Sensitivity Analysis of an Optimal Routing Policy in an *Ad Hoc* Wireless Network

Tara Javidi and Demosthenis Teneketzis

***Abstract*—We examine the sensitivity of optimal routing policies in *ad hoc* wireless networks with respect to estimation errors in channel quality. We consider an *ad hoc* wireless network where the wireless links from each node to its neighbors are modeled by a probability distribution describing the local broadcast nature of wireless transmissions. These probability distributions are estimated in real-time. We investigate the impact of estimation errors on the performance of a set of proposed routing policies.**

fo

I. Introduction

As THE SIZE of communication networks increases and the applications of such networks spreads to various fields, resource allocation issues such as connection admission control, routing, etc., become an increasingly important component of communication research. The research in *ad hoc* networks consists of a large body of work which addresses these issues in the context of networks with no central controller and an unspecified connectivity topology, where each node can itself act as a store-and-forward router (see [1]). The general routing problem in an *ad hoc* network is to define a policy which, given the trajectory histories of all the messages, chooses the nodes to transmit the messages next. Such a policy must be implemented in a distributed fashion. There is an extremely rich literature on routing in *ad hoc* networks (for a summary, see [1]), where several centralized and distributed routing algorithms are proposed and notions of *shortest path* and *minimum energy routing* have been established. Various objectives can be considered in a network routing optimization, including capacity, timeliness, and energy consumption. Further challenges are posed by wireless networks, where similar goals must be achieved with unreliable, time-varying channels, and where new concerns, such as energy consumption and channel interference impose additional constraints. It is shown (see [2] and [3]) that in wireless networks due to the unreliability of channels retransmission mechanisms have a nonnegligible impact on the energy performance of routing protocols. In other words, the proposed routing policies (e.g., MRPC in [3]) achieve a better performance by including the channel quality and expected number of retransmissions, both functions of probability of successful transmission, in the definition of each link's *length* or each node's *transmission cost*. Furthermore, in [4]–[6], it is shown that the local broadcast nature of wireless transmission, which usually causes interference among neighboring nodes, can be used to our advantage. This can be done if successful transmissions are acknowledged by the nodes which receive the message. Using the acknowledgments, sample-path-dependent optimal routes can be constructed (see Section II). The construction of sample path dependent routing as well as MRPC relies on the estimates of channel quality to achieve an improved performance in *ad hoc* routing. In other words, the routing in wireless systems is best addressed by a stochastic model of the system. In general, service provisioning and resource allocation issues (such as admission control, routing, etc.) in wireless networks are best modeled as stochastic scheduling and stochastic control problems, where the wireless links are described by stochastic processes. The statistic of any wireless link depends on the physical channel (additive noise, path loss, shadowing, fading, etc. [7]), the number of users that use the link simultaneously, and the users' transmission strategies. Generally, the overall structure and statistical behavior of the system, e.g., the marginal and joint distributions of the processes involved, is studied and modeled off-line, while the particular parameters of such models, e.g., mean and covariance, are left to be estimated in a real-time measurement-based fashion. For instance, the channel quality of a single-hop wireless link over time might be modeled as an independent identically distributed binary symmetric channel, whose transmission error probability p_e is estimated online. The existence of online estimation adds a new dimension to the question of optimal routing.

Ideally, routing as well as other resource allocation mechanisms can potentially provide information on the statistics of the wireless channels since any resource allocation mechanism regulates the communications in the system. In fact, there exists a tradeoff between identification and learning via communication versus minimizing the communication costs. Hence, even in situations where the wireless system is controlled in a centralized manner, the estimation problem combined with the control issues, should be ideally studied as a stochastic control problem with imperfect information. Stochastic control problems with imperfect information are dual control problems that address joint estimation and control issues (see [8]). The information state [8] for these problems lies in an infinite dimensional space even when the state-space and action space are finite. This feature makes such dual control problems analytically and computationally difficult.

Manuscript received November 6, 2002; revised October 15, 2003. Recommended by Associate Editor C. D. Charalambous. This work was supported in part by the following Grants: ARO DAAHI4-96-1-0377, ONR N000-14-03-1-0232, NSF CCR-0082784, and NSF ECS-9 979 347.

T. Javidi is with the Electrical Engineering Department, the University of Washington, Seattle, WA 98102-2500 USA (e-mail: tara@ee.washington.edu).

D. Teneketzis is with the Electrical Engineering and Computer Science Department, the University of Michigan, Ann Arbor, MI 48109-2122 USA (e-mail: teneket@eecs.umich.edu).

Digital Object Identifier 10.1109/TAC.2004.832204

An alternative approach to the aforementioned dual estimation/control problem formulation is to decouple the estimation and control issues in these problems. Such an approach provides a parameter estimation algorithm which operates independently of the control decisions and feeds the estimated parameters into a controller designed under the perfect information assumption. Following such an approach, wireless *ad hoc* routing can be addressed by the following three step procedure: 1) Offline modeling of the overall statistical behavior of the wireless links; 2) specification of the channel quality via real-time and measurement-based estimation of probabilities of successful transmissions at all nodes; and 3) determination of optimal decision (e.g., routing) strategies assuming that the results of steps 1) and 2) describe the true stochastic behavior of the broadcast channels (referred to as broadcast model). The construction of optimal sample path dependent routing strategies as well as MRPC (see [2], and [4]–[6]) are examples of the solution to step 3) of the wireless *ad hoc* routing problem.

In the approach described previously, there are errors associated with the estimation techniques used in 2), and the accuracy of the estimated parameters is limited to the error margin of the employed estimation algorithm. Furthermore, the optimal decision (e.g., routing) strategy resulting from 3) is guaranteed to be optimal only for the particular values of parameters given by 2); and it generally varies when these parameters change. Hence, it is vital to perform sensitivity analysis in order to quantify the loss in performance of the proposed decision strategy, in the presence of the aforementioned estimation errors. This fact implies that a major component of design and analysis in any wireless system is the sensitivity analysis with respect to errors in channel modeling and channel estimation.

In this paper, we present a sensitivity analysis of a known optimal (with respect to an energy consumption criterion) routing policy in a stochastic *ad hoc* network. Our analysis is based on the model and results of [4], [5], and [6]. In these papers, the authors investigate a network routing problem where a probabilistic model for wireless local broadcasts is used. Under the assumption that the transmission probabilities of the local broadcast model for each node are time-invariant and precisely known, the existence of an optimal priority policy with time-invariant indices is shown in [4]–[6]. As expected, these indices depend on the parameters of the local broadcast model. We investigate the sensitivity of this priority policy with respect to errors in the knowledge of the aforementioned transmission probabilities, and analytically determine the impact of errors in the broadcast model on the performance of the optimal policy. We quantify this impact as follows: 1) We first establish appropriate distance measures between two probabilistic broadcast models, and between two policies in terms of their performance; 2) we construct policies $\tilde{\pi}$ and π^* that are optimal for the true broadcast model P and the estimated broadcast model Q, respectively; and 3) we bound the distance between the performance of the two policies $\tilde{\pi}$ and π^* by a term proportional to the distance between the broadcast models P and Q (estimation error).

As noted earlier, the key feature of an *ad hoc* network is that there exists no central control or computation unit to supervise the implementation and calculation of routing decisions. This feature underlines the importance of providing a distributed algorithm for the computation and implementation of an optimal policy. The authors in [4] and [5] provide algorithms in which each node computes its optimal local routing actions in a distributed fashion, i.e., each node only uses the local information available to it to make routing decisions. It is shown that under certain technical conditions (see [4] and [5]), which hold almost in all real scenarios, these algorithms all converge to an optimal stationary policy which is consistent with the optimal index policy computed centrally. Combining this result with our sensitivity analysis for the centralized control problem, we extend our sensitivity results to optimal routing strategies that are computed in a distributed fashion.

The remainder of this paper is organized as follows. In Section II, we present a stochastic dynamic routing problem and the corresponding result that have appeared in [4]–[6] and are relevant to this paper. Specifically, in Section II, we first present the formulation of the routing problem in an *ad hoc* networks, provide some useful notation and definitions and state the result given by [4] and [5] on the structure of the optimal sample path dependent routing (SPDR) strategy. Sections III and IV contain the main contribution of our paper. Specifically, in Sections III-A and III-B, we formulate the problem of sensitivity of routing policies with respect to errors in channel estimation and establish the desirable goals of such study. In Section III-C, we construct examples to illustrate that such goals may not always be achievable. In Section IV, we present the main results of our sensitivity analysis. In Section IV-A, we establish the mathematical relationships between loss of performance and the error in the estimation of the local broadcast models. In Section IV-B, we discuss the essence of our sensitivity results through examples and numerically compare the robustness of optimal SPDR with that of other known routing strategies. In Section IV-C, we extend our results to the optimal routing policies which are computed and constructed in a distributed fashion. In Section V, we conclude the paper.

II. Background

In this section, we revisit a stochastic dynamic routing problem, formulated in [4]–[6], and state the result developed in [4]–[6] that are necessary for our sensitivity analysis.

A. Model $(\mathbf{M})$, Notation, and Preliminaries

Model $(\mathbf{M})$, as presented in [4], is described by: 1) of a set of nodes; and 2) the connectivity among nodes, indicated by the probability of successful transmission of a message from one node to its "neighbors" (defined later). The network is used for message routing; message routing can be viewed as a stochastic control problem where the objective is to determine optimal routing strategies that minimize the average energy required for the transmission of a set of messages from their sources to their destinations.

A description of the elements of the network is given as follows:

N	number of nodes in the network;
$\Omega = \{1, \ldots, N\}$	set of all nodes; so $\lvert\Omega\rvert = N$;

$S \subseteq \Omega$ — state of the system, defined as the set of nodes which have received the message. S_t refers to the state at time t.

We define $\mathcal{S} := \{S : S \subset \Omega\} = 2^{\Omega}$.

Model $(\mathbf{M})$, as described in [4], has the following properties: 1) Transmissions over time at each node are independent identically distributed (i.i.d); 2) successful transmissions are independent among nodes; and 3) the network has the decoupling property defined as follows.

Definition 1 (Decoupling Property): $(\mathbf{M})$ is said to have the *decoupling property* if successful transmission from a node to a set of neighbors at a given time is unaffected by which other nodes already have the message.

As a result of the aforementioned properties we can define the following notions.

We write $P^i(S' \mid S)$ to indicate the probability of reaching state S' from state S when choosing node i for transmission, $i \in S$. We write $P^i(S \mid i)$ as shorthand for $P^i(S \mid \{i\})$.

We refer to $P = \{P^i(S' \mid S)\}_{i,S',S}$ as the broadcast model.

We define $P_{ij} := \sum_{S: i,j \in S} P^i(S \mid i)$.

Node j is called a *neighbor* of node i if $P_{ij} > 0$. Note that $P_{ij} \neq P_{ji}$ is permitted.

Given the local broadcast model $P, \mathcal{N}_P(i)$ is the set of all neighbors of i, together with i itself.

As a consequence of properties 1)–3), determining the optimal routing of a set of messages is equivalent to determining the optimal route for each message. It is shown in [4]–[6] that the optimal routing of each message can be treated as a Markov decision problem with state–space $\mathcal{S} = 2^{\Omega}$, action space $\Omega \cup \{r\}$, (r indicates retirement), transition probabilities given by $P^i(S' \mid S)$ defined above, reward function R: $2^{\Omega} \to \mathbb{R}^+$, and transmission cost c_i at node i. A stationary Markov policy π is then defined as a function on the state space onto the action space, i.e., π: $2^{\Omega} \to \Omega \cup \{r\}$.

In this paper, we use the following shorthands: $R_i := R(\{i\})$ denotes the reward when retiring at node i. Also $R_{\max} := \max_{i \in \Omega} R_i$.

We write $\pi(S) = i$ to indicate policy π transmits at node i when in state S.

We write $\pi(S) = r$ to indicate policy π retires and receives reward $R(S)$ when in state S. For convenience we write $\pi(S) = r_i$ as shorthand that policy π retires and receives $R_i, i \in S$. When $\pi(S) = r_i$ we say that policy π retires and *receives the reward* of node i.

By $\pi(S) \neq i, r_i$, we mean both $\pi(S) \neq i$ and $\pi(S) \neq r_i$.

By $\pi(S) = \tilde{\pi}(S)$, we mean either $\pi(S) = \tilde{\pi}(S) = i$, or $\pi(S) = \tilde{\pi}(S) = r_i$, for some i.

We also assume that $(\mathbf{M})$ has the following *increasing* property.[1]

Definition 2 (Increasing Property): $(\mathbf{M})$ is said to have the *increasing property* if for any system realization under any policy we have $S_{t_2} \supseteq S_{t_1}, \forall t_1, \forall t_2 > t_1$.

[1]The *increasing property* is only needed to prove the structure of an optimal dynamic routing policy. It is not necessary for the implementation of an optimal centralized routing algorithm as well as for the distributed implementations of the algorithm that are proposed in [4]–[6]. This can be seen from the result in [4]–[6] and will also become apparent in Section II-D of this paper.

We next present the centralized version of the stochastic routing problem with time-invariant parameters as formulated in [4]–[6].

B. Statement of Problem $(\mathbf{P1})$

$(\mathbf{P1})$ We consider the transmission of a single message, from a given initial state S_0 (i.e., a given set of nodes) to a set of destination states, in a wireless *ad hoc* network of N nodes described by $(\mathbf{M})$ in which the transition probabilities are given by the broadcast model P. Transmission instances occur at discrete time points. Each transmission from a given node i incurs a fixed cost $c_i > 0$. According to $(\mathbf{M})$: 1) At each transmission instance, the transmitting node is chosen among all those who have received the message by a central controller that always knows the current state of the system (i.e., the set of nodes that have the message); 2) node transmissions are local broadcasts, that is, multiple neighbor nodes may all simultaneously receive the message; 3) given a node i is chosen to transmit, the probability $P^i(S \mid i)$ that a given set S of nodes receives the message is known and fixed; 4) The central controller is informed, without any cost, as to which nodes receive the message. Control information flow between the nodes and the controller is considered free of energy and instantaneous in time; 5) each transmission event is assumed independent of those before and after; and 6) a reward function R is specified. At any instance, the central controller can terminate the transmission process or choose to continue transmitting. The objective is to choose: 1) The node to transmit at each transmission instance, and 2) the instance to terminate the transmission process, so as to maximize over all Markov policies

$$J_P^{\pi}(S_0) = E^{\pi}\left\{R(S_f) - \sum_{t=1}^{\tau-1} c_{i(t)}\right\} \tag{1}$$

where π is the transmission/termination policy the controller follows, τ is the time when the transmission process is terminated under policy π, S_f is the state at $\tau, i(t)$ is the node chosen by the transmission policy at time t, and $J_P^{\pi}(S)$ is the expected reward when starting in state S under policy π under local broadcast model P.

Restriction to Markov policies does not entail any loss of optimality because $(\mathbf{P1})$ is a stochastic control problem with perfect observations [8].

Mathematically, $(\mathbf{P1})$ is parameterized by a tuple $(N, P, \underline{c}, R)$, where $\underline{c} := (c_1, c_2, \ldots, c_N)$.

C. The Transmission Control Problem

Consider an *ad hoc* network in which control of transmission type (in terms of power, antenna directionality, and addressing) is allowed. In such a network, at each time step the central controller chooses a node for transmission, among the nodes with the message, and a transmission type, among a finite set of allowable types, is chosen for that node. To each node i and transmission type k, a transmission cost $c_{(i,k)}$ and a probability distribution, denoted by $P^{(i,k)}(S \,|\, i)$, describing the probability that a given set of nodes receive the message are assigned. The objective for the controller in such a network is to determine a policy which maximizes a total reward similar to that of (1). Such a policy specifies the optimal number and coverage of hops, along each realization of the operation of network.

The authors in [4]–[6] have shown that such a network can be modeled by $(\mathbf{M})$ and can be formulated as $(\mathbf{P1})$ with a particular structure on the probability of successful transmission, i.e., a particular structure on the broadcast model. We summarize their argument and results here. It is possible to study the networks with transmission control as a special case of $(\mathbf{M})$ since in $(\mathbf{M})$ no particular assumption regarding the statistical correlation among the transition probabilities has been made. We seek to construct a particular network described by $(\mathbf{M})$, which satisfies the conditions required by the addition of multiple transmission types resulting from the use of multiple power levels. In order to do so we represent each node i as a set of sister nodes with cardinality W_i, where W_i refers to the number of transmission types available at node i. Each sister node in such a set represents a transmission type for node i, and is identified by the pair (i,k), where $i \in \Omega$ and $k = 1, 2, \ldots, W_i$. We define Ω_p to be the collection of these sets of sister nodes. Transmissions in the Ω_p space are based on the corresponding events in Ω, as follows. Each transmission at node $i \in \Omega$ with transmission type k corresponds to a control decision u which chooses node $(i,k) \in \Omega_p$. Such a transmission incurs a cost $c_{(i,k)}$. If such a transmission leads to a set of nodes, say $S_1 \in \Omega$, receiving the message, all nodes $(l,k) \in \Omega_p$ such that $l \in S_1$ receive the message. This implies that all sister nodes receive the message simultaneously, or in other words, message receptions for sister nodes in Ω_p are deterministically coupled. Finally, each sister node of i receives the same reward described by the reward function.

The problem of optimal routing in the *ad hoc* wireless network described by Ω_p is a special case of $(\mathbf{P1})$, when transmission probabilities have a particular coupled structure to accommodate different transmission types as sister nodes at which the message receptions are deterministically coupled (for more details, see [4] and [5]). Hence, the sensitivity results derived in this paper apply to a wireless *ad hoc* network with power control and multiple transmission types.

In the previous section, we presented the stochastic model of an *ad hoc* network proposed in [4]–[6] and the stochastic dynamic routing with transmission control problem formulated in [4]–[6]. We refer the reader to [5] for a careful critique of $(\mathbf{M})$ and $(\mathbf{P1})$.

D. Preliminary Results: Optimal Routing in $(\mathbf{P1})$

We conclude Section II by summarizing the results in [4]–[6] that are relevant to our work. For that matter, we need the following definitions.

Definition 3: A Markov policy π is a priority policy if there is a strict priority ordering of the nodes s.t. $\forall i \in \Omega$ we have $\pi(S \cup \{i\}) = \pi(\{i\}) = i$ or $r_i, \forall S \subset \Omega_i^\pi$, where Ω_i^π is the set of nodes of priority lower than i under π.

Definition 4: For priority policy π, we write $i >_\pi j$ when i has higher priority than j under π.

Definition 5: For priority policy π and node i, we denote by $U_P^\pi(i)$ the class of higher priority subsets of neighbors of i, i.e., $U_P^\pi(i) := \{S : i \in S \subset \mathcal{N}_P(i), \pi(S) \neq i\} = \{S : i \in S \subset \mathcal{N}_P(i)\} - 2^{\{j : j \leq i\}}$. Similarly, we define $L_P^\pi(i) := \{S : i \in S \subset \mathcal{N}_P(i), \pi(S) = i\} = \{S : i \in S \subset \mathcal{N}_P(i)\} \cap 2^{\{j : j \leq i\}}$.

Now, we state the following facts from [4] (for more details, see [5]).

Fact 1: For priority policy π we have $J_P^\pi(S) = J_P^\pi(\{\pi(S)\}) = J_P^\pi(\{i\})$, when $i >_\pi j \; \forall j \in S - \{i\}$.

This fact holds by the decoupling property and the definition of a priority policy.

Fact 2: For priority policy π and any state S where $\pi(S) = i$ or r_i we can write the expected reward as shown in (2), as shown at the bottom of the page.

Fact 3: There is an optimal Markov policy π^* for $(\mathbf{P1})$ which is a priority policy and whose expected reward has the following property:

$$J_P^{\pi^*}(S) = \max_{i \in S} J_P^{\pi^*}(\{i\}). \tag{3}$$

Fact 4: Under the optimal Markov policy π^* the expected reward for each node i defines an index shown in (4) as shown at the bottom of the page. This index, in turn, defines an optimal ordering of the nodes and the actions taken at these nodes, i.e.,

$$\begin{aligned} J_P^{\pi^*}(\{i\}) > J_P^{\pi^*}(\{j\}) &\Longrightarrow i >_{\pi^*} j \\ i >_{\pi^*} j &\Longrightarrow J_P^{\pi^*}(\{i\}) \geq J_P^{\pi^*}(\{j\}). \end{aligned} \tag{5}$$

All these facts which are proved in [4] establish that, under the assumption that the network parameters are fixed and known,

$$J_P^\pi(\{i\}) = \begin{cases} -c_i + \sum_{S'} P^i(S' \,|\, i) J_P^\pi(\{\pi(S')\}), & \text{if } \pi \text{ transmits} \\ R_i, & \text{if } \pi \text{ retires.} \end{cases} \tag{2}$$

$$J_P^{\pi^*}(\{i\}) = \max\left\{ \frac{-c_i + \sum_{S \in U_P^{\pi^*}(i)} P^i(S \,|\, i) J_P^\pi(\{\pi(S)\})}{\sum_{S \in U_P^{\pi^*}(i)} P^i(S \,|\, i)}, R_i \right\}. \tag{4}$$

there exists an optimal routing priority ordering of nodes. The authors in [4] propose a centralized algorithm (see [5, Alg. 1]) with complexity $O(N^2)$ which computes the optimal priority listing of nodes in $(\mathbf{P1})$ when all the network parameters are known and available at the same location (this algorithm is similar in nature to standard Dijkstra in open shortest path first (OSPF) [9]). Furthermore, three distributed algorithms to compute the optimal priority listing of the neighbors at each node are proposed in [5]. Note that due to the nature of the optimal priority policy, there is a natural distributed implementation of such policy; such an implementation requires that: 1) successful receptions at neighboring nodes are acknowledged to the transmitting node, and 2) each node keeps a priority list of the node itself and its neighbors. A node transmits until another node of higher priority successfully receives the message. Thus, the network layer does not dictate the route, and in fact there is no *one* route, but the actual route a message takes between source and destination is sample path dependent. This characteristic of the solution is distinctly different from that of other proposed algorithms (see [1]–[3], [9], and [10]).

III. Sensitivity Analysis: Problem Formulation

The model and result presented in Section II assume perfect knowledge of the transmission probabilities P. To actually implement the algorithms, methods to estimate these probabilities should be employed (see the discussion in [5]). On the other hand, such methods introduce various levels of error. Thus, in reality P is not known but has to be estimated. The presence of estimation errors raises the important issue of the sensitivity of the results in [4]–[6], with respect to (small) variations in P. This motivates the formulation of Problem $(\mathbf{P2})$ as follows.

A. Statement of Problem $(\mathbf{P2})$

Consider $(\mathbf{P1})$ associated with two sets of system parameters, $(N, P, \underline{\mathbf{c}}, R)$ and $(N, Q, \underline{\mathbf{c}}, R)$, describing the true and estimated models of the system, respectively. According to the results given in Section II-D there exists an index policy π^* which is an optimal routing policy for $(\mathbf{P1})$ with parameters $(N, P, \underline{\mathbf{c}}, R)$. At the same time, the optimal solution to the estimated model, $(N, Q, \underline{\mathbf{c}}, R)$, is an index policy $\tilde{\pi}$ that is not optimal for the true model $(N, P, \underline{\mathbf{c}}, R)$, in general. Policy $\tilde{\pi}$ is applied to the system with the true broadcast model (distribution) P. We are interested in: 1) Determining/quantifying the difference between the performance of policy $\tilde{\pi}$ in such a system and the best possible performance, achieved by π^*. 2) Relating the aforementioned difference to a quantity describing the estimation error in the (true) broadcast model.

To quantify the difference specified in 1) we define an appropriate metric on the space of all routing policies. We define *the distance between policies* π_1 *and* π_2 *in the context of distribution* P as

$$d_P(\pi_1, \pi_2) := \max_S \left| J_P^{\pi_1}(S) - J_P^{\pi_2}(S) \right|. \tag{6}$$

To relate the difference specified in 1) to the estimation error in the (true) broadcast model we first quantify this error by defining a distance measure between the true broadcast model P and the estimated model Q. We use the total variation metric for this purpose (see [11] and [12]). The total variation distance between two local broadcast models, P and Q, describing the probabilities of transmission success for node i, is defined as

$$\sigma(P_i, Q_i) = \sup_{A \subset 2^\Omega} \left| \sum_{S' \in A} \left(P^i(S' \mid i) - Q^i(S' \mid i) \right) \right|. \tag{7}$$

We extend this measure to define the total variation distance between broadcast models P and Q as

$$\sigma(P, Q) = \max_i \sigma(P_i, Q_i). \tag{8}$$

Note that if at each node transmissions to different neighbors are independent and the maximum error on each link is δ, i.e., for $\forall i, j, k, P^i(\{j,k\} \mid i) = P_{ij}.\, P_{ik}$ and $|P_{ij} - Q_{ij}| < \delta$, then $\sigma(P, Q) < \delta$. Hence, the total variation distance is an appropriate metric to specify the estimation errors.

Based on this, we formulate the following sensitivity analysis problem.

$(\mathbf{P2})$ Consider $(\mathbf{P1})$ for two sets of parameters, $(N, P, \underline{\mathbf{c}}, R)$ and $(N, Q, \underline{\mathbf{c}}, R)$, describing the true and estimated models of the system, respectively. Let π^* be an optimal routing policy for $(\mathbf{P1})$ with parameters $(N, P, \underline{\mathbf{c}}, R)$, and $\tilde{\pi}$ be an optimal routing policy for $(\mathbf{P1})$ with parameters $(N, Q, \underline{\mathbf{c}}, R)$. The objective is to determine the distance between policies $\tilde{\pi}$ and π^* in the context of P, and relate this distance to the total variation distance between the estimated model Q and the true system model P, i.e., $\sigma(P, Q)$.

Routing policy π^* is said to be robust with respect to errors in channel quality estimates, if $d_P(\pi^*, \tilde{\pi})$ is bounded by a finite term proportional to error $\sigma(P, Q)$.

B. Basic Assumption on Convergence Rate of Estimation Algorithm

As mentioned before, the estimated local broadcast model Q is constructed through an online parameter estimation algorithm. Denote by τ_e the time scale for operation of the estimation algorithm.

We assume throughout the following analysis that $\tau_e + \tau_c \ll \tau_n$, where τ_c is the time scale for computing the indices as well as for disseminating the updated (optimal) ranking of the nodes through the network, and τ_n is the time scale for significant variation in network topology. This implies that our result can be extended to a dynamic setting if the changes in topology occur at a much less rate and larger time scale relative to the rate of channel estimation, communication, and computation of priority rankings in the network.

C. Example

The following example shows that, in general, optimal routing can be extremely sensitive to estimation errors, i.e., there exist scenarios where a small error in estimation can unboundedly deteriorate the performance of the constructed priority policy.

Consider the simple network given by Fig. 1. We assume that transmission success probabilities are given by the true model P and the estimated values of these probabilities are given by model Q. The value of these transmission probabilities are $P^1(\{1,2,3\}\,|\,1) = Q^1(\{1,2,3\}\,|\,1) = 0$, $P^1(\{1,2\}\,|\,1) = Q^1(\{1,2\}\,|\,1) = p, P^2(\{1,2\}\,|\,2) = Q^2(\{1,2\}\,|\,2) = p, P^2(\{1,2,3\}\,|\,2) = Q^2(\{1,2,3\}\,|\,2) = 0$, and finally $P^2(\{2,3\}\,|\,2) = Q^2(\{2,3\}\,|\,2) = p$. Furthermore, we assume $P^1(\{1,3\}\,|\,1) = 0$ while $Q^1(\{1,3\}\,|\,1) = \delta$. Assume that node i has transmission costs c_i; $i = 1,2$. Rewards are zero for the first two nodes, and it is equal to $R > (c_i/\delta)$ at the destination. The cost of transmission at node 2 is much larger than the cost of transmission at node 1, i.e., $c_2 \gg c_1$.

In this example, the total variation distance between the two broadcast models P and Q, i.e., $\sigma(P,Q) = \max_i \sigma(P_i,Q_i)$ is δ. There are two priority policies π^* and $\tilde{\pi}$ possible in this network. Under these policies, we have $1 <_{\pi^*} 2$ and $2 <_{\tilde{\pi}} 1$. The distance between these two policies in the context of P is infinite, since $J_P^{\tilde{\pi}}(\{1\}) = -\infty$. We show that even for a small distance between models P and Q, i.e., small δ, policy $\tilde{\pi}$ can be selected as the optimal policy (due to its optimality in the context of Q). To prove this, we write the expected rewards

$$J_Q^{\pi^*}(\{2\}) = R - \frac{c_2}{p} \quad J_Q^{\pi^*}(\{1\}) = R - \frac{c_1 + c_2}{\delta + p}$$
$$J_Q^{\tilde{\pi}}(\{1\}) = R - \frac{c_1}{\delta} \quad J_Q^{\tilde{\pi}}(\{2\}) = R - \frac{c_2}{2p} - \frac{c_1}{2\delta}.$$

Since $c_2 \gg c_1$, there exists a (small) δ for which $\tilde{\pi}$ is selected as the optimal priority policy when the estimated distribution Q is assumed to describe the transmission success. On the other hand, $J_P^{\tilde{\pi}}(\{1\}) = -\infty$. Therefore, we have $d(\pi^*,\tilde{\pi}) = \infty$ even though $\sigma(P,Q) = \delta$.

This example illustrates that in general a small error in channel estimation can cause an unbounded decrease in performance.

IV. Sensitivity Analysis: Results

A. Analysis of $(\mathbf{P2})$

In this section, our goal is to bound the distance between two policies $\tilde{\pi}$ and π^* by a term proportional to the distance between the broadcast models P and Q. As illustrated in Example III.C, this is not possible in general. Thus, to obtain a positive result, we make the following assumption on the nature of the estimation error.

Assumption 1: For any node i such that $\tilde{\pi}(i) = i$, that is, $\tilde{\pi}$ does not retire at node i, there exists $\Delta_i \le 1$ such that

$$\frac{\sigma(P_i,Q_i)}{\sum_{S \in U_Q^{\tilde{\pi}}(i)} Q^i(S\,|\,i)} \le \Delta_i. \tag{9}$$

Intuitively Assumption 1 implies that the magnitude of the estimation error at each node can not be larger than the total estimated probability of routing the message "closer" to the destination. In other words, even under the suboptimal routing policy $\tilde{\pi}$ the transmission at each node has a true (according to P) positive probability of reaching a node of higher priority.

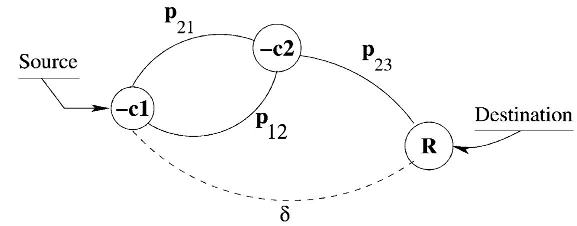

Fig. 1. Example 1.

Under the aforementioned assumption, we seek to bound the loss of performance by a term proportional to the estimation error. To do so, we first present the following definition which will simplify our notation.

Definition 6: For any measurable function $\nu: \mathcal{S} \longmapsto \mathbb{R}$ which is $P^i(\cdot\,|\,S)$-integrable, we define the *Markov operator*

$$P\nu(S,i) := \sum_{S' \in \mathcal{S}} P^i(S'\,|\,S)\nu(S'). \tag{10}$$

Using this definition, we can write the expected reward for priority policy π at state S, assuming $\pi(S) = i$ or r_i, as

$$J_P^{\pi}(\{i\}) = \begin{cases} -c_i + PJ_P^{\pi}(S,i), & \text{if } \pi \text{ transmits} \\ R_i, & \text{if } \pi \text{ retires.} \end{cases} \tag{11}$$

We begin by establishing Lemma 1 which relates the overall distance between two policies $\tilde{\pi}$ and π^* to the distance between the policies at each singleton $\{j\}$.

Lemma 1: $d_P(\pi^*,\tilde{\pi}) \le \max_j\{J_P^{\pi^*}(\{j\}) - J^{\tilde{\pi}})_Q(\{j\})\} + \max_j |J^{\tilde{\pi}})_Q(\{j\}) - J_P^{\tilde{\pi}}(\{j\})|$.

Proof: Let $i = \pi^*(S)$ and $j = \tilde{\pi}(S)$. Then

$$\begin{aligned} d_P(\pi^*,\tilde{\pi},S) &= \left|J_P^{\pi^*}(S) - J_P^{\tilde{\pi}}(S)\right| \\ &= J_P^{\pi^*}(S) - J_P^{\tilde{\pi}}(S) \\ &= J_P^{\pi^*}(\{\pi^*(S)\}) - J_P^{\tilde{\pi}}(\{\tilde{\pi}(S)\}) \\ &= J_P^{\pi^*}(\{i\}) - J_P^{\tilde{\pi}}(\{j\}) \\ &\le J_P^{\pi^*}(\{i\}) - J_Q^{\tilde{\pi}}(\{i\}) \\ &\quad + \left|J^{\tilde{\pi}})_Q(\{j\}) - J_P^{\tilde{\pi}}(\{j\})\right|. \end{aligned} \tag{12}$$

The second equality in (12) is true since π^* is optimal in the context of P, and the first inequality holds because policy $\tilde{\pi}$ is optimal in the context of Q.

The proof is complete after using the fact that $d_P(\pi^*,\tilde{\pi}) = \max_S d_P(\pi^*,\tilde{\pi},S)$. ■

To bound the performance loss, we develop upper bounds on each of the maxima that appear in Lemma 1. Bounds on the first term, i.e., $\max_j\{J_P^{\pi^*}(\{j\}) - J_Q^{\tilde{\pi}}(\{j\})\}$, are obtained via Lemma 2. Bounds on the second term, i.e., $\max_j |J_Q^{\tilde{\pi}}(\{j\}) - J_P^{\tilde{\pi}}(\{j\})|$, are obtained via Lemmas 3 and 4. The proofs of Lemmas 2 and 3 are given in Appendix I.

Lemma 2 establishes at each node j a relationship between $J_P^{\pi^*}(\{j\}) - J_Q^{\tilde{\pi}}(\{j\})$ and the distance between broadcast models P and Q.

Lemma 2: Assume that $\{\xi_1, \xi_2, \ldots, \xi_N\}$ is the nodes' strict priority ordering under policy π^*, i.e., $\xi_j >_{\pi^*} \xi_{j+1}$. Then, we have

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \alpha(\xi_j) \max_{k \leq j} \sigma\left(P_{\xi_k}, Q_{\xi_k}\right) \tag{13}$$

where $\alpha(\xi_j)$ is increasing in j and satisfies the recursion $\alpha(\xi_1) = 0$

$$\alpha(\xi_i) = \begin{cases} \frac{R_{\max}}{\sum_{S \in U_P^{\pi^*}(\xi_i)} P^{\xi_i}(S \mid \xi_i)} + \alpha(\xi_{i-1}), & \pi^*(\xi_i) = \xi_i \\ \alpha(\xi_{i-1}), & \pi^*(\xi_i) = r_{\xi_i}. \end{cases}$$

Lemma 3 establishes, at each node η_j, a relationship between $|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})|$ and the distance between the broadcast models P and Q.

Lemma 3: Assume that $\{\eta_1, \eta_2, \ldots, \eta_N\}$ is the nodes' strict priority ordering under policy $\tilde{\pi}$, i.e., $\eta_j >_{\tilde{\pi}} \eta_{j+1}$. We have

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \leq \gamma(\eta_j) \max_{k \leq j} \sigma\left(P_{\eta_k}, Q_{\eta_k}\right) \tag{14}$$

where $\gamma(\eta_j)$ is increasing in j and satisfies the recursion $\gamma(\eta_1) = 0$

$$\gamma(\eta_i) = \begin{cases} \frac{R_{\max}}{\sum_{S \in U_P^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S \mid \eta_i)} + \gamma(\eta_{i-1}), & \tilde{\pi}(\eta_i) = \eta_i \\ \gamma(\eta_{i-1}), & \tilde{\pi}(\eta_i) = r_{\eta_i}. \end{cases}$$

Remark: In Example III.C, we have $\sum_{S \in U_P^{\tilde{\pi}}(3)} P^3(S \mid 3) = 0$ which implies the unboundedness of the right-hand side of (14). This is consistent with the result provided in Section III-C, establishing that $|J_P^{\tilde{\pi}}(\{3\}) - J_Q^{\tilde{\pi}}(\{3\})| = \infty$. Note that Lemmas 2 and 3 do not require Assumption 1; hence, they can be applied to Example III.C.

Lemma 4: Under Assumption 1, we have

$$\left|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})\right| \leq \phi(\eta_j) \max_{k \leq j} \sigma\left(P_{\eta_k}, Q_{\eta_k}\right) \tag{15}$$

where $\phi(\eta_j)$ is increasing in j and satisfies the recursion $\phi(\eta_1) = 0$ as shown in the equation at the bottom of the page.

Proof:

Case 1) If $\tilde{\pi}(\eta_j) = r_{\eta_j}$, then $J_P^{\tilde{\pi}}(\{\eta_j\}) = J_Q^{\tilde{\pi}}(\eta_j) = R_{\eta_j}$. Hence

$$\left|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})\right| = 0. \tag{16}$$

On the other hand, the right-hand side of (15) is always a positive number. Hence

$$\left|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})\right| \leq \phi(\eta_j) \max_{k \leq j} \sigma(P_{\eta_k}, Q_{\eta_k}). \tag{17}$$

This completes the proof in Case 1).

Case 2) If $\tilde{\pi}(\eta_j) = \eta_j$, then Assumption 1 implies that

$$\frac{\sigma\left(P_{\eta_j}, Q_{\eta_j}\right)}{\sum_{S \in U_Q^{\tilde{\pi}}(\eta_j)} Q^{\eta_j}(S \mid \eta_j)} \leq \Delta_{\eta_j}. \tag{18}$$

On the other hand, we have

$$\begin{aligned} &\sum_{S \in U_P^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S \mid \eta_i) \\ &= \sum_{S \in U_P^{\tilde{\pi}}(\eta_i) \cup U_Q^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S \mid \eta_i) \geq \sum_{S \in L_Q^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S \mid \eta_i) \\ &= \sum_{S \in L_Q^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S \mid \eta_i) - \sum_{S \in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S \mid \eta_i) \\ &\quad + \sum_{S \in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S \mid \eta_i) \geq \sum_{S \in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S \mid \eta_i) \\ &\quad - \left| \sum_{S \in L_Q^{\tilde{\eta_i}}(\eta_i)} P^{\eta_i}(S \mid \eta_i) - \sum_{S \in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S \mid \eta_i) \right| \\ &\geq \sum_{S \in U_Q^{(}\tilde{\pi})(\eta_i)} Q^{\eta_i}(S \mid \eta_i) - \sigma(P_{\eta_i}, Q_{\eta_i}) \\ &= \sum_{S \in U_Q^{(}\tilde{\pi})(\eta_i)} Q^{\eta_i}(S \mid \eta_i) \left(1 - \frac{\sigma(P_{\eta_j}, Q_{\eta_j})}{\sum_{S \in U_Q^{\tilde{\pi}}(\eta_j)} Q^{\eta_j}(S \mid \eta_j)}\right) \\ &\geq \sum_{S \in U_Q^{(}\tilde{\pi})(\eta_i)} Q^{\eta_i}(S \mid \eta_i)(1 - \Delta_{\eta_i}) \\ &= (1 - \Delta_{\eta_i}) \sum_{S \in U_Q^{(}\tilde{\pi})(\eta_i)} Q^{\eta_i}(S \mid \eta_i) \end{aligned} \tag{19}$$

where the first equality holds since $P^{\eta_i}(S \mid \eta_i) = 0$ for $\forall S \in U_Q^{\tilde{\pi}}(\eta_i) - U_P^{\tilde{\pi}}(\eta_i)$. This is due to the fact that $U_Q^{\tilde{\pi}}(\eta_i) - U_P^{\tilde{\pi}}(\eta_i) = (\{S\colon \eta_i \in S \subset N_Q(\eta_i)\} - \{S\colon \eta_i \in S \subset N_P(\eta_i))\}) - 2^{\{\eta_i, \ldots, \eta_N\}}$, hence, for $\forall S \in U_Q^{\tilde{\pi}}(\eta_i) - U_P^{\tilde{\pi}}(\eta_i)$, then $S \not\subset \mathcal{N}_P(i)$, which implies $P^i(S \mid i) = 0$. The third inequality follows the definition of total variation metric, and the last inequality holds because of (18). The assertion of the lemma follows from (19) and Lemma 3. ■

Combining Lemmas 1, 2, and 4 we establish the following theorem, which summarizes one of the two main results of this paper.

Theorem 1: Under Assumption 1, we have

$$d_P(\pi^*, \tilde{\pi}) \leq R_{\max}(\Upsilon(P, \underline{0}) + \Upsilon(Q, \underline{\Delta})) \max_j \sigma(P_j, Q_j) \tag{20}$$

where the function $\Upsilon(A, \underline{\epsilon})$ denotes the optimal probabilistic connectivity for a broadcast model A, its corresponding optimal

$$\phi(\eta_i) = \begin{cases} \frac{R_{\max}}{(1 - \Delta_{\eta_i}) \sum_{S \in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S \mid \eta_i)} + \phi(\eta_{i-1}), & \text{if } \tilde{\pi}(\eta_i) = \eta_i \\ \phi(\eta_{i-1}), & \text{if } \tilde{\pi}(\eta_i) = r_{\eta_i}. \end{cases}$$

priority policy π, and the vector of bounds on estimation error denoted by $\underline{\epsilon} = [\epsilon_1, \epsilon_2, \ldots, \epsilon_N]$ and is defined as

$$\Upsilon(A, \underline{\epsilon}) = \sum_{j \in \Omega, \pi(j)=j} \frac{1}{(1-\epsilon_j)\sum_{S \in U_P^{\pi}(j)} A^j(S \mid j)}. \tag{21}$$

Proof: Combining Lemmas 1, 2, and 4, we have

$$\begin{aligned} d_P(\pi^*, \tilde{\pi}) &\leq \max_j \left\{ J_P^{\pi^*}(\{j\}) - J^{\tilde{\pi}})_Q(\{j\}) \right\} \\ &\quad + \max_j \left| J_Q^{\tilde{\pi}}(\{j\}) - J_P^{\tilde{\pi}}(\{j\}) \right| \\ &\leq \max_j \left\{ \alpha(\xi_j) \max_{k \leq j} \sigma\left(P_{\xi_k}, Q_{\xi_k}\right) \right\} \\ &\quad + \max_j \left\{ \phi(\eta_j) \max_{k \leq j} \sigma\left(P_{\eta_k}, Q_{\eta_k}\right) \right\} \\ &= (\alpha(\xi_N) + \phi(\eta_N)) \max_k \sigma(P_k, Q_k) \\ &= R_{\max}(\Upsilon(P, \underline{0}) + \Upsilon(Q, \underline{\Delta})) \max_k \sigma(P_k, Q_k) \end{aligned} \tag{22}$$

where the first inequality holds because of Lemma 1, the second inequality is a result of Lemmas 2 and 4, the first equality result from the definition of functions α_i and ϕ_i and their monotonicity in the index i, and the last equality follows from the definition of the function Υ. ■

The term $\Upsilon(P, \underline{0}) + \Upsilon(Q, \underline{\Delta})$ in (20) depends on the topology of the network under the true and the estimated broadcast models. This dependency provides insight into the study of sensitivity of the optimal routing policies with respect to the estimation error under various topological structures. It can be seen that loss of performance for networks where each transmission reaches a large set of higher priority nodes is smaller than for networks with "hot links," where the condition of a few links is critical in determining the ability of the routing policy to transfer the message to the destination. This is an important feature of the proposed optimal priority routing policy. We will elaborate on this in Section IV-B.

On the other hand, sensitivity analysis may be used to provide guidelines in designing online estimation algorithms with acceptable margin of error. In such applications, the dependency of our bounds on the true and estimated structures and topology of the network can create difficulty. In general and in a practical setting, such models are not known and cannot be exploited to design appropriate algorithms. Furthermore, in an *ad hoc* network it is undesirable to assume any particular topological structure. For these applications, we provide Theorem 2 to eliminate the dependency of our bound on the particular (and unknown) topology of the network. To do so, we need Lemma 5 and Corollaries 1 and 2 that follow.

Lemma 5 is a direct consequence of the results given in [4].

Lemma 5: Let π^* and $\tilde{\pi}$ be two optimal priority routing policies under broadcast models P and Q, respectively. Assume that $\pi^*(\{i\}) = i$ (respectively, $\tilde{\pi}(\{i\}) = i$), that is, policies π^* and $\tilde{\pi}$ do not retire when the state is $\{i\}$. Then

$$\sum_{S \in U_P^{\pi^*}(i)} P^i(S \mid i) \geq \frac{c_i}{R_{\max} - R_i} > 0 \tag{23}$$

(respectively, $\sum_{S \in U_Q^{\tilde{\pi}}(i)} Q^i(S \mid i) \geq (c_i)/(R_{\max} - R_i) > 0$).

Proof: $\pi^*(\{i\}) = i$ implies that

$$R_i \leq \frac{-c_i + \sum_{S \in U_P^{\pi^*}(i)} P^i(S \mid i) J_P^{\pi^*}(\{\pi^*(S)\})}{\sum_{S \in U_P^{\pi^*}(i)} P^i(S \mid i)} \tag{24}$$

On the other hand, for $\forall S \in \mathcal{S}$ and any policy π we have $J_P^{\pi}(S) \leq R_{\max}$. Hence

$$R_i \leq \frac{-c_i}{\sum_{S \in U_P^{\pi^*}(i)} P^i(S \mid i)} + R_{\max}. \tag{25}$$

This implies that $\sum_{S \in U_P^{\pi^*}(i)} P^i(S \mid i) \geq (c_i)/(R_{\max} - R_i) > 0$.

Similarly, if $\tilde{\pi}(\{i\}) = i$, $\sum_{S \in U_Q^{\tilde{\pi}}(i)} Q^i(S \mid i) \geq (c_i)/(R_{\max} - R_i) > 0$. The proof of Lemma 5 is now complete. ■

Corollary 1 of Lemma 2: Assume that $\{\xi_1, \xi_2, \ldots, \xi_N\}$ is the nodes' strict priority ordering under policy π^*. Then, we have

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \beta(\xi_j) \max_{k \leq j} \sigma(P_{\xi_k}, Q_{\xi_k}) \tag{26}$$

where $\beta(\xi_j)$ is increasing in j and satisfies the recursion

$$\beta(\xi_1) = 0 \quad \beta(\xi_i) = \frac{R_{\max}(R_{\max} - R_{\xi_i})}{c_{\xi_i}} + \beta(\xi_{i-1}). \tag{27}$$

Proof: The assertion of the corollary follows directly from Lemmas 2 and 5. ■

Corollary 2 of Lemma 2: Under Assumption 1, we have

$$\left| J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\}) \right| \leq \theta(\eta_j) \max_{k \leq j} \sigma\left(P_{\eta_k}, Q_{\eta_k}\right) \tag{28}$$

where $\theta(\eta_j)$ is increasing in j and satisfies the recursion

$$\theta(\eta_1) = 0 \quad \theta(\eta_i) = \frac{R_{\max}(R_{\max} - R_{\eta_i})}{c_{\eta_i}(1 - \Delta_{\eta_i})} + \theta(\eta_{i-1}). \tag{29}$$

Proof: The assertion of the corollary follows directly from Lemmas 4 and 5. ■

We now use Corollaries 1 and 2 to prove Theorem 2, which provides a bound on the error independently of the topology.

Theorem 2: Under Assumption 1, we have

$$d_P(\pi^*, \tilde{\pi}) \leq K \max_j \sigma(P_j, Q_j) \tag{30}$$

where

$$K = \sum_{j=1}^{N} \frac{R_{\max}(R_{\max} - R_j)(2 - \Delta_j)}{c_j(1 - \Delta_j)}. \tag{31}$$

Proof: Combining Lemma 1 and Corollaries 1 and 2, we have

$$\begin{aligned} d_P(\pi^*, \tilde{\pi}) &\leq \max_j \left\{ \beta(\xi_j) \max_{k \leq j} \sigma\left(P_{\xi_k}, Q_{\xi_k}\right) \right\} \\ &\quad + \max_j \left\{ \theta(\eta_j) \max_{k \leq j} \sigma\left(P_{\eta_k}, Q_{\eta_k}\right) \right\} \end{aligned}$$

$$\begin{aligned}
&\leq (\beta(\xi_N) + \theta(\eta_N)) \max_k \sigma(P_k, Q_k) \\
&\leq \left(\sum_{i=1}^{N} \frac{R_{\max}(R_{\max} - R_i)(2 - \Delta_i)}{c_i(1 - \Delta_i)} \right) \max_k \sigma(P_k, Q_k) \\
&= K \max_k \sigma(P_k, Q_k) \qquad (32)
\end{aligned}$$

where the first inequality holds because of Lemma 1, the second inequality is a result of Corollaries 1 and 2, and the first and second equalities result from the definition of functions β_i and θ_i and their monotonicity in the index i. ■

1) Sufficient Conditions for Assumption 1 to Hold: Assumption 1 is the minimum requirement needed to guarantee and obtain a finite bound on the sensitivity of the optimal index routing policy to the estimation error in the broadcast model. This assumption depends on the nature of the optimal policies (under the true model P and estimated model Q), hence, the topology of the network. Nevertheless, there are stronger conditions which are independent of the network topology, which are sufficient to guarantee Assumption 1, are easy to verify, and are given here.

Condition 1: For any node i, there exists $M_i < \infty$ such that

$$Q^i(S \,|\, i) - P^i(S \,|\, i) \leq M_i P^i(S \,|\, i). \qquad (33)$$

Condition 1 guarantees that there exists $\Delta_j = (M_j)/(1 + M_j)$ for which Assumption 1 is satisfied. Intuitively, Condition 1 has two significant implications. First, it implies that the network topology under the estimated broadcast model Q does not contain links which do not really exist, i.e., $\forall i \in \{1, 2, \ldots, N\}, \mathcal{N}_Q(i) \subset \mathcal{N}_P(i)$. Second, the condition implies that, whenever there is a link between nodes i and j, i.e., $P_{ij} > 0$, there is a finite bound M_i on the percentage of error in over-estimation of the quality of the link, i.e., $(Q_{ij} - P_{ij})/(P_{ij}) \leq M_i$. In other words, M_i specifies the maximum error percentage in the estimation of the quality of links connected to node i.

Condition 2: The estimation error is bounded by the following expression:

$$\sigma(P_k, Q_k) \leq \Delta_k \frac{c_k}{R_{\max} - R_k}. \qquad (34)$$

From Lemma 5 it follows that Condition 2 guarantees that Assumption 1 holds. Although Condition 2 is too restrictive, it can be checked without any knowledge of the probabilistic topology of the network and/or the structure of the optimal policies. Intuitively, this is a sufficient condition to guarantee a Lipschitz-type continuity of the performance, which in turn implies a bounded gradient around the true model P. Unlike Condition 1, Condition 2 bounds the absolute value of error in estimation rather than the error percentage.

B. Examples and Discussion

In this section we present three networks $(\Omega^1, \Omega^2, \Omega^3)$ given by Figs. 2–4 to illustrate how topology affects the sensitivity of the optimal routing policy with respect to the estimation error. Consider these three networks. We assume that successful transmissions along different links are independent. Hence, $(\mathbf{M})$ for network $\Omega^\omega, \omega = 1, 2, 3$, can be defined by $(N^\omega, Q^\omega, \underline{\mathbf{c}}^\omega, R^\omega)$ where Q^ω is a transition matrix whose (i, j)th element represents the probability of successful transmission from node i to j. We assume $R^1 = R^2 = R^3 = 300, N^1 = 5, N^2 = 7, N^3 = 9, \underline{\mathbf{c}}^1 = [1, 1, 1, 1, 1]$, $\underline{\mathbf{c}}^2 = [\underline{\mathbf{c}}^1, 1.5, 3]$, and $\underline{\mathbf{c}}^3 = [\underline{\mathbf{c}}^2, 2.2, 4]$. Consider routing a message from nodes 1 to 5 in networks $\Omega^1, \Omega^2, \Omega^3$. We assume that the estimated value of the transmission probabilities are

$$Q^1 = \begin{pmatrix} 1 & 0.3 & 0 & 0 & 0 \\ 0.1 & 1 & 0.5 & 0 & 0 \\ 0 & 0.1 & 1 & 0.6 & 0 \\ 0 & 0 & 0.2 & 1 & 0.5 \\ 0 & 0 & 0 & 0.2 & 1 \end{pmatrix}$$

$$Q^2 = \left(\begin{array}{ccccc|cc} & & & & & 0.6 & 0.5 \\ & & & & & 0 & 0 \\ & & Q^1 & & & 0.1 & 0.1 \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 & 1.0 \end{array} \right)$$

and

$$Q^3 = \left(\begin{array}{ccccccc|cc} & & & & & & & 0 & 0 \\ & & & & & & & 0.25 & 0.5 \\ & & & & & & & 0 & 0 \\ & & & Q^2 & & & & 0 & 0 \\ & & & & & & & 0 & 0 \\ & & & & & & & 0.45 & 0 \\ & & & & & & & 0 & 0.45 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 1.0 \end{array} \right).$$

Furthermore, we assume that the probabilities of successful transmission between each pair of nodes are known except for the link between nodes 2 and 3 in each network where there is an ϵ error in the estimation of the quality of the link. Hence, for any error value ϵ and $\omega = 1, 2, 3$, we have $P^\omega(\epsilon) = Q^\omega - \epsilon 1_{23}$, where P^i represents the true broadcast model for Network i, and 1_{ij} is a matrix whose only nonzero element is the (i, j)th element which is equal to 1. Notice that $\sigma(P^\omega, Q^\omega) = \sigma(P_2^\omega, Q_2^\omega) = \epsilon$. We vary the error ϵ from 0% to 100% of the estimated value of the link's transmission probability $Q(2, 3)$, and compare (1) the loss of performance in the three different networks, and (2) the bounds provided by Theorem 1 in each case. Furthermore, for comparison purposes, we provide the performance of a OSPF-type routing algorithm (see [1]–[3], [9], and [10]) where some form of shortest route is established as the minimum energy route. In this OSPF-type routing algorithm a full route with minimum expected energy is identified and set up, and all information between source and destination is transmitted on this fixed route. It can be shown [2] that, under the estimated model Q, the "minimum energy" route from node 1 to node 5 in all three networks is constructed as $(1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5)$. Note that as the error percentage at link $(2 \rightarrow 3)$ increases, the true cost of routing via this "minimum energy" route increases.

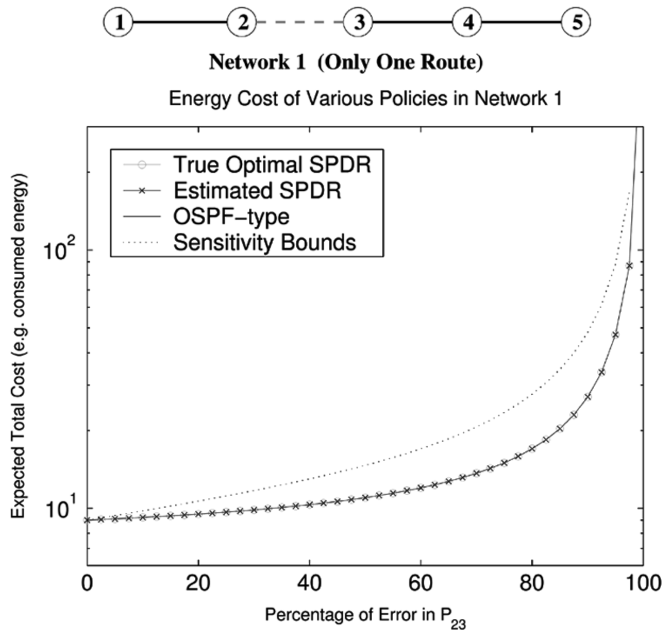

Fig. 2. Network Ω^1.

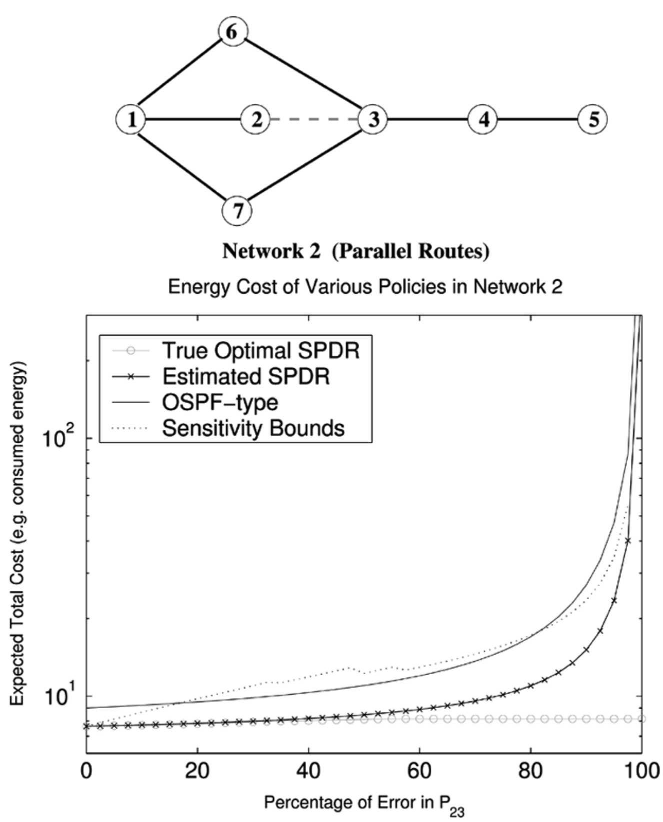

Fig. 3. Network Ω^2.

Before discussing the sensitivity of routing policies in these examples, we would like to point out a known advantage of using the sample-path-dependent routing policy (SPDR) proposed by [4]. As mentioned before, an important feature of the proposed optimal routing policy given in [4] is the fact that the route a message takes between source and destination depends

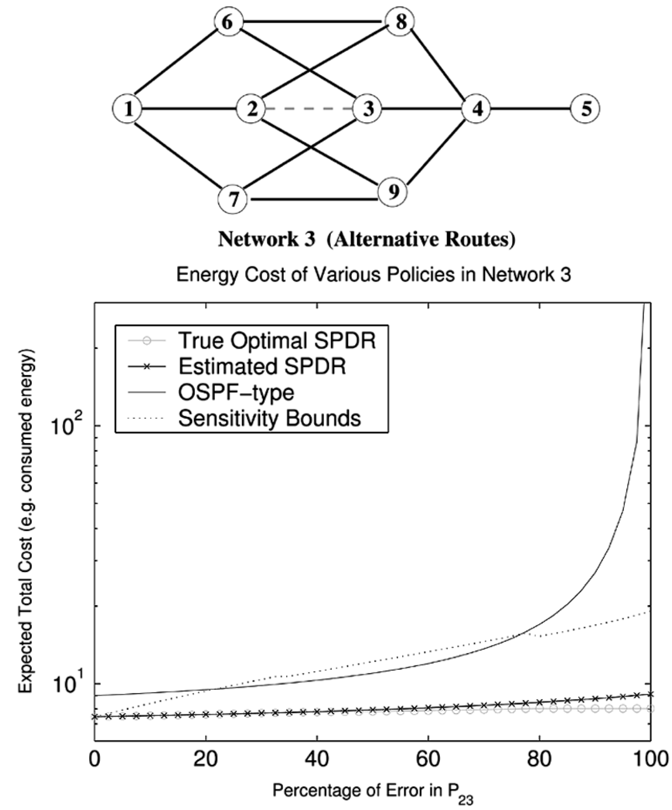

Fig. 4. Network Ω^3.

on the particular realizations of channels and transmission success. In other words, when implementing the proposed SPDR, the network layer avoids establishing a fixed route. It is known (see [4]) that due to this property, in well-connected networks the optimal SPDR policy shows an advantage in achieving lower expected cost over strategies which set up fixed routes. This is mainly because in well-connected networks, due to the presence of multiple alternative routes between source and destination, the optimal SPDR strategy avoids unnecessary retransmissions by choosing the best neighbor among those which have received the message. The graphs provided by Figs. 2–4 illustrate the aforementioned advantage.

We use networks $\Omega^1 - \Omega^3$ to discuss the nature of our sensitivity analysis. We have set up these examples to illustrate the effect of the network topology on the sensitivity of the performance of the optimal SPDR policy (network Ω^1 has the lowest connectivity, while Ω^3 is the most connected). In each case, we study the performance of the optimal SPDR strategy when the estimated probabilistic model includes an $\epsilon\%$ estimation error over a particular link on the minimum energy route. In addition, we assume that such error occurs in the form of over-estimation. For each network Ω^ω we plot the performance of policy $\tilde{\pi}^\omega$, which is the optimal (SPDR) priority policy associated with the estimated model $(N^\omega, Q^\omega, \underline{\mathbf{c}}^\omega, R^\omega)$. We provide the optimal cost, had the true model been known, as a bench-mark. Furthermore, we plot the bounds provided by Theorem 2.

The results (summarized in Figs. 2–4) can be summarized as follows. Network Ω^1 consists of a single route between the source and the destination, hence it is expected for all policies to demonstrate identical performance and robustness. In such a scenario, all routing policies rely on the failing link for their routing decisions. As the quality of the link decreases all policies fail to successfully transport the message from the source to the destination. Hence, as the quality of the link drops the cost of routing the message from nodes 1 to 5 grows due to the retransmission efforts. Notice that due to the retiring option the optimal cost, had the true model been known, is slightly below the cost of other policies. In network Ω^2, there are more than one route from source to destination. This feature of the network has no effect on the performance of OSPF-type routing strategies, since such strategies always select the route which, under model Q, is assumed to be the minimum energy route, i.e., $(1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5)$. The optimal priority (SPDR) policy shows better performance even in the presence of estimation error. Notice there is still only one route from node 2 to the destination, hence as the error goes to 100%, Δ_2 goes to zero and the right hand side of (20) becomes unbounded. In network Ω^3 where there exist alternative routes around the failing link, the optimal SPDR policy shows a high degree of robustness. In addition to capturing the effect of topology on the sensitivity of the optimal policy, these examples provide a rough idea on the tightness of the bounds provided by the right-hand side of (20). It can be seen that the obtained bound becomes tighter as $\max_i \Delta_i$ increases.

C. Distributed Computation of the Optimal Policy

As mentioned in Section I, the key feature of an *ad hoc* network is the absence of a central control or computation unit, and this underlines the importance of providing distributed algorithms for computation and implementation of an optimal routing policy. The authors in [4] and [5] provide various algorithms in which each node uses its local information in order to compute a local priority list of its neighbors. To a great extent these algorithms are, in their information structure, similar to the distributed Bellman–Ford algorithm (see [13] and [14]), and can be thought of the sample-path dependent extensions of the distributed Bellman–Ford algorithm. In all the distributed algorithms presented in [4] and [5], the priority list for each node is determined based on the node's local information. The local information at each node consists of the local broadcast model for the node, an (updated) estimate of its own expected reward, and (updated) estimates of its neighbors' optimal expected rewards. Each node communicates asynchronously and infinitely often with its neighbors. Through this communication process, each node transmits to its neighbors its own newly recomputed estimate of its optimal expected reward and generates its own priority list. The computation of the node's estimate of its own optimal expected reward is performed according to an update equation specified by the distributed algorithm. Thus, according to all the distributed algorithms in [4] and [5], a node can compute its (local) priority list based on the estimates it receives from its neighbors provided that it has an estimate of its local broadcast model. We are interested in studying the loss in performance of these algorithms when there are estimation errors at the local broadcast model.

In [4] and [5], it is shown that under the proposed distributed algorithms the estimates of the optimal expected reward of each node converge to their true values. Furthermore, it is proved that almost in all practical scenarios (i.e., when Assumption 3 is satisfied) this convergence occurs in finite time. Therefore, in almost all practical cases a stationary and optimal local index policy which is consistent with the optimal index policy described in Section II-D can be constructed in finite time. This implies that, given a sufficiently long time horizon, all the algorithms of [4] and [5] demonstrate identical performance loss in the presence of estimation errors in the broadcast model. To see why this is true, assume that the true system model is $(N, P, \underline{\mathbf{c}}, R)$ as before, and the vector $\Pi_t^a = \{\pi_t^1 \pi_t^2 \dots \pi_t^N\}$ represents a collection of optimal local routing decisions at nodes $1, 2, \dots, N$ at time t under distributed algorithm a (one of the three algorithms provided in [4]). Furthermore, assume that each node i estimates its local broadcast model, $\hat{P}(S/i)$, for each $S \subset \mathcal{N}(i)$ in a local fashion, based on all its communications, both control signals and messages, as well as its channel measurements. Construct an overall local broadcast model Q such that for $\forall i \in \{1, 2, \dots, N\}$ and $\forall S \subset \Omega$ we have

$$Q^i(S/i) = \begin{cases} \hat{P}(S/i), & \text{if } S \subset \mathcal{N}_{\hat{P}}(i) \\ 0, & \text{otherwise.} \end{cases} \qquad (35)$$

Assume that vector $\hat{\Pi}_t^a = \{\hat{\pi}_t^1 \hat{\pi}_t^2 \dots \hat{\pi}_t^N\}$, represents a collection of local routing decisions at nodes $1, 2, \dots, N$ at time t under distributed algorithm a and under the estimated model $\hat{P}$. For almost all practical cases (where Assumption 3 is satisfied) there exists a large horizon T_1 such that for $\forall t' > T_1$ $\hat{\Pi}_{t'}^a = \tilde{\pi}$, where $\tilde{\pi}$ is a stationary policy that is optimal for the centralized problem with model $(N, Q, \underline{\mathbf{c}}, R)$. This implies that, in the context of true model P, the performance of policy $\{\hat{\Pi}_{t'}^a\}_{t'=T_1}^{\infty}$, constructed by distributed algorithm a is independent of a and is equal to the performance of the stationary policy $\tilde{\pi}$. On the other hand, due to the optimality of the distributed algorithm a, there exists a finite horizon T_2 such that policy $\{\Pi_t^a\}_{t=T_2}^{\infty}$ is optimal in the context of the true model P; i.e., its performance is identical to that of the stationary policy π^*, where π^* is an optimal index policy for the centralized problem with the true model $(N, P, \underline{\mathbf{c}}, R)$. Hence, past horizon $T = \max\{T_1, T_2\}$, under any of the distributed algorithms of [4] and [5], the loss in performance due to estimation errors in the broadcast model is equal to the loss in performance of policy $\tilde{\pi}$. Such a loss has been calculated in Section IV-A. It is shown that under Assumptions 2 and 3, the performance loss is bounded by a term proportional to the estimation error, i.e., the distance between the true model P and estimated model Q.

Theorem 3, which follows, summarizes this discussion in a precise fashion under the following assumptions.

Assumption 2: For any node i such that $\tilde{\pi}(i) = i$, there exists $\Delta_i \leq 1$ such that

$$\frac{\sigma(P_i, Q_i)}{\sum_{S \in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S \mid \eta_i)} \leq \Delta_i. \qquad (36)$$

This assumption is identical to Assumption 1.

Assumption 3: For any pair of nodes i and j such that $i \in \mathcal{N}_{\hat{P}}(j)$, $J_{\hat{P}}^{\tilde{\pi}}(\{i\}) \neq J_{\hat{P}}^{\tilde{\pi}}(\{j\})$.

This assumption implies that no two neighboring nodes have the same expected reward (i.e., optimal index) to route a message to the destination. It holds in almost all practical scenarios. Under this assumption, the policies constructed by any of the three distributed algorithms in [4] and [5] converge to a stationary optimal index policy in finite time, i.e., $\hat{\Pi}_t^a = \tilde{\pi}$ for all t sufficiently large (see [5]).

Theorem 3: If Assumptions 2 and 3 hold and T is sufficiently large, then for $\forall t > T$ and for any of the distributed algorithms of [4], say a

$$d_P\left(\Pi_t^a, \hat{\Pi}_t^a\right) \leq R_{\max}(\Upsilon(P, \underline{0}) + \Upsilon(Q, \underline{\Delta})) \max_j \sigma(P_j, \hat{P}_j) \tag{37}$$

where the function $\Upsilon(A, \underline{\epsilon})$ is defined by (21).

For the proof of this theorem, see [17].

Notice that T in Theorem 3 is the time to compute and disseminate the local optimal policies. In other words $T \approx \tau_c$ where τ_c is defined in Section III-B. In general, T depends on the values of P, Q, and algorithm a. Therefore, the convergence rate of an algorithm a varies with the estimation error. This raises the issue of sensitivity of the convergence rate of various algorithms with respect to estimation errors in the local broadcast model. The answer to this issue combined with the result of Theorem 3 can shed light on the transient behavior of the distributed algorithms proposed by [4] and [5]. Hence, the effect of estimation errors on the convergence rate can be considered an important goal of future research.

V. Conclusion

In this paper, we examined the sensitivity of optimal routing policies in *ad hoc* wireless networks with respect to estimation errors in channel quality. We considered an *ad hoc* wireless network where the wireless links from each node to its neighbors are modeled by a probability distribution describing the local broadcast nature of wireless transmissions. These probability distributions are estimated in real-time. We investigated the impact of estimation errors on the performance of a set of proposed routing policies. Our results can be used as a guideline to design online estimation algorithms with acceptable margin of error. At the same time, our results provide a method to study the effect of the network topology on the robustness of a routing strategy with respect to errors in channel estimation. We provided a few numerical examples to illustrate such robustness issues. We believe that such results combined with a study of the statistical distribution of "typical" link estimation error can provide a guideline in designing topologies with desirable robustness to estimation errors and link failures.

In summary, the proposed sensitivity analysis provides a bound on the loss of performance as a result of a time-invariant error in the estimation of the quality of broadcast channels. In general, the estimation error is a dynamic variable whose variation depends on the dynamics of the network topology, the rate at which optimal routes are updated, and the convergence rate and error margin of the online estimation algorithm. We provided intuitive conditions on the time characteristics of the network's topological changes and the rate of convergence of the estimation algorithm, hence the dynamics of the estimation errors, to allow for our sensitivity result to be extended to more realistic cases.

We extended our sensitivity analysis to the case of the distributed computation of the optimal routing policy, when the distributed algorithms converge to a stationary policy. A further study should be conducted to investigate the impact of estimation error on the convergence rate of various distributed algorithms.

We would like to point out that in scenarios where changes in the topology and structure of the network are relatively rare, which implies a sufficiently fast recovery of information and reconstruction of an optimal routing strategy, the result of our sensitivity analysis can be used to intuitively predict the transient behavior of various routing strategies when implemented in real systems. In other words, in the case of networks with rare topological changes, the result of sensitivity analysis can provide a bound on the transient performance of proposed routing strategies.

Appendix I
Proof of Lemmas 2 and 3

Now, we provide the full proof for Lemma 2; the proof of Lemma 3 is very similar (for the complete proof, see [17]).

Lemma 2: Assume that $\{\xi_1, \xi_2, \ldots, \xi_N\}$ is the strict priority ordering under policy π^*, i.e., for $\forall j \ \xi_j >_{\pi^*} \xi_{j+1}$. Then, we have

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \alpha(\xi_j) \max_{k \leq j} \sigma\left(P_{\xi_k}, Q_{\xi_k}\right) \tag{38}$$

where $\alpha(\xi_j)$ is increasing in j and satisfies the recursion $\alpha(\xi_1) = 0$

$$\alpha(\xi_i) = \begin{cases} \frac{R_{\max}}{\sum_{S \in U_P^{\pi^*}(\xi_i)} P^{\xi_i}(S \mid \xi_i)} + \alpha(\xi_{i-1}), & \pi^*(\xi_i) = \xi_i \\ \alpha(\xi_{i-1}), & \pi^*(\xi_i) = r_{\xi_i}. \end{cases}$$

Proof: We prove the lemma by induction on the index i of the strict ordering $\{\xi_1, \xi_2, \ldots, \xi_N\}$.

__Basis of Induction:__ Since π^* is an optimal routing policy for a problem associated with the tuple $(N, P, \underline{c}, R)$ we have $J_P^{\pi^*}(\{\xi_1\}) = R_{\max}$. Similarly, $\tilde{\pi}$ is an optimal routing policy under Q and $J_Q^{\tilde{\pi}}(\{\xi_1\}) = R_{\max}$ (see [4, Sec. 3.3.2]). Hence, we have $J_P^{\pi^*}(\{\xi_1\}) - J_Q^{\tilde{\pi}}(\{\xi_1\}) = 0$.

__Induction Step:__ We assume that

$$J_P^{\pi^*}(\{\xi_{j-1}\}) - J_Q^{\tilde{\pi}}(\{\xi_{j-1}\}) \leq \alpha(\xi_{j-1}) \max_{k \leq j-1} \sigma(P_{\xi_k}, Q_{\xi_k}). \tag{39}$$

We need to show that

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \alpha(\xi_j) \max_{k \leq j} \sigma(P_{\xi_k}, Q_{\xi_k}) \tag{40}$$

where $\alpha(\xi_j) = (R_{\max})/(\sum_{S \in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S \mid \xi_j)) + \alpha(\xi_{j-1})$.

Case 1) If $\pi^*(\xi_j) = r_{\xi_j}$, then $J_P^{\pi^*}(\{\xi_j\}) = R_{\xi_j}$. Hence

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) = R_{\xi_j} - \max\left\{R_{\xi_j}, -c_{\xi_j} + QJ_Q^{\tilde{\pi}}(\{\xi_j\},\xi_j)\right\} \le 0. \quad (41)$$

On the other hand, the right-hand side of (40) is always a positive number. Hence

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \le \alpha(\xi_j) \max_{k\le j} \sigma(P_{\xi_k}, Q_{\xi_k}). \quad (42)$$

This completes the induction step in Case 1).

Case 2) If $\pi^*(\xi_j) = \xi_j$, then $J_P^{\pi^*}(\{\xi_j\}) = -c_{\xi_j} + PJ_P^{\pi^*}(\{\xi_j\},\xi_j)$. Hence

$$\begin{aligned}
&J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\})\\
&= -c_{\xi_j} + PJ_P^{\pi^*}(\{\xi_j\},\xi_j)\\
&\quad - \max\left\{R_{\xi_j}, -c_{\xi_j} + QJ_Q^{\tilde{\pi}}(\{\xi_j\},\xi_j)\right\}\\
&\le PJ_P^{\pi^*}(\{\xi_j\},\xi_j) - QJ_Q^{\tilde{\pi}}(\{\xi_j\},\xi_j)\\
&= PJ_P^{\pi^*}(\{\xi_j\},\xi_j) - PJ_Q^{\tilde{\pi}}(\{\xi_j\},\xi_j) + PJ_Q^{\tilde{\pi}}(\{\xi_j\},\xi_j)\\
&\quad - QJ_Q^{\tilde{\pi}}(\{\xi_j\},\xi_j)\\
&\le (\sup_S J_Q^{\tilde{\pi}}(S) - \inf_S J_Q^{\tilde{\pi}}(S))\sigma(P_{\xi_j}, Q_{\xi_j}) + PJ_P^{\pi^*}(\{\xi_j\},\xi_j)\\
&\quad - PJ_Q^{\tilde{\pi}}(\{\xi_j\},\xi_j)\\
&\le R_{\max}\sigma(P_{\xi_j}, Q_{\xi_j})P\left(J_P^{\pi^*}(\{\xi_j\},\xi_j)J_Q^{\tilde{\pi}}(\{\xi_j\},\xi_j)\right)\\
&= R_{\max}\sigma(P_{\xi_j}, Q_{\xi_j})\\
&\quad + \sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j)\left(J_P^{\pi^*}(\{\pi^*(S)\}) - J_Q^{\tilde{\pi}}(\{\tilde{\pi}(S)\})\right)\\
&\quad + \sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j)\left(J_P^{\pi^*}(\{\pi^*(S)\}) - J_Q^{\tilde{\pi}}(\{\tilde{\pi}(S)\})\right)\\
&\le R_{\max}\sigma(P_{\xi_j}, Q_{\xi_j})\\
&\quad + \sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j)\left(J_P^{\pi^*}(\{\pi^*(S)\}) - J_Q^{\tilde{\pi}}(\{\pi^*(S)\})\right)\\
&\quad + \sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j)\left(J_P^{\pi^*}(\{\pi^*(S)\}) - J_Q^{\tilde{\pi}}(\{\pi^*(S)\})\right)\\
&\le R_{\max}\sigma(P_{\xi_j}, Q_{\xi_j})\\
&\quad + \left(J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\})\right) \sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j)\\
&\quad + \sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j)\alpha(\xi_{j-1}).\max_{k\le j-1}\sigma(P_{\xi_k}, Q_{\xi_k})\\
&\le R_{\max}\sigma(P_{\xi_j}, Q_{\xi_j})\\
&\quad + \left(J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\})\right) \sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j)\\
&\quad + \alpha(\xi_{j-1})\max_{k\le j-1}\sigma(P_{\xi_k}, Q_{\xi_k}) \sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j) \quad (43)
\end{aligned}$$

where the second inequality is proved in [17, Fact 7, App. II] , the third inequality holds since an the expected reward of an optimal policy at each node is bounded in the interval $[0, R_{\max}]$ and we assume that $\tilde{\pi}$ is an optimal policy in the context of Q, the fourth inequality is true since $\tilde{\pi}$ is an optimal routing policy associated with the tuple $(N, Q, \underline{\mathbf{c}}, R)$ (i.e., $J_Q^{\tilde{\pi}}(\{\tilde{\pi}(S)\}) \ge J_Q^{\tilde{\pi}}(\{\pi^*(S)\})$), the fifth inequality holds due to the induction hypothesis and the fact that for all S such that $\xi_j \in S$ and $\pi^*(S) \ne \xi_j$ we have $\pi^*(S) \in \{\xi_1, \xi_2, \ldots, \xi_{j-1}\}$, and the last inequality holds since $\alpha(\xi_l)$ is increasing in l.

Now, we solve (43) for $J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\})$, and replacing $1 - \sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j)$ by $\sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S\,|\,\xi_j)$, we have

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\})\alpha(\xi_j)\max_{k\le j}\sigma(P_{\xi_k}, Q_{\xi_k}). \quad (44)$$

The proof of the induction step in Case 2) is now complete. Hence, the assertion of the lemma is true. ■

REFERENCES

[1] C. E. Perkins, *Ad Hoc Networking*. Reading, MA: Addison Wesley, 2001.

[2] S. Benarjee and A. Misra, "Minimum energy paths for reliable communication in multi-hop wireless networks," Dept. Comput. Sci., Univ. Maryland, College Park, MD, Tech. Rep. CS-TR 4315, Dec. 2001.

[3] A. Misra and S. Banerjee, "MRPC: Maximizing network lifetime for reliable routing in wireless environment," in *Proc. 2002 IEEE Wireless Communication and Networking Conference (WCNC02)*, vol. 2, 2002, pp. 800–806.

[4] C. G. Lott and D. Teneketzis, "Optimal routing algorithms in *ad hoc* wireless networks," in *Proc. 38th Allerton Conf. Communication, Control, and Computing*, vol. 3, 2000, pp. 123–132.

[5] C. Lott and D. Teneketzis, "Stochastic routing in *ad hoc* wireless networks," Commun. Signal Processing Lab (CSPL), EECS Dept., Univ. Michigan, Ann Arbor, MI, Group Rep. CSPL-TR-331, Feb. 2001.

[6] C. G. Lott and D. Teneketzis, "Stochastic routing in *ad hoc* wireless networks," in *Proc. IEEE Conf. Decision and Control*, vol. 3, 2000, pp. 2302–2307.

[7] J. G. Proakis and M. Salehi, *Communication Systems Engineering*. Upper Saddle River, NJ: Prentice-Hall, 2002.

[8] R. Kumar and P. Varaiya, *Stochastic Control*. New York: Prentice-Hall, 1986.

[9] J. Kurose and K. Ross, *Computer Networking*. Reading, MA: Addison-Wesley, 2000.

[10] J. Gomez, M. Naghshineh, A. Campbell, and C. Bisdikian, "PARO: A power-aware routing optimization scheme for mobile *ad hoc* networks," *Wireless Networks*, vol. 9, no. 5, Sept. 2002.

[11] A. Müller, "How does the value function of a Markov decision process depend on the transition probabilities?," *Math. Oper. Res.*, vol. 22, no. 4, pp. 872–885, Nov. 1997.

[12] G. B. Folland, *Real Analysis*. New York: Wiley, 1984.

[13] D. T. Bertsekas and R. Gallager, *Data Networks*. Upper Saddle River, NJ: Prentice-Hall, 1987.

[14] J. M. Jaffe and F. Moss, "A responsive distributed routing algorithm for computer networks," *IEEE Trans. Commun.*, vol. 30, pp. 1758–1762, July 1982.

[15] S. T. Rachev, *Probability Metrics and the Stability of Stochastic Models*. New York: Wiley, 1991.

[16] M. Shaked and J. G. Shanthikumar, *Stochastic Orders and Their Applications*. New York: Academic, 1994.

[17] T. Javidi and D. Teneketzis, "Sensitivity analysis for optimal routing in *ad hoc* wireless networks," EECS Dept., Univ. Michigan, Ann Arbor, MI, CSPL Group Rep. CSPL-TR-332, Jan. 2002.

Tara Javidi studied electrical engineering at the Sharif University of Technology, Tehran, Iran, from 1992 to 1996. She received the M.S. degrees in electrical engineering: systems and applied mathematics: stochastics, and the Ph.D. degree in electrical engineering and computer science, all from the University of Michigan, Ann Arbor, in 1998, 1999, and 2002, respectively.

She is currently an Assistant Professor in the Electrical Engineering Department, the University of Washington, Seattle. Her research interests are in communication networks, stochastic resource allocation, and wireless communication.

Dr. Javidi is a recipient of the 2004 National Science Foundation CAREER Award.

Demosthenis Teneketzis received the diploma in electrical engineering from the University of Patras, Patras, Greece, and the M.S., E.E., and Ph.D. degrees, all in electrical engineering, from the Massachusetts Institute of Technology, Cambridge, in 1974, 1976, 1977, and 1979, respectively.

He is currently a Professor of Electrical Engineering and Computer Science at the University of Michigan, Ann Arbor. In winter and spring 1992, he was a Visiting Professor at the Swiss Federal Institute of Technology, (ETH), Zurich, Switzerland. Prior to joining the University of Michigan, he worked for Systems Control, Inc., Palo Alto, CA, and Alphatech, Inc., Burlington, MA. His research interests are in stochastic control, decentralized systems, queueing and communication networks, stochastic scheduling and resource allocation problems, mathematical economics, and discrete-event systems.