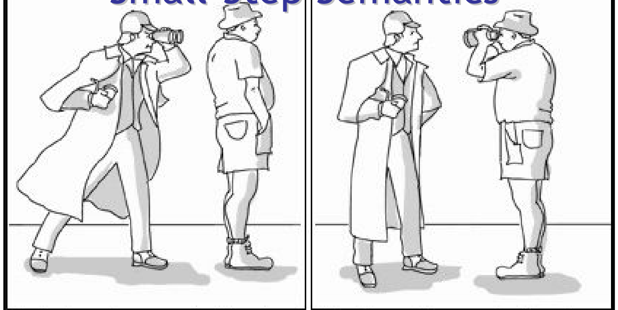
Operational Semantics
Small-Step Semantics



Sherlock saw the man using binoculars. Sherlock saw the man using binoculars.

Today's Cunning Plan

- Review, Truth, and Provability
- Large-Step Opsem Commentary
- Small-Step Contextual Semantics
  - Reductions, Redexes, and Contexts
- Applications
- (Induction)

Summary - Semantics

- A formal semantics is a system for assigning meanings to programs.
- For now, programs are IMP commands and expressions
- In operational semantics the meaning of a program is "what it evaluates to"
- Any opsem system gives rules of inference that tell you how to evaluate programs

Summary - Judgments

- Rules of inference allow you to derive judgments ("something that is knowable") like

$$\langle e, \sigma \rangle \Downarrow n$$

  - In state σ, expression e evaluates to n

$$\langle c, \sigma \rangle \Downarrow \sigma'$$

  - After evaluating command c in state σ the new state will be σ'
- State σ maps variables to values ($\sigma : L \to Z$)
- Inferences equivalent up to variable renaming:

$$\langle c, \sigma \rangle \Downarrow \sigma' \quad === \quad \langle c', \sigma_7 \rangle \Downarrow \sigma_8$$

Summary - Rules

- Rules of inference list the hypotheses necessary to arrive at a conclusion

$$\frac{}{\langle x, \sigma \rangle \Downarrow \sigma(x)} \qquad \frac{\langle e_1, \sigma \rangle \Downarrow n_1 \quad \langle e_2, \sigma \rangle \Downarrow n_2}{\langle e_1 - e_2, \sigma \rangle \Downarrow n_1 \text{ minus } n_2}$$

- A derivation involves interlocking (well-formed) instances of rules of inference

$$\frac{\dfrac{\langle 4, \sigma_3 \rangle \Downarrow 4 \quad \langle 2, \sigma_3 \rangle \Downarrow 2}{\langle 4*2, \sigma_3 \rangle \Downarrow 8} \qquad \langle 6, \sigma_3 \rangle \Downarrow 6}{\langle (4*2) - 6, \sigma_3 \rangle \Downarrow 2}$$

Provability

- Given an opsem system, $\langle e, \sigma \rangle \Downarrow n$ is provable *if there exists* a well-formed derivation with $\langle e, \sigma \rangle \Downarrow n$ as its conclusion
  - "well-formed" = "every step in the derivation is a valid instance of one of the rules of inference for this opsem system"
  - "$\vdash \langle e, \sigma \rangle \Downarrow n$" = "it is provable that $\langle e, \sigma \rangle \Downarrow n$"
- We would *like* truth and provability to be closely related

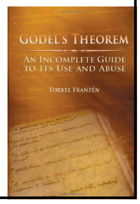
Truth?

- "A Vorlon said understanding is a three-edged sword. Your side, their side and the truth."
  - Sheridan, *Into The Fire*
- We will not formally define "truth" yet
- Instead we appeal to your intuition
  - $<2+2, \sigma> \Downarrow 4$ -- *should be* true
  - $<2+2, \sigma> \Downarrow 5$ -- *should be* false

Completeness

- A proof system (like our operational semantics) is complete if every true judgment is provable.
- If we ***replaced*** the subtract rule with:

$$\frac{<e_1, \sigma> \Downarrow n \qquad <e_2, \sigma> \Downarrow 0}{<e_1 - e_2, \sigma> \Downarrow n}$$



- Our opsem would be incomplete:
  $<4\text{-}2, \sigma> \Downarrow 2$ -- true but not provable

Consistency

- A proof system is consistent (or sound) if every provable judgment is true.
- If we ***replaced*** the subtract rule with:

$$\frac{<e_1, \sigma> \Downarrow n_1 \qquad <e_2, \sigma> \Downarrow n_2}{<e_1 - e_2, \sigma> \Downarrow n_1 + 3}$$

- Our opsem would be inconsistent (or unsound):
  - $<6\text{-}1, \sigma> \Downarrow 9$ -- false but provable

"A foolish consistency is the hobgoblin of little minds, adored by little statesmen and philosophers and divines." -- Ralph Waldo Emerson, *Essays. First Series. Self-Reliance.*

Desired Traits

- Typically a system (of operational semantics) is always complete (unless you forget a rule)
- If you are not careful, however, your system may be unsound
- Usually that is *very bad*
  - A paper with an unsound type system is usually rejected
  - Papers often prove (sketch) that a system is sound
  - Recent research (e.g., Engler, ESP) into useful but unsound systems exists, however
- In this class your work should be complete and consistent (e.g., on homework problems)

Dr. Peter Venkman: I'm a little fuzzy on the whole "good/bad" thing here. What do you mean, "bad"?
Dr. Egon Spengler: Try to imagine all life as you know it stopping instantaneously and every molecule in your body exploding at the speed of light.

With That In Mind

- We now return to opsem for IMP

$$\frac{<e, \sigma> \Downarrow n}{<x := e, \sigma> \Downarrow \sigma[x := n]}$$

Def: $\sigma[x{:=}n](x) = n$
$\sigma[x{:=}n](y) = \sigma(y)$

$$\frac{<b, \sigma> \Downarrow \text{false}}{<\text{while } b \text{ do } c, \sigma> \Downarrow \sigma}$$

$$\frac{<b, \sigma> \Downarrow \text{true} \qquad <c; \text{while } b \text{ do } c, \sigma> \Downarrow \sigma'}{<\text{while } b \text{ do } c, \sigma> \Downarrow \sigma'}$$

Command Evaluation Notes

- The order of evaluation is important
  - c_1 is evaluated before c_2 in $c_1; c_2$
  - c_2 is not evaluated in "if true then c_1 else c_2"
  - c is not evaluated in "while false do c"
  - b is evaluated first in "if b then c_1 else c_2"
  - this is explicit in the evaluation rules
- Conditional constructs (e.g., $b_1 \vee b_2$) have multiple evaluation rules
  - but only one can be applied at one time

Command Evaluation Trials

- The evaluation rules are not syntax-directed
  - See the rules for while, ∧
  - The evaluation might not terminate
- Recall: the evaluation rules suggest an interpreter
- Natural-style semantics has two big disadvantages (continued …)

Disadvantages of Natural-Style Operational Semantics

- It is hard to talk about commands whose evaluation does not terminate
  - i.e., when there is no σ' such that $<c, \sigma> \Downarrow \sigma'$
  - But that is true also of ill-formed or erroneous commands (in a richer language)!
- It does not give us a way to talk about intermediate states
  - Thus we cannot say that on a parallel machine the execution of two commands is interleaved (= no modeling threads)

Semantics Solution

- Small-step semantics addresses these problems
  - Execution is modeled as a (possible infinite) sequence of states
- Not quite as easy as large-step natural semantics, though
- Contextual semantics is a small-step semantics where the atomic execution step is a rewrite of the program

Contextual Semantics

- We will define a relation $<c, \sigma> \to <c', \sigma'>$
  - c' is obtained from c via an atomic rewrite step
  - Evaluation terminates when the program has been rewritten to a terminal program
    - one from which we cannot make further progress
  - For IMP the terminal command is "skip"
  - As long as the command is not "skip" we can make further progress
    - some commands *never* reduce to skip (e.g., "while true do skip")

Contextual Derivations

- In small-step contextual semantics, derivations are not tree-structured
- A contextual semantics derivation is a sequence (or list) of atomic rewrites:

$$<x+(7-3),\sigma> \to <x+(4),\sigma> \to <5+4,\sigma> \to <9,\sigma>$$

$\sigma(x)=5$

What is an Atomic Reduction?

- What is an atomic reduction step?
  - Granularity is a choice of the semantics designer
- How to select the next reduction step, when several are possible?
  - This is the order of evaluation issue

Redexes

- A redex is a syntactic expression or command that can be reduced (transformed) in one atomic step
- Redexes are defined via a grammar:

$$
\begin{array}{ll}
r ::= x & (x \in L) \\
\quad \mid n_1 + n_2 & \\
\quad \mid x := n & \\
\quad \mid \text{skip; } c & \\
\quad \mid \text{if true then } c_1 \text{ else } c_2 & \\
\quad \mid \text{if false then } c_1 \text{ else } c_2 & \\
\quad \mid \text{while } b \text{ do } c &
\end{array}
$$

- For brevity, we mix exp and command redexes
- Note that (1 + 3) + 2 is not a redex, but 1 + 3 is

Local Reduction Rules for IMP

- One for each redex: **$\langle r, \sigma\rangle \to \langle e, \sigma'\rangle$**
  - means that in state σ, the redex r can be *replaced in one step* with the expression e

$\langle x, \sigma\rangle \to \langle \sigma(x), \sigma\rangle$

$\langle n_1 + n_2, \sigma\rangle \to \langle n, \sigma\rangle$ where $n = n_1$ plus n_2

$\langle n_1 = n_2, \sigma\rangle \to \langle \text{true}, \sigma\rangle$ if $n_1 = n_2$

$\langle x := n, \sigma\rangle \to \langle \text{skip}, \sigma[x := n]\rangle$

$\langle \text{skip; } c, \sigma\rangle \to \langle c, \sigma\rangle$

$\langle \text{if true then } c_1 \text{ else } c_2, \sigma\rangle \to \langle c_1, \sigma\rangle$

$\langle \text{if false then } c_1 \text{ else } c_2, \sigma\rangle \to \langle c_2, \sigma\rangle$

$\langle \text{while } b \text{ do } c, \sigma\rangle \to$
$\quad \langle \text{if } b \text{ then } c\text{; while } b \text{ do } c \text{ else skip}, \sigma\rangle$

Not happy? I'll explain with pictures soon!

The Global Reduction Rule

- General idea of contextual semantics
  - Decompose the current expression into the redex-to-reduce-next and the remaining program
    - The remaining program is called a context
  - Reduce the redex "r" to some other expression "e"
  - The resulting (reduced) expression consists of "e" with the original context

As A Picture (1)

(Context)
...
x := 2+2
...

Step 1: Find The Redex

As A Picture (2)

(Context)
...
x := 2+2 (redex)
...

Step 1: Find The Redex

Step 2: Reduce The Redex

As A Picture (3)

(Context)
...
x := 2+2 (redex) → 4 (reduced)
...

Step 1: Find The Redex

Step 2: Reduce The Redex

As A Picture (4)

(Context)

…

x := 4

…

Step 1: Find The Redex

Step 2: Reduce The Redex

Step 3: Replace It In The Context

Contextual Analysis

- We use H to range over contexts
- We write H[r] for the expression obtained by placing redex r in context H
- Now we can define a small step

If <r, σ> → <e, σ'>

then <H[r], σ> → <H[e], σ'>

Contexts

- A context is like an expression (or command) with a marker • in the place where the redex goes
- Examples:
  - To evaluate "(1 + 3) + 2" we use the redex 1 + 3 and the context "• + 2"
  - To evaluate "if x > 2 then c_1 else c_2" we use the redex x and the context "if • > 2 then c_1 else c_2"

Context Terminology

- A context is also called an "expression with a hole"
- The marker • is sometimes called a hole
- H[r] is the expression obtained from H by replacing • with the redex r

"Avoid context and specifics; generalize and keep repeating the generalization."
-- Jack Schwartz

Contextual Semantics Example

- x := 1 ; x := x + 1 with initial state [x:=0]

<Comm, State>	Redex •	Context
<x := 1; x := x+1, [x := 0]>	x := 1	•; x := x+1
<skip; x := x+1, [x := 1]>	skip; x := x+1	•
<x := x+1, [x := 1]>	x	x := • + 1
What happens next?		

Contextual Semantics Example

- x := 1 ; x := x + 1 with initial state [x:=0]

<Comm, State>	Redex •	Context
<x := 1; x := x+1, [x := 0]>	x := 1	•; x := x+1
<skip; x := x+1, [x := 1]>	skip; x := x+1	•
<x := x+1, [x := 1]>	x	x := • + 1
<x := 1 + 1, [x := 1]>	1 + 1	x := •
<x := 2, [x := 1]>	x := 2	•
<skip, [x := 2]>		

More On Contexts

- Contexts are defined by a grammar:

  H ::= • | n + H
  | H + e
  | x := H
  | if H then c_1 else c_2
  | H; c
- A context has exactly one • marker
- A redex is never a value

What's In A Context?

- Contexts specify precisely how to find the next redex
  - Consider $e_1 + e_2$ and its decomposition as H[r]
  - If e_1 is n_1 and e_2 is n_2 then H = • and r = $n_1 + n_2$
  - If e_1 is n_1 and e_2 is not n_2 then H = $n_1 + H_2$ and $e_2 = H_2[r]$
  - If e_1 is not n_1 then H = $H_1 + e_2$ and $e_1 = H_1[r]$
  - In the last two cases the decomposition is done recursively
  - Check that in each case the solution is unique

Unique Next Redex: Proof By Handwaving Examples

- e.g. c = "c_1; c_2" - either
  - c_1 = skip and then c = H[skip; c_2] with H = •
  - or $c_1 \neq$ skip and then c_1 = H[r]; so c = H'[r] with H' = H; c_2
- e.g. c = "if b then c_1 else c_2"
  - either b = true or b = false and then c = H[r] with H = •
  - or b is not a value and b = H[r]; so c = H'[r] with H' = if H then c_1 else c_2

Context Decomposition

- Decomposition theorem:

  If c is not "skip" then there exist unique H and r such that c is H[r]
  - "Exist" means progress
  - "Unique" means determinism

Short-Circuit Evaluation

- What if we want to express short-circuit evaluation of $\wedge$?
  - Define the following contexts, redexes and local reduction rules

    H ::= ... | H $\wedge$ b_2
    r ::= ... | true $\wedge$ b | false $\wedge$ b
    <true $\wedge$ b, σ> → <b, σ>
    <false $\wedge$ b, σ> → <false, σ>
  - the local reduction kicks in before b_2 is evaluated

Contextual Semantics Summary

- Can view • as representing the program counter
- The advancement rules for • are non-trivial
  - At each step the entire command is decomposed
  - This makes contextual semantics inefficient to implement directly
- The major advantage of contextual semantics: it allows a mix of local and global reduction rules
  - For IMP we have only local reduction rules: only the redex is reduced
  - Sometimes it is useful to work on the context too
  - We'll do that when we study memory allocation, etc.

Reading Real-World Examples

- Cobbe and Felleisen, POPL 2005
- Small-step contextual opsem for Java
- Their rule for object field access:

$$P \vdash \langle \mathsf{E}[obj.fd], \mathcal{S}\rangle \hookrightarrow \langle \mathsf{E}[\mathcal{F}(fd)], \mathcal{S}\rangle \quad \text{where } \mathcal{F} = \mathit{fields}(\mathcal{S}(obj)) \text{ and } fd \in \mathrm{dom}(\mathcal{F})$$

$P \vdash <E[obj.fd],S> \rightarrow <E[F(fd)],S>$
- where $F=fields(S(obj))$ and $fd \in dom(F)$

- They use "E" for context, we use "H"
- They use "S" for state, we use "σ"

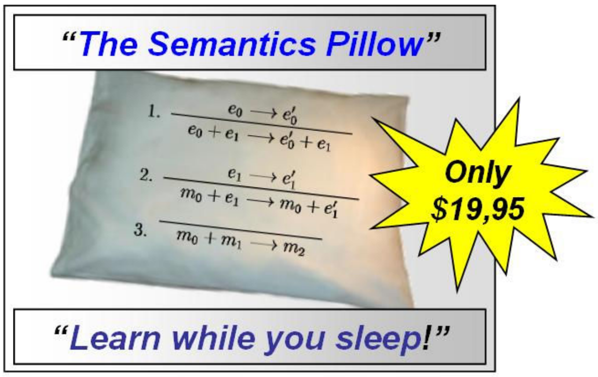
Lost In Translation

- $P \vdash <H[obj.fd],\sigma> \rightarrow <H[F(fd)],\sigma>$
  - Where $F=fields(\sigma(obj))$ and $fd \in dom(F)$
- They have "$P \vdash$", but that just means "it can be proved in our system given P"
- $<H[obj.fd],\sigma> \rightarrow <H[F(fd)],\sigma>$
  - Where $F=fields(\sigma(obj))$ and $fd \in dom(F)$

Lost In Translation 2

- $<H[obj.fd],\sigma> \rightarrow <H[F(fd)],\sigma>$
  - Where $F=fields(\sigma(obj))$ and $fd \in dom(F)$
- They model objects (like obj), but we do not (yet) – let's just make fd a variable:
- $<H[fd],\sigma> \rightarrow <H[F(fd)],\sigma>$
  - Where $F=\sigma$ and $fd \in L$
- Which is just our variable-lookup rule:
- $<H[fd],\sigma> \rightarrow <H[\sigma(fd)],\sigma>$ (when $fd \in L$)

"Sleep On It"



Homework

- Straw Poll
- Homework 2 Out Today
  - Due Next Week
- Read Winskel Chapter 3
- Want an extra opsem review?
  - *Natural deduction* article
  - Plotkin Chapter 2
- Optional Philosophy of Science article