\ \}$
  - After $x_1 := e_1$, $\Sigma_1 = \{ x_1 \to e_1 \}$
  - After $x_2 := e_2$, $\Sigma_2 = \{x_1 \to e_1,\ x_2 \to e_2[e_1/x_1] \}$
  - Note that we have applied Σ_1 as a substitution to right-hand side of assignment $x_2 := e_2$

#28

Simple Assembly Language

- Consider the language of instructions:

  I ::= x := e | f() | if e goto L | goto L |
  L: | return | inv e

- The "inv e" instruction is an annotation
  - Says that boolean expression e holds at that point
- Each function f() comes with Pre_f and Post_f annotations (pre- and post-conditions)
- New Notation (yay!): I_k is the instruction at address k

#29

Symex States

- We set up a symbolic execution state:

Σ : Var $\to$ SymbolicExpressions

$\Sigma(x)$ = the symbolic value of x in state Σ

$\Sigma[x{:=}e]$ = a new state in which x's value is e

- We use states as substitutions:

$\Sigma(e)$ - obtained from e by replacing x with $\Sigma(x)$

- Much like the opsem so far …

#30

Symex Invariants

- The symbolic executor tracks invariants passed
- A new part of symex state: $Inv \subseteq \{1 \ldots n\}$
- If $k \in Inv$ then I_k is an invariant instruction that we have already executed
- Basic idea: execute an inv instruction only twice:
  - The first time it is encountered
  - Once more time around an arbitrary iteration

#31

Symex Rules

- Define a VC function as an interpreter:

$VC : Address \times SymbolicState \times InvariantState \rightarrow Assertion$

$VC(k, \Sigma, Inv) =$

$VC(L, \Sigma, Inv)$	if I_k = goto L
$e \Rightarrow VC(L, \Sigma, Inv) \quad \wedge$ $\neg e \Rightarrow VC(k+1, \Sigma, Inv)$	if I_k = if e goto L
$VC(k+1, \Sigma[x:=\Sigma(e)], Inv)$	if I_k = x := e
$\Sigma(Post_{current\text{-}function})$	if I_k = return
$\Sigma(Pre_f) \quad \wedge$ $\forall a_1..a_m.\Sigma'(Post_f) \Rightarrow$ $VC(k+1, \Sigma', Inv)$ (where $y_1, \ldots, y_m$ are modified by f) and $a_1, \ldots, a_m$ are fresh parameters and $\Sigma' = \Sigma[y_1 := a_1, \ldots, y_m := a_m]$	if I_k = f()

#32

Symex Invariants (2a)

Two cases when seeing an invariant instruction:

1. We see the invariant for the first time
   - I_k = inv e
   - $k \notin Inv$ (= "not in the set of invariants we've seen")
   - Let $\{y_1, \ldots, y_m\}$ = the variables that could be modified on a path from the invariant back to itself
   - Let $a_1, \ldots, a_m$ be fresh new symbolic parameters

$VC(k, \Sigma, Inv) =$

$\Sigma(e) \wedge \forall a_1 \ldots a_m.\ \Sigma'(e) \Rightarrow VC(k+1, \Sigma', Inv \cup \{k\})$

with $\Sigma' = \Sigma[y_1 := a_1, \ldots, y_m := a_m]$

(like a function call)

#33

Symex Invariants (2b)

2. We see the invariant for the second time
   - I_k = inv E
   - $k \in Inv$

$VC(k, \Sigma, Inv) = \Sigma(e)$

(like a function return)

- Some tools take a more simplistic approach
  - Do not require invariants
  - Iterate through the loop a fixed number of times
  - PREfix, versions of ESC (DEC/Compaq/HP SRC)
  - Sacrifice completeness for usability

#34

Homework

- Homework 3 Due Today
- Homework 4 Out Today
- Read Winskel 7.4-7.6 (on VC's)
- Read Dijkstra article

- Bonus Lecture Shortly

#35

Old Questions Answered

- Denotational Semantics class question:
- "What's up with the continuity requirement?"
- A function $F : S^m \rightarrow S^n$ is continuous if for every chain $W \subset S^m$
  - $F(W)$ has a LUB $= \sqcup F(W)$
  - and $F(\sqcup W) = \sqcup F(W)$
- See the Ed Lee paper on the lectures page.

#36