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Measurement Scheduling for Recursive Team Estimation[1]

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Abstract. We consider a decentralized LQG measurement scheduling problem in which every measurement is costly, no communication between observers is permitted, and the observers' estimation errors are coupled quadratically. This setup, motivated by considerations from organization theory, models measurement scheduling problems in which cost, bandwidth, or security constraints necessitate that estimates be decentralized, although their errors are coupled. We show that, unlike the centralized case, in the decentralized case the problem of optimizing the time integral of the measurement cost and the quadratic estimation error is fundamentally stochastic, and we characterize the ϵ-optimal open-loop schedules as chattering solutions of a deterministic Lagrange optimal control problem. Using a numerical example, we describe also how this deterministic optimal control problem can be solved by nonlinear programming.

Key Words. Measurement scheduling, decentralized estimation, team theory, chattering controls.

1. Introduction

In real-world estimation problems, every measurement has an intrinsic cost. When a variety of measurements are possible, one would like to schedule the measurements, on-line or off-line, to minimize an objective modeling the tradeoff between measurement error and measurement cost. It is well

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known that open-loop measurement schedules are optimal over the class of closed-loop schedules for single-observer centralized problems in which the dynamics is linear, the objective is quadratic, and the noise is Gaussian (LQG problems); see Refs. 1–4. This result is an immediate consequence of the fact that, for open-loop schedules, the error covariance of the underlying Kalman–Bucy estimator is data independent. It follows that the conditional error covariance and measurement cost incurred by any realization of any closed-loop measurement schedule is the same as the unconditional error covariance and measurement cost incurred by some open-loop schedule. Hence, some open-loop schedule must perform at least as well as every closed-loop schedule. A similar result holds for multi-observer LQG problems in which limited continuous communication between observers and an estimate fusion center is permitted [i.e., quasi-decentralized problems (Ref. 5)].

In this paper, we consider a multi-observer LQG measurement scheduling problem in which no communication is permitted, but the observers' estimation errors are coupled quadratically [a dynamic team decision problem (Refs. 6, 7)]. This setup, motivated by considerations from organization theory (Ref. 8), models measurement scheduling problems in which cost, bandwidth, or security constraints necessitate that estimates be decentralized, although their errors are coupled. Such coupling arises, for instance, in decentralized tracking problems in which the cost and accuracy of local estimates must be balanced against the cost of systematic errors, e.g., all estimators simultaneously misjudging target position in the same direction. We show that, unlike the centralized and quasi-decentralized cases, the decentralized scheduling problem is fundamentally stochastic, and we characterize the ϵ-optimal open-loop schedules as chattering solutions of a deterministic Lagrange optimal control problem. Using a numerical example, we also describe how this deterministic optimal control problem can be solved by nonlinear programming. Our results follow from the recursive structure of team-optimal estimators (Ref. 9) and two properties of relaxed controllers (Ref. 10). The fundamental stochastic nature of the decentralized problem is a consequence of the fact that the team-optimal estimation error, although Gaussian, is not orthogonal to any of the observers' observations.

2. Problem Statement

Consider a linear dynamic system with a n-dimensional state process $\{x_t, t \geq t_0\}$ that satisfies

$$dx_t = A_t x_t \, dt + d\xi_t, \qquad x_{t_0} = x_0. \tag{1}$$

Here, $\{\xi_t, t \geq t_0\}$ is a n-dimensional zero mean Wiener process with covariance

$$E[\xi_t \xi_s^T] := \int_{t_0}^{\min\{t,s\}} \Sigma_{\xi\xi;r}\, dr, \tag{2}$$

x_0 is a zero mean Gaussian n-vector with covariance $\Sigma_{x_0x_0}$ that is uncorrelated with $\{\xi_t, t \geq t_0\}$, and A_t and $\Sigma_{\xi\xi;t}$ are appropriately dimensioned, continuously differentiable, real-valued, matrix functions. Under these conditions, (1) admits a unique, mean-square continuous, Gaussian solution (Ref. 11, Theorems 8.1.5, 7.1.2, and 8.2.10).

Suppose that $\{x_t, t \geq t_0\}$ is monitored by N observers and that, for all $t \geq t_0$, the kth observer must use one of M^k distinct, costly, noisy, linear, measurement devices, indexed by v_t^k, to make its m^k-dimensional observation,

$$dy_t^k = C_t^k(v_t^k)x_t\, dt + d\omega_t^k(v_t^k), \qquad t \geq t_0. \tag{3}$$

Here, $C_t^k(v_t^k)$ models the dynamics of the kth observer's v_t^kth measurement; it is assumed to be a continuously differentiable, real-valued, matrix function of appropriate dimension; $\{\omega_t^k, t \geq t_0\}$ models the m^k-dimensional zero mean Wiener noise process associated with this measurement. The noises are uncorrelated with x_0 and have cross covariance

$$E[\omega_t^i(v_t^i)\omega_s^{jT}(v_s^j)] := \int_{t_0}^{\min\{t,s\}} \Sigma_{\omega^i\omega^j;r}(v_r^i v_r^j)\, dr, \tag{4}$$

where $\Sigma_{\omega^i\omega^j;t}(v_t^i, v_t^j)$ is an appropriately dimensioned continuously differentiable, real-valued, matrix function that is positive definite for all $i=j$.

It is assumed that the observers do not share any information, but have perfect recall of all that they have observed. Specifically, the information sets of the kth observer just prior to and just after its measurement at time $t \geq t_0$ are assumed to be

$$\Lambda_{t-}^k := \{dy_s^k : s \in [t_0, t)\}, \qquad \Lambda_t^k := \{dy_s^k : s \in [t_0, t]\}. \tag{5}$$

Let Γ_t^k and Φ_t^k denote the sets of $\sigma(\Lambda_{t-}^k)$-measurable measurement schedules $\gamma_t^k, \gamma_t^k(\Lambda_{t-}^k) \in \{1, \ldots, M^k\}$ and $\sigma(\Lambda_t^k)$-measurable estimators ϕ_t^k, $\phi_t^k(\Lambda_t^k) \in \mathbb{R}^n$, available to the kth observer at time $t \geq t_0$. The objective is to identify collective measurement scheduling and estimation policies,

$$\gamma := \{\gamma_t := (\gamma_t^1, \ldots, \gamma_t^N) : t \in [t_0, t_f]\} \in \Gamma := \left\{\Gamma_t := \prod_{i=1}^N \Gamma_t^i : t \in [t_0, t_f]\right\}, \tag{6}$$

$$\phi := \{\phi_t := (\phi_t^1, \ldots, \phi_t^N) : t \in [t_0, t_f]\} \in \Phi := \left\{\Phi_t := \prod_{i=1}^N \Phi_t^i : t \in [t_0, t_f]\right\}, \tag{7}$$

that minimize the expected value of a cost function that is the sum of a term,

$$J_m(\gamma) := \int_{t_0}^{t_f} \sum_{i=1}^{N} q_t^i(v_t^i)\, dt, \tag{8}$$

modeling the observers' collective measurement cost under measurement schedule γ, and a term,

$$J_e(\gamma, \phi) := \int_{t_0}^{t_f} \begin{bmatrix} x_t - \hat{x}_t^1 \\ \vdots \\ x_t - \hat{x}_t^N \end{bmatrix}^T Q \begin{bmatrix} x_t - \hat{x}_t^1 \\ \vdots \\ x_t - \hat{x}_t^N \end{bmatrix} dt, \tag{9}$$

modeling the observers' collective estimation error when the observers' estimators are $\hat{x}_t^i = \phi_t^i(\Lambda_t^i)$, for $i = 1, \ldots, N$. Here, the measurement cost $q_t^i(k)$ is a real, nonnegative, continuously differentiable time function for all $i = 1, \ldots, N$ and $k = 1, \ldots, M^i$, and the coupling matrix $Q \in \mathbb{R}^{Nn \times Nn}$ is assumed to be symmetric and positive definite.

An ϵ-optimal solution to this partially-nested dynamic team problem (Refs. 6, 12) is a pair $\{\gamma^\epsilon \in \Gamma, \phi^\epsilon \in \Phi\}$ such that, for $\epsilon > 0$,

$$E^{\gamma^\epsilon, \phi^\epsilon}[J_m(\gamma^\epsilon) + J_e(\gamma^\epsilon, \phi^\epsilon)] < \inf_{\Gamma, \Phi} E^{\gamma, \phi}[J_m(\gamma) + J_e(\gamma, \phi)] + \epsilon. \tag{10}$$

We accept ϵ-optimal solutions because, in general, infimal solutions may correspond to inadmissible mixtures of pure schedules, i.e., schedules that chatter between several measurement configurations arbitrarily fast (Ref. 10).

To interpret the coupling introduced by Q, note that, when Q is partitioned into N^2 submatrices $Q^{ij} \in \mathbb{R}^{n \times n}$, (9) can be rewritten as

$$\int_{t_0}^{t_f} \sum_{i,j=1}^{N,N} ([x_t - \hat{x}_t^i]^T Q^{ij} [x_t - \hat{x}_t^j])\, dt. \tag{11}$$

Hence the Q^{ij} can be viewed as weighting the relative importance of local errors versus system-wide errors. For instance, when the off-diagonal Q^{ij} terms are identity matrices, the error terms are simply dot products, and systematic errors (errors in the same half-plane) are discouraged (Ref. 7). When the off-diagonal Q^{ij} terms are zero, the observer's estimates are uncoupled, and the problem decomposes into N single-observer problems.

The motivation for considering such problems comes from organization theory (Ref. 8). In many organizations, the compartmentalization of information forbids information sharing; yet, there is need for coordinated decision-making. For instance, as noted in the introdution, in decentralized tracking problems, it may be important to balance the cost and accuracy of local estimates against the cost of all estimators simultaneously misjudging the target position in the same direction. In the present problem, it is the quadratic coupling of the observers' estimation errors that models this tradeoff.

3. Team Estimation

To clarify the issues underlying the determination of optimal decentralized estimation and measurement scheduling policies, we consider first the quadratic team estimation problem that arises when the observers' measurement scheduling policies are open-loop and fixed. The results obtained are essentially those of Barta (Ref. 9). We summarize them here, in our notation, because they will be needed later, and because Ref. 9 is unpublished.

Let Θ denote the set of open-loop measurement scheduling policies in Γ. For fixed $\theta \in \Theta$, the observers estimation problem is

$$\inf_{\Phi} E^{\theta,\phi}[J_e(\theta, \phi)] = \inf_{\Phi} E^{\theta,\phi}\left[\int_{t_0}^{t_f} \begin{bmatrix} x_t - \hat{x}_t^1 \\ \vdots \\ x_t - \hat{x}_t^N \end{bmatrix}^T Q \begin{bmatrix} x_t - \hat{x}_t^1 \\ \vdots \\ x_t - \hat{x}_t^N \end{bmatrix} dt\right]. \tag{12}$$

Because x_t is uncontrolled, and because the mean-square continuity of x_t ensures that the expectation in (12) is finite for all nontrivial ϕ, an interchange of $\inf E^{\theta,\phi}$ and $\int$ is justified by the Fubini theorem. Thus, we have the following simple result.

Lemma 3.1. Solving (12) is equivalent to solving

$$\inf_{\Phi_t} E^{\theta,\phi}\left[\begin{bmatrix} x_t - \hat{x}_t^1 \\ \vdots \\ x_t - \hat{x}_t^N \end{bmatrix}^T Q \begin{bmatrix} x_t - \hat{x}_t^1 \\ \vdots \\ x_t - \hat{x}_t^N \end{bmatrix}\right], \qquad \text{for all } t \in [t_0, t_f]. \tag{13}$$

Because $(x_t^T, \ldots, x_t^T)^T$ is a Nn-dimensional functional in the Hilbert space $\mathscr{H}^{Nn}$ of Nn-vectors of square-integrable random variables with inner product

$$(x, y) := E[x^T Q y], \tag{14}$$

solving (13) is in turn equivalent to finding the Hilbert space projection of $(x_t^T, \ldots, x_t^T)^T$ on the subspace $\mathscr{Y}_t^{Nn}$ of $\mathscr{H}^{Nn}$ containing those vectors whose $(k-1)n+1$ to kn elements are measurable functionals of Λ_t^k (Ref. 7). The Hilbert space projection theorem ensures that this problem admits a unique solution, while the problem's partial nestedness and jointly Gaussian observations ensure, by the partial nestedness theorem in Ref. 6 and the quadratic team decision theorem in Ref. 7, that this solution is linear in the observers' observations. Hence, we have the following result.

Theorem 3.1. (Ref. 6, Theorem 2). The optimization problem (12) admits a unique solution $\phi^* = \{\phi_t^{*T} = (\phi_t^{*1T}, \ldots, \phi_t^{*NT}) : t \in [t_0, t_f]\}$ in which each estimator ϕ_t^{*k} is a linear functional of Λ_t^k.

As detailed in Ref. 9, one approach to deriving recursive expressions for these linear estimators is to attempt to parallel the innovations derivation of the classical Kalman–Bucy filter (Ref. 13). To this end, it is useful to note that (13) is a special case of the more general quadratic team estimation problem

$$\inf_{\Psi_t} \operatorname{tr} E^{\theta,\psi}[[X_t - \hat{X}_t]Q[X_t - \hat{X}_t]^T], \tag{15}$$

where the $(Nn^2 \times Nn)$-dimensional state process $\{X_t, t \geq t_0\}$ satisfies

$$dX_t = \mathscr{A}_t X_t\, dt + d\Xi_t, \qquad X_{t_0} = X_0, \tag{16}$$

for

$$X_0 := \operatorname{diag}[x_0, \ldots, x_0]_{Nn^2 \times Nn}, \tag{17}$$

$$\mathscr{A}_t := \operatorname{diag}[A_t, \ldots, A_t]_{Nn^2 \times Nn^2}, \tag{18}$$

$$\Xi_t := \operatorname{diag}[\xi_t, \ldots, \xi_t]_{Nn^2 \times Nn}, \tag{19}$$

and the estimate $\hat{X}_t$ is determined by a functional ψ_t in Ψ_t, the space of linear functionals of

$$dY_t = \mathscr{C}_t(\theta_t) X_t\, dt + d\Omega_t, \tag{20}$$

where

$$\mathscr{C}_t(\theta_t) := \operatorname{diag}[\underbrace{C_t^1(\theta_t^1), \ldots, C_t^1(\theta_t^1)}_{n \text{ times}}, \ldots, \underbrace{C_t^N(\theta_t^N), \ldots, C_t^N(\theta_t^N)}_{n \text{ times}}]_{n\sum_{i=1}^N m^i \times Nn^2}, \tag{21}$$

$$\Omega_t(\theta_t) := \operatorname{diag}[\underbrace{\omega_t^1(\theta_t^1), \ldots, \omega_t^1(\theta_t^1)}_{n \text{ times}}, \ldots, \underbrace{\omega_t^N(\theta_t^N), \ldots, \omega_t^N(\theta_t^N)}_{n \text{ times}}]_{n\sum_{i=1}^N m^i \times Nn}, \tag{22}$$

for $\theta_t := (\theta_t^1, \ldots, \theta_t^N)$.

To see the relation between (15) and (13), note that solving (15) is equivalent to finding the projection of the $(Nn^2 \times Nn)$-dimensional functional $X_t \in \mathscr{H}^{Nn^2 \times Nn}$ on $\mathscr{Y}_t^{Nn^2 \times Nn}$, where $\mathscr{H}^{Nn^2 \times Nn}$ denotes the Hilbert space of $(Nn^2 \times Nn)$-dimensional matrices of square-integrable random variables with inner product

$$\langle X, Y \rangle := \operatorname{tr} E[XQY^T], \tag{23}$$

and $\mathscr{Y}_t^{Nn^2 \times Nn}$ denotes the subspace of $\mathscr{H}^{Nn^2 \times Nn}$ for which columns $(k-1)n+1$ to kn of X are measurable functionals of Λ_t^k (Ref. 9). By the Hilbert space projection theorem, the infimum in (15) over $\mathscr{Y}_t^{Nn^2 \times Nn}$ is achieved by the unique $\hat{X}_t^* \in \mathscr{Y}_t^{Nn^2 \times Nn}$ satisfying the orthogonality condition

$$\langle X_t - \hat{X}_t^*, D_t \rangle = 0, \qquad \text{for all } D_t \in \mathscr{Y}_t^{Nn^2 \times Nn}. \tag{24}$$

Because $KD_t \in \mathscr{Y}_t^{Nn^2 \times Nn}$ for any real $(Nn^2 \times Nn^2)$-dimensional matrix K, (24) implies that

$$\begin{aligned} \langle X_t - \hat{X}_t^*, KD_t \rangle &:= \operatorname{tr} E^{\theta,\psi^*}[[X_t - \hat{X}_t^*]Q[KD_t]^T] \\ &= \operatorname{tr} E^{\theta,\psi^*}[K^T[X_t - \hat{X}_t^*]QD_t^T] \\ &= 0. \end{aligned} \tag{25}$$

But by (25) and the Hilbert space projection theorem, if $\hat{X}_t^*$ achieves the infimum of (15), then $Z_t = K^T \hat{X}_t^*$ achieves the infimum of

$$\inf_{\Psi_t} \operatorname{tr} E^{\theta,\psi}[[K^T X_t - Z_t]Q[K^T X_t - Z_t]^T]. \tag{26}$$

Setting $K = [S^T | 0]$, where

$$S := [\underbrace{I_{1\cdot}^n \ldots I_{n\cdot}^n | \ldots \ldots | I_{1\cdot}^n \ldots I_{n\cdot}^n}_{I_1^n \ldots I_n^n \text{ repeated } N \text{ times}}], \tag{27}$$

and $I_{i\cdot}^n$ denotes the ith row of a $n \times n$ identity matrix, we find that problem (13) is a special case of problem (26). Thus, we have obtained the following result.

Lemma 3.2. If $\hat{X}_t^*$ achieves the infimum of (15), then $(\hat{x}_t^{*1T}, \ldots, \hat{x}_t^{*NT}) = S\hat{X}_t^*$ achieves the infimum of (12); moreover,

$$\inf_{\Phi_t} E^{\theta,\phi}\left[\begin{bmatrix} x_t - \hat{x}_t^1 \\ \vdots \\ x_t - \hat{x}_t^N \end{bmatrix}^T Q \begin{bmatrix} x_t - \hat{x}_t^1 \\ \vdots \\ x_t - \hat{x}_t^N \end{bmatrix}\right] = SE^{\theta,\psi^*}[[X_t - \hat{X}_t^*]Q[X_t - \hat{X}_t^*]^T]S^T. \tag{28}$$

The primary advantage of working in the higher-dimensional space of problem (15) is that, in this space, orthogonality implies a simple Wiener–Hopf-like condition. Specifically, since (25) holds for all real $K \in \mathbb{R}^{Nn^2 \times Nn^2}$, the optimal estimate $\hat{X}_t^*$ must satisfy

$$E^{\theta,\psi^*}[K^T[X_t - \hat{X}_t^*]QD_s^T] = 0, \tag{29}$$

for all $s \in [t_0, t]$ and $D_s \in \mathscr{Y}_s^{Nn^2 \times Nn}$, or more generally, for any $(M \times Nn)$-dimensional D_s, such as Y_s, for which columns $(k-1)n+1$ to kn are measurable functionals of Λ_s^k. But since

$$\hat{X}_t = \mathscr{G}[Y_s: t_0 \le s \le t],$$

where $\mathscr{G}$ is a functional with kernel $\mathscr{G}(t, s)$, (29) implies the following result.

Lemma 3.3. (Ref. 9, Theorem 5.1). The optimal functional $\mathscr{G}^*$ for problem (15) must satisfy

$$E^{\theta,\mathscr{G}^*}\left[\left[X_t - \int_{t_0}^{t} \mathscr{G}^*(t, r)\, dY_r\right] QY_s^T\right] = 0, \qquad \text{for all } t_0 \le s \le t. \tag{30}$$

To derive a recursive expression for $\hat{X}_t^*$, fix $\theta \in \Theta$, let

$$\Sigma_{X_0X_0}^Q := E[X_{t_0} Q X_{t_0}^T] = Q \otimes \Sigma_{x_0x_0}, \tag{31}$$

$$\Sigma_{\Xi\Xi,t}^Q := E[\Xi_t Q \Xi_t^T] = Q \otimes \Sigma_{\xi\xi,t}, \tag{32}$$

and let

$$\Sigma_{\Omega\Omega,t}^Q(\theta_t) := E^\theta[\Omega_t Q \Omega_t^T]$$

$$= \begin{bmatrix} Q^{11} \otimes \Sigma_{\omega^1\omega^1,t}(\theta_t^1, \theta_t^1) & \dots & Q^{1N} \otimes \Sigma_{\omega^1\omega^N,t}(\theta_t^1, \theta_t^N) \\ \vdots & \ddots & \vdots \\ Q^{N1} \otimes \Sigma_{\omega^N\omega^1,t}(\theta_t^N, \theta_t^1) & \dots & Q^{NN} \otimes \Sigma_{\omega^N\omega^N,t}(\theta_t^N, \theta_t^N) \end{bmatrix}, \tag{33}$$

where $\otimes$ denotes the Kronecker product, i.e.,

$$X_{m \times m} \otimes Y_{n \times n} := \begin{bmatrix} X_{11}Y & \dots & X_{1m}Y \\ \vdots & \ddots & \vdots \\ X_{m1}Y & \dots & X_{mm}Y \end{bmatrix}_{mn \times mn}. \tag{34}$$

When Y_t is replaced by a Q-orthogonal increments process V_t, i.e., a process satisfying

$$E[V_t Q V_s^T] = \int_{t_0}^{\min\{t,s\}} \Sigma_{VV,r}^Q\, dr, \tag{35}$$

for some symmetric, positive definite $(n\sum_{i=1}^{N} m^i \times n\sum_{i=1}^{N} m^i)$-dimensional $\Sigma^Q_{\dot{V}V,r}$ (Ref. 9), Eq. (30) and the basic properties of stochastic integrals (Ref. 11, Theorem 5.5.1) imply that

$$\hat{X}_t^* = \int_{t_0}^{t} \left(\left(\frac{\partial}{\partial s}\right) E[X_s Q V_s^T]\right) [\Sigma^Q_{\dot{V}V,s}]^{-1}\, dV_s. \tag{36}$$

In Ref. 9, it is shown by a Gohberg–Krein factorization argument (Ref. 13) that, for all $t \geq t_0$, the process $\{V_t, t \in [t_0, t_f]\}$, with dynamics

$$dV_t = dY_t - \mathscr{C}_t(\theta_t)\hat{X}_t^*\, dt, \tag{37}$$

has covariance

$$E^{\theta}[V_t Q V_s^T] = \int_{t_0}^{\min\{t,s\}} \Sigma^Q_{\Omega\Omega,r}\, dr. \tag{38}$$

It is also shown that the subspaces generated in $\mathscr{H}^{Nn^2 \times Nn}$ by respectively all square-integrable linear functionals of V_t, and all square-integrable linear functionals of Y_t are identical. It follows that we can substitute (37) in (36) and parallel the arguments in Ref. 13 to obtain the following theorem.

Theorem 3.2. (Ref. 9, Theorem 5.5). For fixed $\theta \in \Theta$, the estimator $\hat{X}_t^*$ achieving the infimum in (15) satisfies the matrix stochastic differential equation

$$d\hat{X}_t^* = \mathscr{A}_t \hat{X}_t^*\, dt + \Sigma_{t|\theta}\mathscr{C}_t(\theta_t)^T [\Sigma^Q_{\Omega\Omega,t}(\theta_t)]^{-1} \times [dY_t - \mathscr{C}_t(\theta_t)\hat{X}_t^*\, dt], \qquad \hat{X}_{t_0}^* = 0, \tag{39}$$

where

$$\Sigma_{t|\theta} := E^{\psi}[[X_t - \hat{X}_t^*] Q [X_t - \hat{X}_t^*]^T | \{\theta_s : s \in [t_0, t]\}] \tag{40}$$

satisfies the matrix Riccati equation

$$\dot{\Sigma}_{t|\theta} = \mathscr{A}_t \Sigma_{t|\theta} + \Sigma_{t|\theta}\mathscr{A}_t^T + \Sigma^Q_{\Xi\Xi,t} - \Sigma_{t|\theta}\mathscr{C}_t(\theta_t)^T [\Sigma^Q_{\Omega\Omega,t}(\theta_t)]^{-1} \mathscr{C}_t(\theta_t)\Sigma_{t|\theta}, \qquad \Sigma_{t_0|\theta} = \Sigma^Q_{X_{t_0}X_{t_0}}. \tag{41}$$

Note that the invertibility of $\Sigma^Q_{\Omega\Omega,t}$ is assured, because Q and $\Sigma_{\omega^i\omega^i,t}(\theta_t^i, \theta_t^i)$ are positive definite for all i, and $t \geq t_0$ (Ref. 7, p. 870).

4. Decentralized Measurement Scheduling

Because the arguments underlying (12)–(28) hold regardless of whether the measurement scheduling policy operates open-loop, the original measurement scheduling problem,

$$\inf_{\Gamma,\Phi} E^{\gamma,\phi}[J_m(\gamma)+J_e(\gamma,\phi)], \tag{42}$$

can be reformulated as

$$\inf_{\Gamma} E^{\gamma,\psi^*}\left[J_m(\gamma)+\int_{t_0}^{t_f} S(X_t-\hat{X}_t^*)Q(X_t-\hat{X}_t^*)^T S^T\,dt\right], \tag{43}$$

where for fixed $\gamma\in\Gamma$, $\{\hat{X}_t^*, t\in[t_0,t_f]\}$ is generated by the estimator that achieves

$$\inf_{\Psi_t} \operatorname{tr} E^{\gamma,\Psi}[[X_t-\hat{X}_t]Q[X_t-\hat{X}_t]^T], \tag{44}$$

for all $t\in[t_0,t_f]$. In view of (43), the results of Section 3 have the following significance. Assume hypothetically that, for all $t\in[t_0,t_f]$, $\gamma\in\Gamma$, and $k=1,\ldots,N$, those terms in

$$S(X_t-\hat{X}_t^*)Q(X_t-\hat{X}_t^*)^T S^T=\sum_{i,j=1}^{N,N}[x_t-\hat{x}_t^{*i}]^T Q^{ij}[x_t-\hat{x}_t^{*j}] \tag{45}$$

involving the kth estimator's error $[x_t-\hat{x}_t^{*k}]$ are independent of the data Λ_t^k. Then (i) some open-loop scheduling policy performs at least as well as every ϵ-optimal closed-loop policy; thus, (ii) the original measurement scheduling problem reduces to a deterministic optimal control problem.

To see this, observe that, for every ϵ-optimal closed-loop policy $\gamma^\epsilon\in\Gamma$, there exists at least one realization λ_{t_f} of Λ_{t_f} such that

$$\begin{aligned}
&E^{\gamma^\epsilon}[J_m(\gamma^\epsilon)|\lambda_{t_f}]+\sum_{i,j=1}^{N,N}\int_{t_0}^{t_f} E^{\gamma^\epsilon,\phi^*}[[x_t-\hat{x}_t^{*i}]^T Q^{ij}[x_t-\hat{x}_t^{*j}]|\lambda_t^i,\lambda_t^j]\,dt\\
&\le E^{\gamma^\epsilon}[J_m(\gamma^\epsilon)]+\sum_{i,j=1}^{N,N}\int_{t_0}^{t_f} E^{\gamma^\epsilon,\phi^*}[[x_t-\hat{x}_t^{*i}]^T Q^{ij}[x_t-\hat{x}_t^{*j}]]\,dt\\
&=E^{\gamma^\epsilon,\psi^*}\left[J_m(\gamma^\epsilon)+\int_{t_0}^{t_f} S(X_t-\hat{X}_t^*)Q(X_t-\hat{X}_t^*)^T S^T\,dt\right]\\
&\le\inf_{\Gamma} E^{\gamma,\psi^*}\left[J_m(\gamma)+\int_{t_0}^{t_f} S(X_t-\hat{X}_t^*)Q(X_t-\hat{X}_t^*)^T S^T\,dt\right]+\epsilon.
\end{aligned} \tag{46}$$

Because the schedule $\{v_t=\gamma_t^\epsilon(\lambda_{t-}), t\in[t_0,t_f]\}$ induced by this realization is indistinguishable from that of the open-loop schedule $\{\theta_t=v_t, t\in[t_0,t_f]\}$, if those terms in (45) involving the kth estimator error $[x_t-\hat{x}_t^{*k}]$ are independent of Λ_t^k, then the following equality holds:

$$E^{\gamma^\epsilon}[J_m(\gamma^\epsilon)|\lambda_{t_f}] + \sum_{i,j=1}^{N,N} \int_{t_0}^{t_f} E^{\gamma^\epsilon,\phi^*}[[x_t - \hat{x}_t^{*i}]^T Q^{ij} [x_t - \hat{x}_t^{*j}]|\lambda_t^i, \lambda_t^j]\, dt$$

$$= E^\theta[J_m(\theta)|\lambda_{t_f}] + \sum_{i,j=1}^{N,N} \int_{t_0}^{t_f} E^{\theta,\phi^*}[[x_t - \hat{x}_t^{*i}]^T Q^{ij} [x_t - \hat{x}_t^{*j}]|\lambda_t^i, \lambda_t^j]\, dt$$

$$= J_m(\theta) + \sum_{i,j=1}^{N,N} \int_{t_0}^{t_f} E^{\theta,\phi^*}[[x_t - \hat{x}_t^{*i}]^T Q^{ij} [x_t - \hat{x}^{*j}{}_t]]\, dt. \tag{47}$$

But the substitution of (47) in (46) would then establish the ϵ-optimality of the open-loop schedule θ. Hence, for every ϵ-optimal closed-loop policy γ^ϵ, we could find an open-loop policy that would perform at least as well; thus, by Lemmas 3.1 and 3.2, and by Theorem 3.2, the original measurement scheduling problem could be reduced to the deterministic problem

$$\inf_{\Theta} \quad \left[J_m(\theta) + \int_{t_0}^{t_f} S\Sigma_{t|\theta} S^T\, dt \right], \tag{48a}$$

$$\text{s.t.} \quad \dot{\Sigma}_{t|\theta} = \mathscr{A}_t \Sigma_{t|\theta} + \Sigma_{t|\theta} \mathscr{A}_t^T + \Sigma^Q_{\Xi\Xi,t} - \Sigma_{t|\theta} \mathscr{C}_t(\theta_t)^T [\Sigma^Q_{\Omega\Omega,t}(\theta_t)]^{-1} \mathscr{C}_t(\theta_t) \Sigma_{t|\theta}, \tag{48b}$$

$$\Sigma_{t_0|\theta} = \Sigma^Q_{X_{t_0} X_{t_0}}. \tag{48c}$$

Unfortunately, those terms in (45) involving the kth estimator error $[x_t - \hat{x}_t^{*k}]$ need not be orthogonal to, let alone independent of, the observations in Λ_t^k. For instance, consider a two observer, scalar system [i.e., $N = 2$, $n = 1$, and $m^i = 1$, for $i = 1, 2$], with open-loop measurement schedule θ. Because the optimal estimator $\hat{X}_t^*$ must satisfy the orthogonality condition (29) for $K = [(1, 1)^T|0]$ (Lemma 3.3), the team optimal estimator $(\hat{x}_t^{*1}, \hat{x}_t^{*2}) = (1, 1)\hat{X}_t^*$ (Lemma 3.2) satisfies

$$E^{\theta,\phi^*}[(x_t - \hat{x}_t^{*1}, x_t - \hat{x}_t^{*2}) Q Y_s^T] = 0, \qquad s \in [t_0, t], \tag{49}$$

or equivalently,

$$E^{\theta,\phi^*}[((x_t - \hat{x}_t^{*1}) Q^{1k} + (x_t - \hat{x}_t^{*2}) Q^{2k}) y_t^k] = 0, \qquad s \in [t_0, t], \qquad k = 1, 2. \tag{50}$$

Since $(x_t - \hat{x}_t^{*1})$, $(x_t - \hat{x}_t^{*2})$, and y_t^k are jointly Gaussian, this in turn implies that

$$(x_t - \hat{x}_t^{*1}) Q^{1k} + (x_t - \hat{x}_t^{*2}) Q^{2k}, \tag{51}$$

and hence the square of (51) times $1/Q^{kk}$,

$$(x_t - \hat{x}_t^{*1})^2 (Q^{1k})^2 / Q^{kk} + 2(x_t - \hat{x}_t^{*1})(x_t - \hat{x}_t^{*2}) Q^{1k} Q^{2k} / Q^{kk} + (x_t - \hat{x}_t^{*2})^2 (Q^{2k})^2 / Q^{kk}, \tag{52}$$

must be independent of Λ_t^k.

Now, suppose that those terms in $S\Sigma_{t|\theta} S^T$ involving the kth estimator, i.e.,

$$(x_t - \hat{x}_t^{*k})^2 Q^{kk} + 2(x_t - \hat{x}_t^{*1})(x_t - \hat{x}_t^{*2}) Q^{21} \tag{53}$$

are also independent of Λ_t^k. Then, the difference between (52) and these terms,

$$(x_t - \hat{x}_t^{*3-k})^2 (Q^{21})^2 / Q^{kk}, \tag{54}$$

as well as the square root of this difference,

$$(x_t - \hat{x}_t^{*3-k}) Q^{21} / \sqrt{Q^{kk}}, \tag{55}$$

must also be independent of Λ_t^k. It follows that $x_t - \hat{x}_t^{*3-k}$ is orthogonal to the observations in Λ_t^k, and by (50), Λ_t^{3-k}. Hence, by the Hilbert space projection theorem

$$\hat{x}_t^{*1} = E^\theta[x_t | \Lambda_t^1] = \hat{x}_t^{*2} = E^\theta[x_t | \Lambda_t^2]. \qquad \text{a.s.} \tag{56}$$

But this is impossible, unless $\Lambda_t^1 = \Lambda_t^2$ a.s. Thus by contradiction, we have the following lemma.

Lemma 4.1. For decentralized LQG measurement scheduling problems, the terms in

$$S(X_t - \hat{X}_t^*) Q (X_t - \hat{X}_t^*)^T S^T = \sum_{i,j=1}^{N,N} [x_t - \hat{x}_t^{*i}]^T Q^{ij} [x_t - \hat{x}_t^{*j}], \tag{57}$$

involving the kth estimator error $[x_t - \hat{x}_t^{*k}]$, need not be independent of the observations in Λ_t^k.

Lemma 4.1 suggests that, unlike centralized LQG measurement scheduling problems, the decentralized problem (43) is fundamentally stochastic. Hence, to identify member-by-member optimal solutions (Ref. 7), let alone optimal solutions, we must solve N coupled, partially observed, stochastic control problems. We could formulate N coupled, dynamic programming equations for these problems, but the information states would be infinite dimensional. Instead, we restrict attention to open-loop measurement schedules. This reduces (43) to the deterministic optimal control problem (48), or equivalently, to a deterministic optimal control problem in which $\Sigma_{t|\theta}$

plays the role of the state matrix and $\theta_t := (\theta_t^1, \ldots, \theta_t^N)$ can be viewed as the measurement pattern selected at time t by a 0-1 switching control.

Formally, we can transform (48) to a Lagrange problem of optimal control as follows. Let ρ denote an arbitrary mapping of the $M = \prod_{i=1}^N M^i$ possible measurement patterns onto the integers 1 to M; let

$$\theta(k) := (\theta^1(k), \ldots, \theta^N(k)) \in \prod_{i=1}^N \{1, \ldots, M^i\}$$

denote the kth of these patterns; and define

$$u_t(k) := \begin{cases} 1, & \text{when } \rho(\theta_t) = \theta(k), \\ 0, & \text{else,} \end{cases} \tag{58}$$

to be the 0-1 switching control that indicates when the kth pattern is active. Then, problem (48) is equivalent to the following Lagrange optimal control problem:

$$\inf_U \int_{t_0}^{t_f} \left[\sum_{i,j=1}^{N,M} q_t^i(\theta^i(j)) u_t(j) + S\Sigma_{t|u} S^T \right] dt, \tag{59a}$$

$$\text{s.t.} \quad \dot{\Sigma}_{t|u} = \mathscr{A}_t \Sigma_{t|u} + \Sigma_{t|u} \mathscr{A}_t^T + \Sigma^Q_{\Xi\Xi,t}$$

$$- \Sigma_{t|u} \left[\sum_{j=1}^{M} \mathscr{C}_t(\theta(j))^T [\Sigma^Q_{\Omega\Omega,t}(\theta(j))]^{-1} \mathscr{C}_t(\theta(j)) u_t(j) \right] \Sigma_{t|u}, \tag{59b}$$

$$\Sigma_{t_0|u} = \Sigma^Q_{X_{t_0} X_{t_0}}, \tag{59c}$$

where U denotes the collection of measurable M-vectors $u_t := (u_t(1), \ldots, u_t(M))$ that remain in the control restraint set

$$U_t := \left\{ (u_t(1), \ldots, u_t(M)) : \sum_{j=1}^{M} u_t(j) = 1, u_t(j) \in \{0, 1\}, j = 1, \ldots, M \right\}, \tag{59d}$$

for all $t \in [t_0, t_f]$; i.e.,

$$U := \{u_t \in U_t : t \in [t_0, t_f]\}.$$

Because U_t is not convex for any t, problem (59) need not have a solution. Instead, schedules achieving the infimum may correspond to a mixture of pure schedules, i.e., a schedule that chatters between several measurement configurations arbitrarily fast (Ref. 10). To guarantee the existence of solutions, we could impose further restrictions on U: a switching cost, an upper bound on the number of measurement patterns enabled over $[t_0, t_f]$, or a minimum time between switchings, for instance. Instead, we relax the restrictions on U in a way that ensures that optimal solutions to the corresponding relaxed version of problem (59) always exist, and so that the performance of these solutions can be approximated arbitrarily closely by some $u \in U$.

Define the relaxed version of problem (59) to be

$$\inf_{\text{co } U} \int_{t_0}^{t_f} \left[\sum_{i,j=1}^{N,M} q_t^i(\theta^i(j)) u_t(j) + S\Sigma_{t|u} S^T \right] dt, \tag{60a}$$

$$\text{s.t.} \quad \dot{\Sigma}_{t|u} = \mathscr{A}_t \Sigma_{t|u} + \Sigma_{t|u} \mathscr{A}_t^T + \Sigma^Q_{\Xi\Xi,t}$$

$$- \Sigma_{t|u} \left[\sum_{j=1}^{M} \mathscr{C}_t(\theta(j))^T [\Sigma^Q_{\Omega\Omega,t}(\theta(j))]^{-1} \mathscr{C}_t(\theta(j)) u_t(j) \right] \Sigma_{t|u}, \tag{60b}$$

$$\Sigma_{t_0|u} = \Sigma^Q_{X_{t_0} X_{t_0}}, \tag{60c}$$

where co U denotes the collection of measurable M-vectors $u_t := (u_t(1), \ldots, u_t(M))$ that remain in the convex hull of the original control restraint set U_t,

$$\text{co } U_t := \left\{ (u_t(1), \ldots, u_t(M)) : \sum_{j=1}^{M} u_t(j) = 1, u_t(j) \geq 0, j = 1, \ldots, M \right\}, \tag{60d}$$

for all $t \in [t_0, t_f]$; i.e.,

$$\text{co } U := \{ u_t \in \text{co } U_t : t \in [t_0, t_f] \}.$$

The relaxed controls can be viewed as probability vectors in a $M-1$ unit simplex. The jth vertex of this simplex, enables the jth measurement pattern. Controls that are convex combinations of multiple vertices enable pattern j with probability $u_t(j)$. Physically, the combinations correspond to schedules that use simultaneously a bit of many patterns, i.e., schedules that chatter.

Because (i) U_t is compact and (ii) the state dynamics and objective in (60) are linear functions of u, with coefficients that are continuously differentiable in $\Sigma_{t|u}$ and t, the existence of a $u^* \in$ co U achieving the infimum in (60) follows from the standard existence theorem for relaxed controllers (Ref. 10, Theorem 4.5) if (iii) the state trajectories generated by arbitrary $u \in$ co U are uniformly bounded for all $t \in [t_0, t_f]$. To see that (iii) is satisfied, note that, because $\Sigma^Q_{X_0 X_0}$ is symmetric and nonnegative definite, the basic properties of the Riccati equation ensure that, independently of u, $E[X_t Q X_t^T]$, $\Sigma_{t|u}$, and $E[X_t Q X_t^T] - \Sigma_{t|u}$ are symmetric and nonnegative definite. Because the principal minors of symmetric, nonnegative-definite matrices are nonnegative,

$$(E[X_t Q X_t^T] - \Sigma_{t|u})_{ii} \geq 0, \tag{61}$$

for all $i \in \{1, \ldots, Nn^2\}$, hence

$$\begin{aligned} \|\Sigma_{t|u}\|_{\max} &:= \max_{ij} |(\Sigma_{t|u})_{ij}| \\ &= \max_{i} |(\Sigma_{t|u})_{ii}| \\ &\leq \max_{i} |(E[X_t Q X_t^T])_{ii}| \\ &= \max_{i} |(Q \otimes E[x_t x_t^T])_{ii}| \\ &= \max_{i} |Q_{ii}| \max_{i} |(E[x_t x_t^T])_{ii}|. \end{aligned} \tag{62}$$

But since x_t is mean-square continuous, $(E[x_t x_t^T])_{ii}$ is uniformly bounded for all $t \in [t_0, t_f]$. Thus, we have proved the following result.

Theorem 4.1. The relaxed version of the open-loop decentralized measurement scheduling problem (60) admits an optimal solution $u^* \in \text{co } U$.

In fact, the performance of the optimal relaxed controller $u^* \in \text{co } U$ can be approximated arbitrarily closely by some $u \in U$. To see this, partition $[t_0, t_f]$ into k equal intervals $I_{[t_{ki}, t_{k(i+1)})}$, where $i = 0, \ldots, k-1$ and $t_0 = t_{k0} < t_{k1} < \cdots < t_{kk} = t_f$; partition each of these k intervals into consecutive subintervals $I_{[t_{ki}, t_{k(i+1)})}(j)$, $j = 1, \ldots, M$, with lengths proportional to $u^*_{t_{ki}}(j)$; and let

$$u_t^k := (\delta_{1j}, \ldots, \delta_{Mj}), \qquad \text{when } t \in \bigcup_{i=0}^{k-1} I_{[t_{ki}, t_{k(i+1)})}(j), \tag{63}$$

where

$$\begin{aligned} \delta_{lm} &= 1, \qquad \text{when } l = m, \\ \delta_{lm} &= 0, \qquad \text{otherwise.} \end{aligned}$$

Then, by construction, the fraction of time that measurement pattern $(\delta_{1j}, \ldots, \delta_{Mj}) \in U_t$ is active during the interval $I_{[t_k, t_{k+1})}$ is precisely $u_t^*(j)$. By the chattering lemma (Ref. 10, p. 267), under hypotheses (i)–(iii) above, $u_t^k \in U_t$ convergers weakly to $u_t^* \in \text{co } U_t$ on $[t_0, t_f]$ as $k \to \infty$; i.e., for every bounded measurable test function g_t,

$$\int_{t_0}^{t_f} g_t u_t^k \, dt \to \int_{t_0}^{t_f} g_t u_t \, dt;$$

see Ref. 10. Moreover, the state trajectory $\Sigma_{t|u^k}$ induced by $u^k \in U$ converges uniformly on $[t_0, t_f]$ to the state trajectory $\Sigma_{t|u^*}$ induced by $u^* \in \text{co } U$. Hence, we have the following result.

Theorem 4.2. For all $\epsilon > 0$, and for every relaxed open-loop measurement schedule $u^* \in \text{co } U$, there exists an open-loop measurement schedule $u \in U$ such that the cost of schedule u is within ϵ of that of u^*.

5. Numerical Solutions

One approach to identifying ϵ-optimal solutions to the relaxed scheduling problem is to restrict u to the class of piecewise constant functions with at most $L-1$ arbitrarily spaced discontinuities, i.e., to let

$$u_t(k) := u_i(k), \qquad \text{when } t \in [t_i, t_{i+1}), \text{ for } t_0 < t_1 < \cdots < t_{L-1} < t_L = t_f, \tag{64}$$

for all $i = 0, \ldots, L-1$, $k = 1, \ldots, M$, and $t \in [t_0, t_f]$. Then, (60) can be viewed as a linearly constrained, nonlinear programming problem of the form

$$\min \quad J(u), \tag{65a}$$

$$\text{s.t.} \quad \sum_{j=1}^{M} u_i(j) = 1, u_i(k) \geq 0, i = 0, \ldots, L-1 \text{ and } k = 1, \ldots, M, \tag{65b}$$

where

$$u := (u_0(1), \ldots, u_0(M), u_1(1), \ldots, u_1(M), \ldots, u_{L-2}(M),$$
$$u_{L-1}(1), \ldots, u_{L-1}(M)), \tag{65c}$$

$$J(u) := \sum_{i=1}^{L-1} \left[\int_{t_i}^{t_{i+1}} \left[\sum_{j,k=1}^{N,M} q_t^j(\theta_t^j(k)) u_i(k) + S \Sigma_{t|u} S^T \right] dt \right], \tag{65d}$$

with $\Sigma_{t|\theta}$ satisfying

$$\dot{\Sigma}_{t|u} = \mathscr{A}_t \Sigma_{t|u} + \Sigma_{t|u} \mathscr{A}_t^T + \Sigma_{\Xi\Xi,t}^{Q}$$
$$- \Sigma_{t|u} \left[\sum_{k=1}^{M} \mathscr{C}_t(\theta(k))^T [\Sigma_{\Omega\Omega,t}^{Q}(\theta(k))]^{-1} \mathscr{C}_t(\theta(k)) u_j(k) \right] \Sigma_{t|u}, \tag{65e}$$

$$\Sigma_{t_0|u} = \Sigma_{X_{t_0} X_{t_0}}^{Q}, \tag{65f}$$

for all $t \in [t_i, t_{i+1})$, $i = 0, \ldots, L-1$.

Given $J(u)$ and its gradient $\nabla_u J(u)$, local optima for (65) can be computed using a variety of descent algorithms. The challenge is the computation of $\nabla_u J(u)$. Because $J(u)$ is parameterized by a matrix differential equation, solving for $\nabla_u J(u)$ is equivalent to solving a two-point boundary-value problem involving the Hamiltonian of $J(u)$,

$$H(\Sigma_{t|u}, P_{t|u}, u) := \left[\sum_{j,k=1}^{N,M} q_t^J(\theta_t^j(k))u_j(k) + S\Sigma_{t|u}S^T\right] + \mathrm{tr}\,[P_{t|u}^T\dot{\Sigma}_{t|u}], \tag{66}$$

for all $t\in[t_i, t_{i+1})$, $i=1, \ldots, L-1$. Formally (see Ref. 14),

$$\nabla_u J(u) = \int_{t_0}^{t_f} [\partial H(\Sigma_{t|u}, P_{t|u}, u)/\partial u]\, dt, \tag{67}$$

or equivalently (differentiating term by term),

$$\begin{aligned}&\partial J(u)/\partial u_i(k)\\ &= \int_{t_i}^{t_{i+1}} \Bigg[\sum_{j=1}^{N} q_t^j(\theta^j(k))\\ &\qquad - \mathrm{tr}\,[P_{t|u}^T\Sigma_{t|u}\mathscr{C}_t(\theta(k))^T[\Sigma_{\Omega\Omega,t}^Q(\theta(k))]^{-1}\mathscr{C}_t(\theta(k))\Sigma_{t|u}]\Bigg]\, dt,\end{aligned} \tag{68}$$

where the state matrix $\Sigma_{t|u}$ satisfies

$$\begin{aligned}&\partial H(\Sigma_{t|u}, P_{t|u}, u)/\partial P_{t|u}\\ &\quad = \dot{\Sigma}_{t|u} = \mathscr{A}_t\Sigma_{t|u} + \Sigma_{t|u}\mathscr{A}_t + \Sigma_{\Xi\Xi,t}^Q\\ &\qquad - \Sigma_{t|u}\left[\sum_{k=1}^{M} \mathscr{C}_t(\theta(k))^T[\Sigma_{\Omega\Omega,t}^Q(\theta(k))]^{-1}\mathscr{C}_t(\theta(k))u_i(k)\right]\Sigma_{t|u},\end{aligned} \tag{69}$$

for $\Sigma_{t_0|u} = \Sigma_{X_{t_0}X_{t_0}}^Q$, and the costate matrix $P_{t|u}$ satisfies

$$\begin{aligned}&\partial H(\Sigma_{t|u}, P_{t|u}, u)/\partial\Sigma_{t|u}\\ &\quad = -\dot{P}_{t|u} = -S^TS - \mathscr{A}_tP_{t|u} - P_{t|u}\mathscr{A}_t^T\\ &\quad + \sum_{k=1}^{M} [\mathscr{C}_t(\theta(k))^T[\Sigma_{\Omega\Omega,t}^Q(\theta(k))]^{-1}\mathscr{C}_t(\theta(k))u_i(k)]\Sigma_{t|u}P_{t|u}\\ &\quad + P_{t|u}\Sigma_{t|u}[\mathscr{C}_t(\theta(k))^T[\Sigma_{\Omega\Omega,t}^Q(\theta(k))]^{-1}\mathscr{C}_t(\theta(k))u_i(k)],\end{aligned} \tag{70}$$

for $P_{t_L|u}=0$, and $\Sigma_{t_L|u}=\Sigma_{t_L|u}$, for all $t\in[t_i, t_{i+1})$, $i=0, \ldots, L-1$.

Example 5.1. To illustrate the approach, consider, for $t\in[0, 2]$ sec, a two-observer problem with scalar state and observation dynamics [see (1) and (3)],

$$dx_t = 2x_t\, dt + d\xi_t, \qquad x_0 \sim N[0, 40], \tag{71}$$

$$dy_t^1 \in \{2x_t\, dt + d\omega_t^1(1), x_t\, dt + d\omega_t^1(2)\}, \tag{72}$$

$$dy_t^2 \in \{2x_t\, dt + d\omega_t^2(1),\, x_t\, dt + d\omega_t^2(2)\}, \tag{73}$$

noise covariance [cf. (2) and (4)]

$$E\begin{bmatrix} \xi_t \\ \omega_t^1(1) \\ \omega_t^1(2) \\ \omega_t^2(1) \\ \omega_t^2(2) \end{bmatrix} [\xi_s, \omega_s^1(1), \omega_s^1(2), \omega_s^2(1), \omega_s^2(2)]$$

$$= \int_0^{\min\{t,s\}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.10 & 0.25 & -0.50 \\ 0 & 0.10 & 2 & -0.50 & 0.25 \\ 0 & 0.25 & -0.50 & 1 & 0.10 \\ 0 & -0.50 & 0.25 & 0.10 & 2 \end{bmatrix} dr, \tag{74}$$

measurement costs [see (8)]

$$q_t^i(1) = 3 \text{ and } q_t^i(2) = 1, \qquad i = 1, 2, \tag{75}$$

and coupling matrix [see (9)]

$$Q = \begin{bmatrix} 1 & 0.75 \\ 0.75 & 1 \end{bmatrix}. \tag{76}$$

Suppose that the maximum switching frequency $f_{\max}$ is 200 Hz, and suppose that we restrict the set of allowable measurement schedules to the class of piecewise constant functions with at most $L-1=9$ uniformly spaced discontinuities, where L is chosen such that the interval $(t_f - t_0)/L$ is large enough that relaxed controls can be approximated reasonably by chattering, e.g.,

$$L = [(t_f - t_0) f_{\max}]/[10(\# \text{ of patterns})]. \tag{77}$$

Solving this relaxed open-loop measurement scheduling problem (60) is equivalent to solving (65) with u constrained to a $M^1 M^2 L = 40$-dimensional unit simplex. Given routines for computing $J(u)$ and $\nabla_u J(u)$ at points in this simplex, we can use any reliable convergent descent algorithm, such as the hybrid reduced-gradient/quasi-Newton algorithm implemented in MINOS 5.0 (Ref. 15), to iterate on u. To evaluate $J(u)$ and $\nabla_u J(u)$, we must solve Eqs. (68)–(70), a two-point boundary-value problem. One approach is to use a differential equation solver, such as the IMSL sixth-order Runge–Kutta–Verner routine DVERK (Ref. 16), to integrate (65d) and (65e) forward in time, to compute $J(u)$ and $\Sigma_{t_f|u}$, and (68)–(70) backward in time, starting from $\Sigma_{t_f|u}$, to compute $\nabla_u J(u)$.

When the DVERK error control tolerance and minimum stepsize are set to 10^{-9} and 10^{-6}, and the MINOS 5.0 optimality tolerance and function precision are set to 10^{-6}, we find, after computing $J(u)$ and $\nabla_u J(u)$ at 178 points (38 seconds of Amdahl 5860 CPU time), that relative to a schedule in which both observers use observation dynamics 1 over the entire interval, the coupled schedule u^{*c} in Table 1 is locally ϵ-optimal. Here, $u^{*c}_{(t_i,t_{i+1})}(j, k)$ denotes the fraction of time that the locally ϵ-optimal, coupled, chattering schedule should simultaneously engage observer 1's jth observation dynamics and observer 2's kth observation dynamics during the interval $[t_i, t_{i+1})$.

When we neglect the coupling between the observers errors (i.e., when we set the off-diagonal elements of Q to zero), after computing $J(u)$ and $\nabla_u J(u)$ at 108 points (23 seconds of Amdahl 5860 CPU time), we find that relative to a schedule in which both observers use observation dynamics 1 over the entire interval, the schedule u^{*uc} in Table 2 is locally ϵ-optimal. Moreover, we find that the coupled cost (the cost for the original Q) increases from 17.40 to 19.40.

Although we cannot prove that the relaxed coupled and uncoupled schedules in Tables 1 and 2 are globally ϵ-optimal, the results are not counterintuitive. In both cases, over the first interval, the large covariance of the initial state relative to the state noise covariance makes it desirable to rely on the accurate 1,1 dynamics despite its cost. Over the next eight intervals, the observers either chatter (see Theorem 4.2) between the 2,1 and 1,2 dynamics (cheap/costly and costly/cheap) or between the 1,1 and 2,2 dynamics (costly/costly and cheap/cheap), depending on whether positively correlated errors are penalized. When these errors are penalized, the negative correlation between $\omega^1_t(i)$ and $\omega^2_t(j)$, for $i \neq j$, helps to reduce the penalty. Over the final interval, the value of accurate, coupled (uncoupled) observations diminishes, because no further observations or estimates will be made; hence, both schedules make greater use of the least expensive observation dynamics 2,2.

6. Conclusions

In this paper, we have shown that decentralized LQG measurement scheduling problems are fundamentally stochastic, and that the problems' ϵ-optimal open-loop schedules are chattering solutions of deterministic Lagrange optimal control problems. Subject to the inherent limitations of LQG models, these results provide a means for determining numerically the ϵ-optimal open-loop measurement schedules for decentralized surveillance

Table 1. Locally ϵ-optimal coupled measurement schedule.

Dynamics used	Time									
$u^{*c}_{[t_i,t_{i+1})}(\#1, \#2)$	[0.0, 0.2)	[0.2, 0.4)	[0.4, 0.6)	[0.6, 0.8)	[0.8. 1.0)	[1.0, 1.2)	[1.2, 1.4)	[1.4, 1.6)	[1.6, 1.8)	[1.8, 2.0)
$u^{*c}_{[t_i,t_{i+1})}(1,1)$	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$u^{*c}_{[t_i,t_{i+1})}(1,2)$	0.00	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.14
$u^{*c}_{[t_i,t_{i+1})}(2,1)$	0.00	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.14
$u^{*c}_{[t_i,t_{i+1})}(2,2)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.72
	Total cost $J(u^{*c}) = 17.40$									

Table 2. Locally ϵ-optimal uncoupled measurement schedule.

Dynamics used	Time									
$u^{*uc}_{[t_i,t_{i+1})}(\#1,\#2)$	[0.0, 0.2)	[0.2, 0.4)	[0.4, 0.6)	[0.6, 0.8)	[0.8, 1.0)	[1.0, 1.2)	[1.2, 1.4)	[1.4, 1.6)	[1.6, 1.8)	[1.8, 2.0)
$u^{*uc}_{[t_i,t_{i+1})}(1,1)$	1.00	0.68	0.60	0.62	0.62	0.62	0.63	0.59	0.74	0.00
$u^{*uc}_{[t_i,t_{i+1})}(1,2)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$u^{*uc}_{[t_i,t_{i+1})}(2,1)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$u^{*uc}_{[t_i,t_{i+1})}(2,2)$	0.00	0.32	0.40	0.38	0.38	0.38	0.37	0.41	0.26	1.00
	Total cost $J(u^{*uc}) = 19.40$									

or tracking systems when the observers actions are coupled through a common quadratic performance measure. The results are of interest, because the coupling present in such systems (the undesirability of systematic tracking errors among observers that do not communicate), for example may be crucial to the systems' performance.

This work can be used to identify ϵ-optimal open-loop schedules under a variety of interesting conditions, e.g., conditions that preclude instantaneous switching, limit the measurements that can be made concurrently, or make measurement availability dependent on past measurement selections. In each case, one simply deletes those simplex vertices corresponding to inadmissible measurement patterns from the switching control restraint set. Discrete-time versions of the results can also be derived. The work is not directly applicable to the problem of selecting schedules for systems in which the decentralized observers exert control over the system state (a dynamic team problem), nor does it provide much insight into the structure of the ϵ-optimal closed-loop schedules.

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