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INFORMATION STRUCTURES, CAUSALITY, AND NONSEQUENTIAL STOCHASTIC CONTROL I: DESIGN-INDEPENDENT PROPERTIES*

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Abstract. In control theory, the usual notion of causality—that, at all times, a system's output (action) only depends on its past and present inputs (observations)—presupposes that all inputs and outputs can be ordered, a priori, in time. In practice, many distributed systems (those subject to deadlock, for instance) are not *sequential* in this sense.

This paper explores the relationship between deadlock freeness, a less restrictive notion of causality, and the properties of a potentially *nonsequential* generic stochastic control problem formulated within the framework of Witsenhausen's intrinsic model. A property of the problem's *information structure* that is necessary and sufficient to ensure deadlock-freeness is identified and shown to be sufficient to ensure that all of the problem's control policies possess expected rewards. It is also shown, by example, that there exist stochastic control problems for which all sequential policies are suboptimal.

These results subsume Witsenhausen's "causality" condition (property C), suggest a framework for the optimization of unconstrained nonsequential stochastic control problems, and provide an intuitive design-independent characterization of the cause/effect notion of causality. The results also have game theoretic implications—they suggest, for instance, necessary and sufficient conditions for a finite game to possess an extensive form.

Key words. information structures, causality, deadlock-freeness, nonsequential stochastic control.

1. Introduction.

In control theory, the usual notion of causality—that, at all times, a system's output (action) only depends on its past and present inputs (observations)—presupposes that all inputs and outputs can be ordered, a priori, in time. As it becomes increasingly attractive to decentralize the control of large systems, it has become clear that many important systems—distributed data [5], communication [13], manufacturing [11], and detection networks (Appendix A), for instance—need not be *sequential* in this sense.

The distinguishing feature of these *nonsequential* systems is the impossibility of ordering their control actions a priori, independently of the set of control laws, called the *design* (or *control policy*), that determines the actions. In the simplest case, a system's actions can be ordered a priori, given any design, but the order varies from design to design. More generally, for at least one design, the order implicitly depends on the system's uncontrolled inputs—e.g., action α may depend on action β under some circumstances while β may depend on α under others. In the worst case, for some design, and for some uncontrolled input, no "causal" ordering of the actions is possible because two or more actions are mutually dependent—e.g., action α depends on action β and vice versa. This last phenomenon, unique to nonsequential systems, is known as *deadlock*.

In this paper we explore the relationship between *deadlock-freeness*, a property that generalizes the usual notion of causality, and nonsequential stochastic control. We begin by defining deadlock-freeness (Definition 1, §3.1). Given this definition

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we consider the following question: Under what conditions is it possible to pose well-defined nonsequential stochastic control problems? This question is of interest because there exist problems for which all *sequential* designs (designs whose actions can be ordered a priori) are suboptimal (see Appendix A).

Witsenhausen's intrinsic model [19], [21] provides the framework for our results. This model, which was originally used to investigate a related causality question, encompasses all systems in which (1) the uncontrolled inputs can be viewed as an element of a measurable space $(\Omega, \mathcal{B})$; (2) the number of actions to be taken is finite, say N; (3) the kth action, $k = 1, 2, \ldots, N$, can be viewed as an element of a measurable space $(U^k, \mathcal{U}^k)$ in which the singletons are measurable; and (4) the possible designs can be viewed as N-tuples $\gamma := (\gamma^1, \gamma^2, \ldots, \gamma^N)$ of $\mathcal{J}^k/\mathcal{U}^k$-measurable functions γ^k, $k = 1, 2, \ldots, N$, where the subfield $\mathcal{J}^k$ of the product field $\mathcal{B} \otimes (\bigotimes_{i=1}^N \mathcal{U}^i)$ denotes the maximal information (knowledge) that can be used to select the kth action.

Within this framework, we identify a property of the information subfields $\mathcal{J}^k, k = 1, 2, \ldots, N$, (property CI, §3.2) that is necessary and sufficient to ensure that every N-tuple γ of $\mathcal{J}^k/\mathcal{U}^k$-measurable functions $\gamma^k, k = 1, 2, \ldots, N$, is deadlock-free. Moreover, we show that this property is sufficient to ensure that an expected reward can be defined for every N-tuple, and consequently, that the problem of maximizing a generic system's expected reward, given a probability measure on $(\Omega, \mathcal{B})$, and a reward function, is well-posed. The property is *design-independent* in the sense that it holds for all designs γ.

These results subsume Witsenhausen's "causality" condition (property C) [19], suggest a framework for the recursive optimization of unconstrained nonsequential stochastic control problems [1], and provide an intuitive characterization of the cause/effect notion of causality. In essence, this characterization says that a system is causal if and only if for each tuple of uncontrolled inputs there exists an ordering of the system's actions such that no information that may be used to determine an action depends on that action or subsequent actions.

There are other approaches to the modeling of nonsequential systems. None, however, are as well suited to examining the relationship between deadlock-freeness and nonsequential control as the intrinsic model. Most game-theoretic models that accommodate nonsequentiality are variations of Kuhn's extensive form [12], a "game tree" representation that precludes deadlock by definition (cf. [19, §2]). The discrete event models that accommodate nonsequentiality are, for the most part, state transition- (e.g., [6], [16]), algebraic equation- (e.g., [9], [10], [15]), or logical calculus- (e.g., [4], [8], [14], [17]) based representations of the action sequences (traces) that a system can generate; consequently, they are incompatible with the usual control theoretic representations of uncertainty and information.

The remainder of the paper is organized as follows. In §2 we introduce Witsenhausen's intrinsic model and formulate our generic nonsequential stochastic control problem. In §3 we define properties DF (deadlock-freeness) and CI (causal implementability), and prove that property CI, a condition that is necessary and sufficient to ensure deadlock-freeness, is sufficient to ensure that unconstrained versions of the generic problem are well defined. In §4 we consider the relationship between property CI and Witsenhausen's "causality" property C. Section 5 contains our conclusions.

2. Problem formulation. To examine the relationship between deadlock-freeness and nonsequential stochastic control it is necessary to represent nonsequential systems in a framework in which each action can be viewed as depending on some system information, for instance, an observation of the system. The "conventional" control theoretic models—controlled difference, or differential equations modeling time-indexed "states" and "observations"—provide such a framework; however, they presuppose a fixed ordering of the system's control actions. In this paper, as in [19] and [21], we relax this assumption.

2.1. Preliminaries. Consider a generic stochastic system in which the number of control actions and uncontrolled inputs are both finite (Fig. 1). From a game-theoretic perspective (cf. [18]), the control actions can be viewed as being the actions of N distinct decision-making *agents* (computers, devices, processes, etc.). Likewise the uncontrolled inputs can be viewed a single action of *nature* (chance).

To couple the agents' actions without preordering their decisions, suppose that nature's action $\omega := (\omega^0, \omega^1, \ldots, \omega^N)$, the kth agent's observation y^k, and the kth agent's action u^k, take values in, respectively, the measurable spaces $(\Omega, \mathcal{B})$, $(Y^k, \mathcal{Y}^k)$, and $(U^k, \mathcal{U}^k)$. Let $U := \prod_{i=1}^N U^i$ and $\mathcal{U} := \bigotimes_{i=1}^N \mathcal{U}^i$; constrain the system's kth observation to be a measurable function

$$h^k : (\Omega \times U, \mathcal{B} \otimes \mathcal{U}) \to (Y^k, \mathcal{Y}^k) \tag{2.1}$$

of the system's *intrinsic variables*, ω and $u := (u^1, u^2, \ldots, u^N)$; and constrain the kth agent's decision policy, to be a measurable function

$$g^k : (Y^k, \mathcal{Y}^k) \to (U^k, \mathcal{U}^k) \tag{2.2}$$

of this observation.

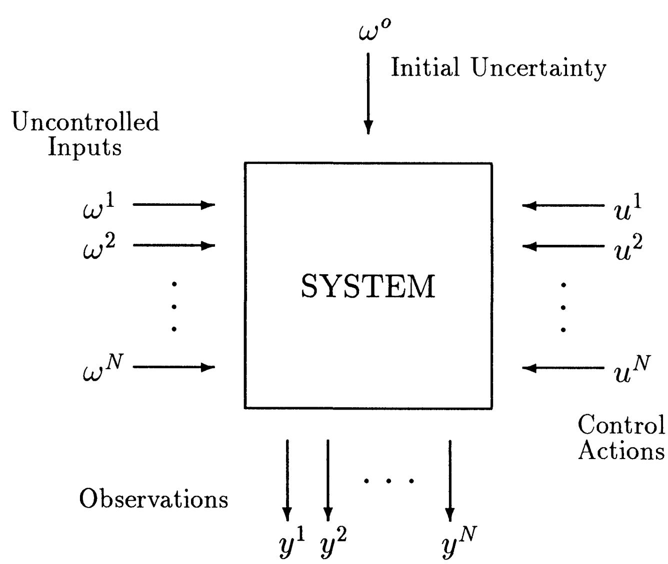

FIG. 1. *A generic stochastic system.*

With respect to the "conventional" discrete-time, finite horizon models of stochastic control, this representation entails no loss of generality. The system's uncontrolled inputs—its initial state, state and observation noises, and so on—can always be viewed as a single uncontrolled input $\omega \in \Omega$. Moreover, the kth observation—normally assumed to be a measurable function of some subset of the system's control actions, states, and random inputs—can always be viewed as a measurable function of the system's intrinsic variables.

The advantage of this representation, as opposed to the conventional models, is that as long as the superscripts on y and u are not assumed to index time, it permits interdependence among a system's control actions—e.g., given a fixed control policy $\gamma := (\gamma^1, \gamma^2, \ldots, \gamma^N)$, u^j may depend on u^k (through y^j) for some ω, and vice versa for other ω. Consequently, it is possible to model nonsequential stochastic control problems, that is, problems in which a causal ordering of the control actions cannot be determined a priori because the ordering is policy, and possibly, ω-dependent.

Witsenhausen's intrinsic model [19], [21] simplifies the preceding representation. The crucial observations are (1) that the system's control actions are solely determined by the intrinsic variables (e.g., $u^k = (g^k \circ h^k)(\omega, u)$ for all $k = 1, 2, \ldots, N$); and (2) that for reasonable observation functions, the kth observation, $k = 1, 2, \ldots, N$, can only affect the kth control action via the information subfield it induces on the space of intrinsic variables (i.e., via $[h^k]^{-1}(\mathcal{Y}^k) \subset \mathcal{B} \otimes \mathcal{U}^1$). Accordingly, it is unnecessary to model the observations explicitly if the control agents' actions are viewed as measurable functions of the intrinsic variables.

2.2. The intrinsic model. Formally, the intrinsic model has three components:

1. An *information structure* $\mathcal{I} := \{(\Omega, \mathcal{B}), (U^k, \mathcal{U}^k), \mathcal{J}^k : 1 \leq k \leq N\}$ specifies the system's allowable decisions and distinguishable events.

(a) $N \in \mathbb{N}$ denotes the number of control actions to be taken.

(b) $(\Omega, \mathcal{B})$ denotes the measurable space from which a random input ω is drawn.

(c) $(U^k, \mathcal{U}^k)$ denotes the measurable space from which u^k, the kth control action, is selected. It is assumed that $\text{card}(U^k)$ is greater than one,[2] and that $\mathcal{U}^k$ contains the singletons of U^k. The product space containing the N-tuple of control actions, $u := (u^1, u^2, \ldots, u^N)$, is denoted by $(U, \mathcal{U}) := (\prod_{i=1}^N U^i, \bigotimes_{i=1}^N \mathcal{U}^i)$.[3]

(d) σ-field $\mathcal{J}^k \subset \mathcal{B} \otimes \mathcal{U}$ characterizes the maximal information that can be used to select the kth control action.

2. A design constraint set Γ_C constrains the set of admissible N-tuples of control laws, $\gamma := (\gamma^1, \gamma^2, \ldots, \gamma^N)$, called *designs*, to a nonempty subset of $\Gamma := \prod_{i=1}^N \Gamma^i$, where Γ^k, $k = 1, 2, \ldots, N$, denotes the set of all $\mathcal{J}^k/\mathcal{U}^k$-measurable functions.

3. A probability measure P on $(\Omega, \mathcal{B})$ specifies the mixed (randomized) decision policy to be used by nature to select ω.

Note that the intrinsic model does not exclude the possibility of an agent employing a mixed decision policy, or a policy that occasionally dictates that the agent not act. To model the mixed policy, randomizing devices can be included as factors in

[1] $[f]^{-1}$ denotes the inverse image of the function f, $[f]^{-1}(\mathcal{C}) := \{[f]^{-1}(A) : A \in \mathcal{C}\}$ denotes the set of inverse images induced by the sets in $\mathcal{C}$. Since inverse images preserve unions and complements, the inverse image of a σ-field is always a σ-field.

[2] Although the assumption $\text{card}(U^k) > 1$ was not made by Witsenhausen, it does not constitute a loss of generality. Any agent k for which $\text{card}(U^k) = 1$ has no decision to make; consequently, that agent can be deleted from the model without effect (naturally, the remaining agents' information fields—defined in 1(d)—must be adjusted to account for the kth agent's deletion—i.e., for all $j \neq k$, $\mathcal{J}^j$ must be replaced by $\mathcal{J}^j|_{u^k}$, the u^k-section of $\mathcal{J}^j$.

[3] $\mathcal{X} \otimes \mathcal{Y}$ denotes the product σ-field of the σ-fields $\mathcal{X}$ and $\mathcal{Y}$—i.e., $\mathcal{X} \otimes \mathcal{Y} := \sigma([\pi_X]^{-1}(\mathcal{X}) \cup [\pi_Y]^{-1}(\mathcal{Y}))$, the smallest σ-field of $X \times Y$ for which the canonical projections $\pi_X(\pi_X(x, y) = x)$ and $\pi_Y(\pi_Y(x, y) = y)$ are both measurable.

$(\Omega, \mathcal{B}, P)$, and the effects of the devices' outputs can be specified in $\mathcal{J}^k$. To model the occasional inaction, the agent can be allowed to make decisions that have no effect.

2.3. A generic problem. Within this framework we can formulate the following generic stochastic control problem.

(P) Given an information structure $\mathcal{I}$, a design constraint set Γ_C, a probability measure P, and a bounded, nonnegative, $\mathcal{B} \otimes \mathcal{U}$-measurable reward function V,

$$\text{Identify a design } \gamma \text{ in } \Gamma_C \text{ that achieves}$$
$$\sup_{\gamma \in \Gamma_C} E_\omega[V(\omega, u_\omega^\gamma)] \text{ exactly, or within } \epsilon > 0.\text{[4]}$$

Is this generic problem well defined? Since the problem may be nonsequential there are two issues: "deadlock-freeness" (Is every $\gamma \in \Gamma_C$ deadlock-free?) and "mathematical wellposedness" (Does every design $\gamma \in \Gamma_C$ possess an expected reward?).

In general, nonsequential problems of the form (P) need not be deadlock-free or well-posed. Suppose, for instance, that for some design $\gamma \in \Gamma_C$, and some random outcome $\omega \in \Omega$, the control actions

$$u^j = \gamma^j(\omega, u^1, \ldots, u^k, \ldots, u^N), \tag{2.3}$$

and

$$u^k = \gamma^k(\omega, u^1, \ldots, u^j, \ldots, u^N), \tag{2.4}$$

are interdependent. Then a deadlock arises, and consequently, the problem is not deadlock-free. Alternatively, suppose that for some design $\gamma \in \Gamma_C$, and some random outcome $\omega \in \Omega$, the *closed-loop* equations

$$u^k = \gamma^k(\omega, u^1, \ldots, u^N), \qquad k = 1, 2, \ldots, N \tag{2.5}$$

fail to possess a unique solution

$$u_\omega^\gamma := (u_\omega^{\gamma^1}, \ldots, u_\omega^{\gamma^N}). \tag{2.6}$$

Then the reward $V(\omega, u_\omega^\gamma)$ induced by ω under γ need not be unique, the expected reward $E_\omega[V(\omega, u_\omega^\gamma]$ need not exist, and consequently, the problem need not be well posed.

The primary objective of this paper is to identify conditions sufficient to ensure that problem (P) is deadlock-free and well-posed. Since there exist problems of the form (P) for which some, but not all, nontrivial designs are deadlock-free and possess expected rewards (Appendix A)—two classes of conditions can be considered: conditions based on the problem's *design-independent* properties (properties that hold for *all* $\gamma \in \Gamma$), and conditions based on the problem's *design-dependent* properties (properties that may only hold for *specific* designs $\gamma \in \Gamma$). In this paper, conditions based on the problem's *design-independent* properties are explored. Conditions based on the problem's *design-dependent* properties are introduced in a companion paper [3].

[4] The notation u_ω^γ indicates that u depends on ω through γ (see §3.1).

3. Design-independent conditions. In this section, necessary and sufficient conditions for problem (P) to be well-posed and deadlock-free are developed under the assumption that the problem's design set is unconstrained (i.e., $\Gamma_C = \Gamma$). The conditions are design-independent in the sense that they are solely based on properties of the problem's information structure $\mathcal{I}$.

3.1. Properties DF, S, and SM. To ensure that problem (P) is deadlock-free it suffices to require that its information structure $\mathcal{I}$ possess *property* DF (deadlock-freeness).

DEFINITION 1. An information structure $\mathcal{I}$ possesses *property* DF (*deadlock-freeness*) if for each $\gamma \in \Gamma$, and for every $\omega \in \Omega$, there exists an ordering of γ's N control laws, say $\gamma^{s_1(\omega)}, \gamma^{s_2(\omega)}, \ldots, \gamma^{s_N(\omega)}$, such that no control action $u^{s_n(\omega)}, n = 1, 2, \ldots N$, depends on itself or the control actions that follow.

Note that the ordering in Definition 1 may depend on the design $\gamma \in \Gamma$ and the random input $\omega \in \Omega$. For instance, for some $\gamma \in \Gamma$ a triggering random event may determine the identity of the initial control action (see Appendix A).

When $\mathcal{I}$ possesses property DF, for each $\gamma \in \Gamma$ and for all $\omega \in \Omega$, γ is deadlock-free in the sense that, given ω, $u^{s_1(\omega)}$ can be determined; given ω and $u^{s_1(\omega)}$, $u^{s_2(\omega)}$ can be determined; given ω, $u^{s_1(\omega)}$ and $u^{s_2(\omega)}$, $u^{s_3(\omega)}$ can be determined; and so on. Hence, property DF generalizes the usual notion of causality in the sense that it does not presuppose that the actions' order is fixed.

To ensure that problem (P) is well-posed, it suffices to require (i) that for each $\gamma \in \Gamma$ and every $\omega \in \Omega$ there exist a unique $u := (u^1, u^2, \ldots, u^N) \in U$ satisfying the system of equations

$$u^k = \gamma^k(\omega, u), \qquad k = 1, 2, \ldots, N, \tag{3.1}$$

and (ii) that each of the *solution maps* $\Sigma^\gamma : \Omega \to U$ induced via these solutions (i.e., $\Sigma^\gamma(\omega) = u^\gamma_\omega$ where $u^\gamma_\omega = \gamma(\omega, u^\gamma_\omega)$) be $\mathcal{B}/\mathcal{U}$-measurable. Then, for each $\gamma \in \Gamma$, $V(\cdot, \Sigma^\gamma(\cdot))$ is $\mathcal{B}$-measurable, and consequently, $E_\omega[V(\omega, \Sigma^\gamma(\omega))]$ exists. Systems that satisfy (i) are said to possess *property* S (solvability) while systems that satisfy (ii) are said to possess *property* SM (solvability/measurability) [19]. In fact, property S often implies property SM [2].

3.2. Property CI. Property SM holds when, for each $\gamma \in \Gamma$, and each uncontrolled input $\omega \in \Omega$, every agent's action is uniquely determined and the actions' ω-dependence is $\mathcal{B}$-measurable. Since property SM does not rule out the possibility that, for some $\omega \in \Omega$, agent N's information depends on agent 1's action, and for all $k = 1, 2, \ldots, N-1$, agent k's information depends on agent $k+1$'s action, property SM is not sufficient to ensure property DF (cf. [19], Thm. 2). That is, although property SM holds, for some $\omega \in \Omega$, every agent's information may depend on every other agents' actions, and consequently, for that ω, no agent can act without precognition.

Property DF suggests that such deadlocks cannot arise if for each $\omega \in \Omega$, the agents can be ordered such that each agent's information only depends on ω and its predecessors' actions. To formalize this observation it is convenient to adopt the notation in [19]. For all $k = 1, 2, \ldots, N$, define S_k to be the set of all k-agent orderings—i.e., all injections of $\{1, 2, \ldots, k\}$ into $\{1, 2, \ldots, N\}$. For all $j = 0, 1, \ldots, N$, and $k = j, j+1, \ldots, N$, let $T^k_j : S_k \to S_j$ denote a truncation map that returns the ordering of the first j agents of a k-agent ordering—i.e., T^k_j restricts $s \in S_k$ to the domain $\{1, 2, \ldots, j\}$ or to $\emptyset$ when $j = 0$. Finally, for all $s := (s_1, s_2, \ldots, s_k) \in S_k$, and

$k = 1, 2, \ldots, N$, define $\mathcal{P}_s$ to be the projection of $\Omega \times U$ onto $\Omega \times (\prod_{i=1}^{k} U^{s_i})$—i.e.,

$$\mathcal{P}_s(\omega, u) := (\omega, u^{s_1}, u^{s_2}, \ldots, u^{s_k}), \qquad \mathcal{P}_{\emptyset}(\omega, u) := (\omega). \tag{3.2}$$

Then, we can characterize deadlock-freeness as follows.

DEFINITION 2. An information structure $\mathcal{I}$ possesses *property* CI (*causal implementability*) when there exists at least one map $\psi : \Omega \times U \to S_N$ such that for all $k = 1, 2, \ldots, N$, and $(\omega, u) \in \Omega \times U$,

$$\mathcal{J}^{s_k} \bigcap [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) \subset \{\emptyset, [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u))\} \tag{3.3}$$

when $s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u)$.

ψ is a function that maps every intrinsic outcome $(\omega, u) \in \Omega \times U$ into an N-agent ordering.

$$[\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) = [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\omega, u^{s_1}, \ldots, u^{s_{k-1}}) \tag{3.4}$$

is the cylinder set induced on $\Omega \times U$, when the intrinsic outcome is (ω, u), by the actions of nature and the first $k - 1$ agents in $s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u)$. Since

$$\mathcal{J}^{s_k} \bigcap [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) \tag{3.5}$$

denotes the *trace* of the s_kth agent's information field on this cylinder set (i.e., $\mathcal{J}^{s_k} \cap C := \{A \cap C : A \in \mathcal{J}^{s_k}\}$), (3.3) constrains the cylinder set to be a subset of all events containing (ω, u) in the s_kth agent's information field $\mathcal{J}^{s_k}$—i.e., no event in $\mathcal{J}^{s_k}$ containing (ω, u) may depend on u^{s_k}, $u^{s_{k+1}}, \ldots$, or u^{s_N}. Accordingly, property CI ensures that for all outcomes $(\omega, u) \in \Omega \times U$, there exists an order $s := (s_1, s_2, \ldots s_N) = \psi(\omega, u)$ such that, for all $k = 1, 2, \ldots, N$, *the s_kth agent's information, at the point (ω, u), only depends on the actions of nature and its predecessors in s.*

Property CI implies property SM and is a necessary and sufficient condition for all $\gamma \in \Gamma$ to be deadlock-free. Theorem 1 states this formally.

THEOREM 1. *Let $\mathcal{I}$ be an arbitrary information structure, then*

(i) *$\mathcal{I}$ possesses property SM if $\mathcal{I}$ possesses property CI, and*

(ii) *$\mathcal{I}$ possesses property DF if and only if $\mathcal{I}$ possesses property CI.*

Proof. See Appendix B. □

Theorem 1 ensures that problem (P) is *well defined* (deadlock-free and well-posed) when it satisfies property CI. Its proof hinges on the following observation. When ψ is an order function such that $\mathcal{I}$ possesses property CI, for arbitrary but fixed $(\omega, u) \in \Omega \times U$, and $k = 1, 2, \ldots, N$, (3.3) and the fact that $\mathcal{U}^k$ contains the singletons of U^k imply that, at the point (ω, u), all $\mathcal{J}^{s_k}/\mathcal{U}^{s_k}$-measurable functions $\gamma^{s_k} \in \Gamma^{s_k}$, $s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u)$, do not depend on the components s_k, $s_{k+1}, \ldots$, and s_N of u. This suggests that, for fixed $\gamma \in \Gamma$, a unique $\mathcal{B}$-measurable solution $\Sigma^\gamma : \Omega \to U$ to the closed-loop equation $u = \gamma(\omega, u)$ can be obtained by the following recursion. Fix $\omega \in \Omega$, let $r \in U$ be an arbitrary reference element, let $L^\gamma : \Omega \times U \to \Omega \times U$ be defined as

$$L^\gamma(\omega, r) := (\omega, \gamma(\omega, r)), \tag{3.6}$$

and let $L^\gamma_k : \Omega \times U \to \Omega \times U$ be a k-fold composition of L^γ—i.e.,

$$L^\gamma_k(\omega, r) := (\underbrace{L^\gamma \circ \cdots \circ L^\gamma}_{k \text{ times}})(\omega, r). \tag{3.7}$$

1. After one iteration, the components of $L_1^\gamma(\omega, r)$ corresponding to agents whose information, at the point (ω, r), does not depend on r, become invariant to subsequent iterations. By property CI, the set $\mathcal{A}_1(\omega) \subset \{1, 2, \ldots, N\}$ indexing (by agent) these components is nonempty since at least agent $(\psi(\omega, r))_1$'s information does not depend on r.

2. After two iterations, the components of $L_2^\gamma(\omega, r)$ corresponding to agents in $\{1, 2, \ldots, N\} \setminus \mathcal{A}_1(\omega)$ whose information, at the point $L_1^\gamma(\omega, r)$, does not depend on the components of agents in $\{1, 2, \ldots, N\} \setminus \mathcal{A}_1(\omega)$, become invariant to subsequent iterations.[5] By property CI, the set $\mathcal{A}_2(\omega)$ indexing (by agent) these components is nonempty when $\operatorname{card}(\mathcal{A}_1(\omega)) < N$ since at least agent $(\psi(L_1^\gamma(\omega, r)))_j$'s information,

$$j = \min\{m \in \{1, 2, \ldots, N\} : (\psi(L_1^\gamma(\omega, r)))_m \notin \mathcal{A}_1(\omega)\}, \tag{3.8}$$

does not depend on the components of agents in $\{1, 2, \ldots, N\} \setminus \mathcal{A}_1(\omega)$.

$$\vdots$$

k. After k iterations, the components of $L_k^\gamma(\omega, r)$ corresponding to agents in $\{1, 2, \ldots, N\} \setminus \bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega)$ whose information, at the point $L_{k-1}^\gamma(\omega, r)$, does not depend on the components of agents in $\{1, 2, \ldots, N\} \setminus \bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega)$, become invariant to subsequent iterations. By property CI, the set $\mathcal{A}_k(\omega)$ indexing (by agent) these components is nonempty when $\operatorname{card}(\bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega)) < N$ since at least agent $(\psi(L_{k-1}^\gamma(\omega, r)))_j$'s information,

$$j = \min\left\{ m \in \{1, 2, \ldots, N\} : (\psi(L_{k-1}^\gamma(\omega, r)))_m \notin \bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega) \right\} \tag{3.9}$$

does not depend on the components of agents in $\{1, 2, \ldots, N\} \setminus \bigcup_{i=1}^{k-1} \mathcal{A}_i(\omega)$.

$$\vdots$$

and so on.

Since property CI ensures that, until all agents' components are invariant, at least one new component becomes invariant after every iteration, the recursive procedure must converge in, at most, N iterations—i.e., the unique solution to the closed-loop equation $u = \gamma(\omega, u)$ is $\pi_U(L_N^\gamma(\omega, r))$, where π_U denotes the canonical projection of $\Omega \times U$ onto U ($\pi_U(\omega, u) = u$) and $r \in U$ is an arbitrary "seed" that starts the recursive solution process. Since π_Ω, π_U, and γ are, respectively, $\mathcal{B} \otimes \mathcal{U}/\mathcal{B}$-, $\mathcal{B} \otimes \mathcal{U}/\mathcal{U}$-, and $\mathcal{B} \otimes \mathcal{U}/\mathcal{U}$-measurable, L^γ, and by composition, L_k^γ and $\pi_U \circ L_N^\gamma$, are, respectively, $\mathcal{B} \otimes \mathcal{U}/\mathcal{B} \otimes \mathcal{U}$-, $\mathcal{B} \otimes \mathcal{U}/\mathcal{B} \otimes \mathcal{U}$-, and $\mathcal{B} \otimes \mathcal{U}/\mathcal{U}$-measurable. It follows, since all u-sections of $\mathcal{B} \otimes \mathcal{U}/\mathcal{U}$-measurable functions are $\mathcal{B}/\mathcal{U}$-measurable, that the induced solution map $\Sigma^\gamma = \pi_U \circ L_N^\gamma|_r$ is necessarily $\mathcal{B}/\mathcal{U}$-measurable.

The above recursion has an obvious physical interpretation. For fixed $\gamma \in \Gamma$ and $\omega \in \Omega$, suppose that we conduct the following thought experiment: decouple the agents and record in succession, $C_1(\omega)$, the indices of those agents that act given ω

[5] For sets $A, B \subset X$, $A \setminus B := \{x \in A : x \notin B\}$.

alone; $C_2(\omega)$, the indices of those agents that act given ω and the actions of agents in $C_1(\omega)$; $C_3(\omega)$, the indices of those agents that act given ω and the actions of agents $C_1(\omega) \cup C_2(\omega)$; and so on. Clearly, $\mathcal{A}_k(\omega) = C_k(\omega)$ for all $k = 1, 2, \ldots, N$. Accordingly, if for all k we ignore all components of $\pi_U(L_k^\gamma(\omega, r))$ but those corresponding to the agents indexed in $\mathcal{A}_k(\omega)$, the preceding recursion outlines the partial ordering of agent actions that a passive observer would record, given ω, if the design γ were implemented in a "maximally" concurrent fashion.

Although the preceding recursion implicitly demonstrates that property CI implies property DF, it is far easier to establish sufficiency by a direct appeal to property CI. For all $(\omega, u) \in \Omega \times U$ and $k = 1, 2, \ldots, N$, property CI implies that at the point (ω, u), all $\mathcal{J}^{s_k}/\mathcal{U}^{s_k}$-measurable functions $\gamma^{s_k} \in \Gamma^{s_k}$, $s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u)$, do not depend on the components s_k, s_{k+1}, ..., and s_N of u. Consequently, no agent's information depends on its own action or the actions of its successors—i.e., the system must be deadlock-free.

The fact that some design $\gamma \in \Gamma$ must deadlock when property CI fails to hold is also a direct consequence of property CI's definition. When property CI fails, for some outcome $(\omega, u) \in \Omega \times U$ and for all N-agent orderings $s \in S_N$, (3.3) fails for at least one $k \in \{1, 2, \ldots, N\}$. Since there are at most $N\text{card}(S_N) = N(N!)$ k, s combinations for which (3.3) can fail, and since all agents may take at least two distinct actions, it is always possible to construct a design γ that possesses all of the interdependencies that cause (3.3) to fail—i.e., a design γ such that for all $s \in S_N$, when the s_kth agent's information depends on the actions of its successors in s, $\gamma^{s_k}(\omega, u)$ depends on the s_kth agent's successors' components of u. Accordingly, it is always possible to construct a design that deadlocks.

4. Property CI's relationship to property C. Witsenhausen was the first to develop conditions sufficient to ensure a system's deadlock-freeness (he termed it "causality"). Specifically, he introduced the following property.

DEFINITION 3 ([19]). An information structure $\mathcal{I}$ possesses *property* C (*causality*) when there exists at least one map $\psi : \Omega \times U \to S_N$ such that for all $s := (s_1, s_2, \ldots, s_k) \in S_k, k = 1, 2, \ldots, N$,

$$\mathcal{J}^{s_k} \bigcap [T_k^N \circ \psi]^{-1}(s) \subset \mathcal{F}(T_{k-1}^k(s)), \tag{4.1}$$

where $\mathcal{F}(s)$ denotes the cylindrical extension of $\mathcal{B} \otimes (\bigotimes_{i=1}^k \mathcal{U}^{s_i})$ to $\Omega \times U$ for all $s \in S_k$, $k = 1, 2, \ldots, N$.[6]

He then proved the following theorem (DF is our terminology).

THEOREM 2 ([19]). *Let $\mathcal{I}$ be an arbitrary information structure; then*

(i) *$\mathcal{I}$ possesses property SM if $\mathcal{I}$ possesses property* C, *and*

(ii) *$\mathcal{I}$ possesses property DF if $\mathcal{I}$ possesses property* C.

Proof. See [19, §§6 and 7]. □

Since property C implies property DF (Theorem 2(ii)), and since property DF implies property CI (Theorem 1(ii)), the following is clear.

COROLLARY 1. *Property* C *implies Property* CI.

Proof. See Appendix C for a direct proof. □

This corollary suggests that the ψ/γ-dependent umpire recursion that Witsenhausen used to prove Theorem 2 ([19, §7]), is not fundamental—i.e., to prove Theorem 2, it suffices to compose $\gamma|_\omega$ with itself N times (i.e., to form $\pi_U \circ L_N^\gamma|_\omega$) as

[6] Here, in contrast to [19], ψ is a mapping from $\Omega \times U$ to S_N and $\mathcal{F}(\emptyset)$ is the cylindrical extension of $\mathcal{B}$ to $\Omega \times U$ (see [21]).

described in §3. The corollary also raises the following question: Are properties C and CI equivalent? Equivalence would imply, by Theorem 1(ii), that $\mathcal{I}$'s possession of property C is both a necessary and sufficient condition for deadlock-freeness. Nonequivalence would imply that property C is, in general, only a sufficient condition for deadlock-freeness.

When $N \leq 2$, properties C and CI are always equivalent. Corollary 2 states this formally.

COROLLARY 2. *Property* CI *implies property* C *when* $N \leq 2$.

Proof. By Theorem 1(i) property CI implies property SM which, in turn, implies property S. The corollary follows since property S implies property C when $N \leq 2$ ([19, Thm. 2]). □

When $N > 2$, it is not known (in general) whether property CI implies property C; the implication, however, holds in at least two important special cases (Thms. 3 and 4).

DEFINITION 4. An information structure $\mathcal{I}$ is said to be *sequential* when property CI holds for some constant order function ψ.

THEOREM 3. *All constant order functions* ψ *such that* $\mathcal{I}$ *possesses property* CI *are order functions such that* $\mathcal{I}$ *possesses property* C; *consequently, property* CI *implies property* C *when* $\mathcal{I}$ *is sequential.*

Proof. See Appendix D. □

Note that an unconstrained problem of the form (P) is sequential (in the sense discussed in §1) if and only if its information structure is sequential. Witsenhausen defines an information structure to be sequential when property C holds with a constant order function ψ ([20, §3]). Accordingly, Theorem 3 ensures that, as far as unconstrained problems of the form (P) are concerned, sequentiality, as defined in this paper, is equivalent to Witsenhausen's sequentiality.

When $\mathcal{I}$ is nonsequential, even if $\mathcal{I}$ possesses property C, order functions for which $\mathcal{I}$ possesses property CI need not be order functions for which $\mathcal{I}$ possesses property C.

Example 1. Consider a nonsequential information structure $\mathcal{I}$ of the following form:

$$\begin{aligned} N &= 3, \\ \Omega &= U^1 = U^2 = U^3 = \{0,1\}, \\ \mathcal{B} &= \mathcal{U}^1 = \mathcal{U}^2 = \mathcal{U}^3 = \Big\{\emptyset, \{0\}, \{1\}, \{0,1\}\Big\}, \\ \mathcal{J}^1 &= \left\{\emptyset, \{(\omega,u) : \omega u^2 = 0\}, \{(\omega,u) : \omega u^2 = 1\}, \Omega \times U\right\}, \\ \mathcal{J}^2 &= \left\{\emptyset, \{(\omega,u) : \overline{\omega} u^1 = 0\}, \{(\omega,u) : \overline{\omega} u^1 = 1\}, \Omega \times U\right\}, \quad \text{and} \\ \mathcal{J}^3 &= \Big\{\emptyset, \{(\omega,u) : \omega = 0\}, \{(\omega,u) : \omega = 1\}, \Omega \times U\Big\}.^7 \end{aligned} \tag{4.2}$$

Although

$$\overline{\psi}(\omega, u^1, u^2, u^3) = \begin{cases} (1,2,3) & \text{when } \omega = 0 \\ (2,1,3) & \text{else} \end{cases} \tag{4.3}$$

[7] $\overline{x}$ denotes the binary complement of $x \in \{0,1\}$—i.e., $\overline{x} := 1 - x$.

is an order function such that $\mathcal{I}$ possesses properties CI and C,

$$\psi(\omega, u^1, u^2, u^3) = \begin{cases} (1,2,3) & \text{when } \omega = 0 \\ (3,2,1) & \text{when } \omega u^3 = 1 \\ (2,1,3) & \text{else} \end{cases} \tag{4.4}$$

is an order function such that $\mathcal{I}$ possesses property CI, but not property C ((3.3) fails when $k = 1$ and $s = 3 \in S_1$, for instance, since $[T_1^3 \circ \psi]^{-1}(3) = \{(\omega, u) : \omega u^3 = 1\} \notin \mathcal{F}(\emptyset) = \mathcal{B} \otimes \{\emptyset, U\}$).

The fact that there exist nonsequential information structures $\mathcal{I}$, and order functions ψ, such that $\mathcal{I}$ possesses property CI, but not property C, implies that general proofs that property CI implies property C (if such exist) must be constructive—i.e., given a ψ such that $\mathcal{I}$ possesses property CI, but not property C, we must be able to construct a new order function $\hat{\psi}$ (obviously distinct from ψ), such that $\mathcal{I}$ possesses property C.

Given the generality of the intrinsic model, such constructions are, at best, tedious. Consider Example 1. By simple combinatorial arguments it can be shown that 3^{16} of the 6^{16} possible order functions for $\mathcal{I}$ are order functions for which $\mathcal{I}$ possesses property CI. Of these 3^{16} order functions, only 25 are order functions for which $\mathcal{I}$ possesses property C.[8] Any proof that property CI implies property C, under conditions satisfied by the Example 1's information structure, must produce, as a byproduct, a construction that maps every one of the 3^{16} order functions for which $\mathcal{I}$ possesses property CI to one of the 25 order functions for which $\mathcal{I}$ possesses property C.

One such construction (Appendix E, (E.6)-(E.11)) can be used to prove the following theorem.

THEOREM 4. *Property* CI *implies property* C *when* Ω, *and* U^k, $k = 1, 2, \ldots, N$, *are countable sets, and* $\mathcal{B}$ *contains the singletons of* Ω.

Proof. See Appendix E. □

Since the success of this construction hinges on the fact that for all $s \in S_k$, $k = 0, 1, \ldots, N$, $\mathcal{F}(s)$ is the cylindrical extension of the power set of $\Omega \times (\prod_{i=1}^{k} U^{s_k})$ (a property that only holds under the conditions of the theorem), other constructions must be developed to establish that property CI implies property C under more general conditions.

5. Conclusions. In this paper we have introduced conditions necessary and sufficient to ensure that a generic stochastic system, represented within the framework of Witsenhausen's intrinsic model, is deadlock-free. The main results concern the fact that $\mathcal{I}$'s possession of property CI is

(1) A necessary and sufficient condition for all $\gamma \in \Gamma$ to be deadlock-free (Theorem 1(ii)); and

(2) A sufficient condition to ensure the existence, for all $\gamma \in \Gamma$, of a unique $\mathcal{B}/\mathcal{U}$-measurable function Σ^γ mapping all $w \in \Omega$ into unique solutions u_w^γ of the closed-loop equation $\gamma(w, u) = u$ (Theorem 1(i)).

[8] There are $(3!)^{16} = 6^{16}$ possible order functions $\psi : \Omega \times U \to S_3$ since $\text{card}(\Omega \times U) = 16$ and $\text{card}(S_3) = 3!$. Only 3^{16} of these satisfy the conditions of property CI since u^1 must precede u^2 when $w = 0$ and vice versa when $w = 1$ (for each (w, u) this rules out half of the 3! possible orders). Only 5^2 of the order functions satisfy the conditions of property C since $[T_1^3 \circ \psi]^{-1}(s)$ must be $\mathcal{F}(\emptyset)$-measurable and $[T_2^3 \circ \psi]^{-1}(s)$ must be $\mathcal{F}(T_1^2(s))$-measurable for all $s \in S_2$ (when $w = 0$, only $\psi|_{w=0} = (3,1,2)$ and $\psi|_{w=0,u^1} \in \{(1,2,3),(1,3,2)\}$ are acceptable; when $w = 1$ only $\psi|_{w=1} = (3,2,1)$ and $\psi|_{w=1,u^2} \in \{(2,1,3),(2,3,1)\}$ are acceptable).

These results subsume the principal result in [19 Theorem 1], and provide a necessary and sufficient condition for unconstrained stochastic control problems of the form (P) to be well-posed and deadlock-free.

The remaining results establish

(3) That $\mathcal{I}$'s possession of property CI ensures, for all $\gamma \in \Gamma$, that the function Σ^γ can be determined recursively, starting from an arbitrary "seed" $r \in U$, by composing $\gamma|_w$ with itself N times (see the discussion following Theorem 1);

(4) That property CI implies property C in a least three special cases (Corollary 2, Theorem 3, and Theorem 4); and

(5) That any general proof that property CI implies property C (i.e., that property C is a necessary condition for causality) must be constructive (see Example 1 and the discussion that follows).

Note that nowhere in the paper was any property of the reward—let alone the implicit assumption that agents cooperate to maximize this reward—ever used to construct a definition or derive a result; consequently, the results of this paper apply to games as well as controlled systems. For instance, by Theorem 1, a game involving a finite number of decisions chosen from decision spaces satisfying the constraints imposed by the intrinsic model, has an *extensive form* (i.e., a "game tree" representation, see [12]) if and only if its information structure possesses property CI.

Appendix A.

A. Decentralized detection: An example. This appendix concerns a decentralized detection network in which the optimal control policies must make explicit use of the fact that the network's control actions can be nonsequential. By example, it is shown that the introduction of nonsequentiality into the network can, under some circumstances, give rise to deadlocks, and under other circumstances, improve network performance.

A.1. The problem. Consider the problem of designing a simple decentralized detection network (Fig. A.1) consisting of two detectors, D1 and D2.

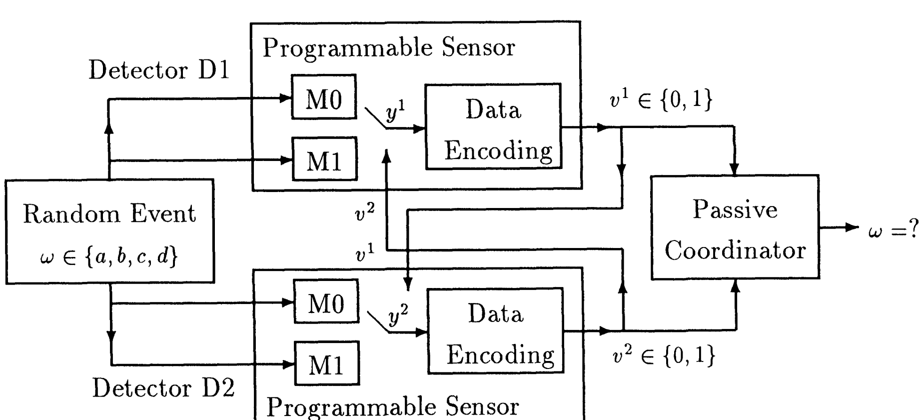

FIG. A.1. *A simple decentralized detection network.*

A.1.1. Observations. Each detector is permitted to make a single noisy observation, $y^k \in \{A, B\}$, $k = 1, 2$, of a random event $\omega \in \{a, b, c, d\}$ using one of two distinct configurations (Fig. A.2) of a programmable sensor possessing two operational modes $m^k \in \{0, 1\}$. Formally, for $k = 1, 2$,

$$y^k = h_{\alpha^k}(\omega, m^k), \tag{A.1}$$

where $\alpha^k \in \{1, 2\}$ indexes detector Dk's sensor configuration.

Configuration 1:

$$h_1(\omega, m^k) = \begin{cases} A & (\omega, m^k) \in \{(a,1), (d,1)\} \\ A & \omega = c \\ B & (\omega, m^k) \in \{(a,0), (d,0)\} \\ B & \omega = b \end{cases}$$

Configuration 2:

$$h_2(\omega, m^k) = \begin{cases} A & (\omega, m^k) \in \{(b,1), (c,1)\} \\ A & \omega = a \\ B & (\omega, m^k) \in \{(b,0), (c,0)\} \\ B & \omega = d \end{cases}$$

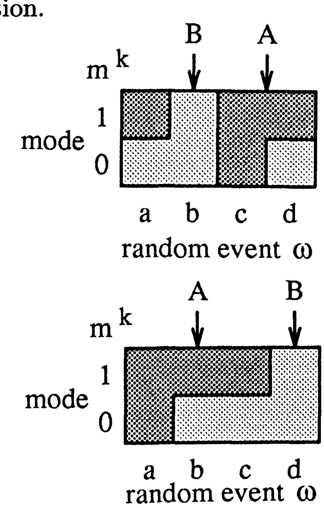

FIG. A.2. *Available sensor configurations.*

A.1.2. Data encoding. Once a detector has made its observation, it transmits a one bit summary, $v^k \in \{0, 1\}$, to a passive coordinator over a noiseless channel. Formally, for $k = 1, 2$,

$$v^k = g^k(y^k), \tag{A.2}$$

where g^k can be any function mapping $\{A, B\}$ to $\{0, 1\}$.

A.1.3. Sensor programming. Each detector can monitor the other's transmissions; accordingly, either may elect to program its sensor (i.e., set $m^k = 0$ or 1) based on the other's summary. Formally, for $k = 1, 2$,

$$m^k = f^k(v^{\bar{k}}), \tag{A.3}$$

where f^k can be any function mapping $\{0, 1\}$ to $\{0, 1\}$. When f^k is a constant function, the sensor programming is *static*—i.e., the mode in which detector Dk's sensor is operated is determined a priori. When f^k is not a constant function, the sensor programming is *dynamic*—i.e., the mode in which detector Dk's sensor is operated may depend on detector D$\bar{k}$'s one bit summary ($\bar{k}$ denotes the binary complement of $k \in \{0, 1\}$). It is the possibility that both detectors' sensors may be programmed dynamically that makes this decentralized detection network nonsequential—i.e., when neither f^1 nor f^2 is constant, the detectors' data summaries may be interdependent.

A.1.4. Passive coordinator. The passive coordinator, given the detectors' data summaries v^1 and v^2, attempts to correctly detect (identify) the uncertain outcome $\omega \in \{a, b, c, d\}$. Formally, the coordinator generates an estimate of ω,

$$\hat{\omega} = \eta(v^1, v^2), \tag{A.4}$$

using any function η mapping $\{0, 1\} \times \{0, 1\}$ to $\{a, b, c, d\}$.

A.1.5. Objective. Given a probability distribution for ω, the objective is to select an estimation policy for the passive observer, and sensor configurations, sensor programming policies, and data encoding policies for the detectors, that collectively maximize the probability that the coordinator can correctly identify ω. Formally, the objective is to

$$\text{Identify a } \textit{design}(\alpha^1, \alpha^2, f^1, f^2, g^1, g^2, h^1, h^2, \eta)$$

$$\text{that achieves} \max_{\substack{\alpha^k, f^k, g^k, h^k, \eta \\ k=1,2}} P\{\omega \in \Omega : \omega = \hat{\omega}\} \text{ exactly.}^9 \tag{A.5}$$

A.2. Deadlock. Clearly the preceding detection network is susceptible to deadlock. Suppose, for instance,

(1) That both detectors' sensors are in configuration 1 (i.e., $\alpha^1 = \alpha^2 = 1$),

(2) That each detector programs its sensor based on the other's data summary (e.g., $m^1 = u^2$, and $m^2 = u^1$), and

(3) That neither detector's data encoding policy is constant.

Then, when $\omega \in \{a, d\}$, detector D1's observation depends on detector D2's data summary and detector D2's observation depends on detector D1's data summary; consequently, neither detector can generate a data summary without precognition—i.e., the network is deadlocked.

A.3. A solution. Although the possibility of deadlock can be completely eliminated by constraining the network's design to be sequential (i.e., by prohibiting at least one detector from programming its sensor based on the other detector's data summary and thereby eliminating the possibility of nonsequentiality), this "fix" ignores the possibility that nonsequentiality may improve network performance. In fact,

(1) There exists a deadlock-free nonsequential design that enables the coordinator to correctly identify, with certainty, all uncertain outcomes $\omega \in \{a, b, c, d\}$, and

(2) No sequential design permits the coordinator to correctly identify, with certainty, more than two of the four uncertain outcomes $\omega \in \{a, b, c, d\}$.

In other words, in this case, *optimal network performance can only be achieved by exploiting the nonsequentiality of the network.*

A.3.1. An optimal nonsequential design. Consider, for instance, the following design:

[9] Note that, although it is tedious, it is not difficult to transform this problem into an unconstrained problem of the form (P) (§2.3). By setting $\omega = \omega \in \Omega := \{a, b, c, d\}, u^1 = \alpha^1 \in U^1 := \{1, 2\}, u^2 = \alpha^2 \in U^2 := \{1, 2\}, u^3 = m^1 \in U^3 := \{0, 1\}, u^4 = m^2 \in U^4 := \{0, 1\}, u^5 = v^1 \in U^5 := \{0, 1\}, u^6 = v^2 \in U^6 := \{0, 1\}, u^7 = \hat{\omega} \in U^7 := \{a, b, c, d\}$; by translating the informational constraints imposed (by the original problem formulation) into constraints on the information subfields $\mathcal{J}^k, k = 1, 2, \ldots, 7$, of $2^{\Omega \times U}$ (e.g., $\mathcal{J}^1 = \{\emptyset, \Omega \times U\}$ since $u^1 = \alpha^1$ must be a constant, $\mathcal{J}^3 = \{\emptyset, \Omega\} \otimes (\bigotimes_{i=1}^{5} \{\emptyset, U^i\}) \otimes 2^{U^6} \otimes \{\emptyset, U^7\}$ since $u^3 = m^1$ can only depend on $u^6 = v^2$, and so on); and by setting $V(\omega, u) = I_{\{\omega = u^7\}}$ (the indicator of the event $\{\omega = u^7\}$), one can transform the original problem into an unconstrained 7-agent problem in which the first two agents' decisions determine the detectors' sensor configurations, the third and forth agents' decisions correspond to the detectors' sensor programming decisions, the fifth and sixth agents' decisions correspond to the detectors' data summaries, the seventh agent's decision corresponds to the passive coordinator's estimate.

Detector D1:

$$\text{(A.6)}\quad \begin{aligned} m^1 &= f^1(v^2) = v^2 && \text{(D1's mode = D2's data summary),} \\ y^1 &= h_1(\omega, m^1) && \text{(D1's sensor in configuration 1), and} \\ v^1 &= g^1(y^1) = \begin{cases} 1 & y^1 = A \\ 0 & y^1 = B \end{cases} && \text{(D1's data summary).} \end{aligned}$$

Detector D2:

$$\text{(A.7)}\quad \begin{aligned} m^2 &= f^2(v^1) = v^1 && \text{(D2's mode = D1's data summary),} \\ y^2 &= h_2(\omega, m^2) && \text{(D2's sensor in configuration 2), and} \\ v^2 &= g^2(y^2) = \begin{cases} 0 & y^2 = A \\ 1 & y^2 = B \end{cases} && \text{(D2's data summary).} \end{aligned}$$

Passive Coordinator:

$$\text{(A.8)}\qquad \hat{\omega} = \eta(v^1, v^2) = \begin{cases} a & (v^1, v^2) = (0,0) \\ b & (v^1, v^2) = (0,1) \\ c & (v^1, v^2) = (1,0) \\ d & (v^1, v^2) = (1,1). \end{cases}$$

It is not difficult to verify that:

When $\omega = a$, D2 transmits $v^2 = 0$ first,
$D1$ transmits $v^1 = 0$ second, and
the passive coordinator sets $\hat{\omega} = a$;

When $\omega = b$, D1 transmits $v^1 = 0$ first,
D2 transmits $v^2 = 1$ second, and
the passive coordinator sets $\hat{\omega} = b$;

When $\omega = c$, D1 transmits $v^1 = 1$ first,
D2 transmits $v^2 = 0$ second, and
the passive coordinator sets $\hat{\omega} = c$;

When $\omega = d$, D2 transmits $v^2 = 1$ first,
D1 transmits $v^1 = 1$ second, and
the passive coordinator sets $\hat{\omega} = d$.

Since the order in which the detectors transmit their data summaries cannot be prespecified, this design is nonsequential. Since both detectors can transmit data summaries, without precognition, for all $\omega \in \{a, b, c, d\}$, the design is also deadlock-free. Finally, since the passive coordinator can correctly identify, with certainty, all uncertain events $\omega \in \{a, b, c, d\}$, the design is optimal.

A.3.2. No sequential design is optimal. Since the selection of an estimation policy $\eta : \{0,1\} \times \{0,1\} \to \{a, b, c, d\}$ and data encoding policies $g^1 : \{A, B\} \to \{0,1\}$ and $g^2 : \{A, B\} \to \{0,1\}$ is equivalent to the selection of a mapping $\zeta : \{A, B\} \times \{A, B\} \to \{a, b, c, d\}$—because

$$\text{(A.9)}\qquad \hat{\omega} = \eta(v^1, v^2) = \eta(g^1(y^1), g^2(y^2))$$

—to establish that no sequential design is optimal, it suffices to show that, as long as the mode of at least one of the detectors' sensors is fixed a priori, there is no way that the other detector can program its sensor (in either configuration) so as to ensure that every uncertain outcome $\omega \in \{a, b, c, d\}$ induces a unique element (y^1, y^2)

TABLE A.1
A (graphical) proof that sequential designs are suboptimal.

▩ =A ▨ =B

	configuration 1 mode 0 a b c d ω	configuration 1 mode 1 a b c d ω	configuration 2 mode 0 a b c d ω	configuration 2 mode 1 a b c d ω
configuration 1 mode 0 a b c d ω	a b d	a d	b d	a b
configuration 1 mode 1 a b c d ω	a d	a c d	c d	a c
configuration 2 mode 0 a b c d ω	b d	c d	b c d	b c
configuration 2 mode 1 a b c d ω	a b	a c	b c	a b c

in $\{A, B\} \times \{A, B\}$. Since each sensor has two configurations and two modes, and since there are two detectors, there are 16 cases to consider (eight if we exploit the fact that the sensors available to each detector are identical).

These 16 cases are succinctly summarized in Table A.1. The table can be read as follows. The rows correspond to the possible sensor configurations and fixed modes of the detector that is constrained to act first. The columns correspond to the possible sensor configurations and modes that can be associated with the first detector's uncertain event (i.e., $\{a, b, d\}$ in row 1, $\{a, c, d\}$ in row 2, etc.) when the second detector's sensor configuration and sensor programming policy are appropriately chosen. The table entries correspond to those uncertain outcomes that cannot be distinguished under the stated conditions (i.e., those outcomes that cannot be associated with a unique element of $\{A, B\} \times \{A, B\}$). For example, suppose

(1) That detector D1 is constrained to use sensor configuration 1 mode 0,

(2) That detector D2 uses sensor configuration 2, and

(3) That the composition of D2's programming policy with D1's encoding policy (i.e., $f^2 \circ g^1$) maps event B (D1's uncertain event) to mode 0.

Then, as one can easily verify, the uncertain outcomes b and d are indistinguishable (row one, column three).

Since there is an entry for every possible combination of sensor configurations and modes, under all circumstances, at least two uncertain outcomes are indistinguishable. It follows that no sequential design permits the coordinator to correctly identify, with certainty, more than two of the four uncertain outcomes $\omega \in \{a, b, c, d\}$.

A.4. Summary. By example, it has been shown that nonsequentiality can, under some circumstances, give rise to deadlocks (§A.3), and under other circumstances, improve network performance (§A.4).

Appendix B: Proof of Theorem 1. *Proof of* (i). Fix $\gamma \in \Gamma$ and suppose that ψ is an order function such that $\mathcal{I}$ possesses property CI. To prove that $\mathcal{I}$ possesses property SM it suffices to show that the closed-loop equation $u = \gamma(\omega, u)$ possesses at least one solution, $u^\gamma_\omega \in U$, for all $\omega \in \Omega$; that for each ω, this solution is unique; and that the mapping $\Sigma^\gamma : \Omega \to U$, induced by these unique solutions (i.e., $\Sigma^\gamma(\omega) = u^\gamma_\omega$) is $\mathcal{B}/\mathcal{U}$-measurable (see §3.1).

Existence. Fix $\omega \in \Omega$ and $r \in U$. Let π_U denote the canonical projection of $\Omega \times U$ onto U, let $L^\gamma : \Omega \times U \to \Omega \times U$ be defined as

$$L^\gamma(\omega, r) := (\omega, \gamma(\omega, r)), \tag{B.1}$$

let $L^\gamma_k : \Omega \times U \to \Omega \times U$ be a k-fold composition of L^γ,

$$L^\gamma_k(\omega, r) := (\underbrace{L^\gamma \circ \cdots \circ L^\gamma}_{\lfloor k\text{times} \rfloor})(\omega, r), \tag{B.2}$$

and let

$$s := (s_1, s_2, \ldots, s_N) = \psi(L^\gamma_N(\omega, r)). \tag{B.3}$$

To establish the existence of a closed-loop solution $u^\gamma_\omega \in U$, it suffices to show that

$$\begin{aligned} \gamma(L^\gamma_N(\omega, r)) &= \pi_U(L^\gamma_N(\omega, r)) \\ &= \pi_U(\omega, \gamma(L^\gamma_{N-1}(\omega, r))) \\ &= \gamma(L^\gamma_{N-1}(\omega, r)), \end{aligned} \tag{B.4}$$

or, equivalently, that

$$\gamma^{s_k}(L^\gamma_N(\omega, r)) = \gamma^{s_k}(L^\gamma_{N-1}(\omega, r)) \tag{B.5}$$

for all $k = 1, 2, \ldots, N$.

Since property CI holds with order function ψ, for all $k = 1, 2, \ldots, N$,

$$\mathcal{J}^{s_k} \bigcap [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(L^\gamma_N(\omega, r))) \subset \{\emptyset, [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(L^\gamma_N(\omega, r)))\}. \tag{B.6}$$

Since $\mathcal{U}^k$ contains the singletons of U^k for all $k = 1, 2, \ldots, N$, (B.6) implies that, at the point $L^\gamma_N(\omega, r) \in \Omega \times U$, all $\mathcal{J}^{s_k}/\mathcal{U}^{s_k}$-measurable functions, including γ^{s_k}, do not depend on components $(s_k+1), (s_{k+1}+1), \ldots$, and (s_N+1) of $L^\gamma_N(\omega, r)$; consequently, to establish (B.5) it suffices to show that components 1, $(s_1 + 1), (s_2 + 1), \ldots$, and $(s_{k-1} + 1)$ of $L^\gamma_N(\omega, r)$ and $L^\gamma_{N-1}(\omega, r)$ are identical—i.e., it suffices to show that

$$\mathcal{P}_{T^N_{k-1}(s)}(L^\gamma_N(\omega, r)) = \mathcal{P}_{T^N_{k-1}(s)}(L^\gamma_{N-1}(\omega, r)). \tag{B.7}$$

When $k = 1, T^N_{k-1}(s) = \emptyset$, and

$$\begin{aligned} \mathcal{P}_\emptyset(L^\gamma_N(\omega, r)) &= \mathcal{P}_\emptyset(\omega, \gamma(L^\gamma_{N-1}(\omega, r))) \\ &= (\omega) \\ &= \mathcal{P}_\emptyset(\omega, \gamma(L^\gamma_{N-2}(\omega, r))) \\ &= \mathcal{P}_\emptyset(L^\gamma_{N-1}(\omega, r)). \end{aligned} \tag{B.8}$$

For $k > 1$, suppose that (B.7) holds. Then, due to (B.6), (B.5) holds; accordingly,

$$\begin{aligned} \mathcal{P}_{T_k^N(s)}(L_N^\gamma(\omega,r)) &= (\mathcal{P}_{T_{k-1}^N(s)}(L_N^\gamma(\omega,r)), \gamma^{s_k}(L_N^\gamma(\omega,r))) \\ &= (\mathcal{P}_{T_{k-1}^N(s)}(L_{N-1}^\gamma(\omega,r)), \gamma^{s_k}(L_{N-1}^\gamma(\omega,r))) \\ &= \mathcal{P}_{T_k^N(s)}(L_{N-1}^\gamma(\omega,r)). \end{aligned} \tag{B.9}$$

It follows, by induction, that (B.7) holds for all $k = 1, 2, \ldots, N$; hence, (B.5) holds for all $k = 1, 2, \ldots, N$, and consequently, (B.4) holds—i.e., $\pi_U(L_N^\gamma(\omega, r))$ satisfies the closed-loop equation.

Uniqueness. Fix $\omega \in \Omega$ and $r \in U$, and once again, let

$$s := (s_1, s_2, \ldots, s_N) = \psi(L_N^\gamma(\omega,r)). \tag{B.10}$$

To establish that $\pi_U(L_N^\gamma(\omega, r))$ is the unique solution to the closed-loop equation $u = \gamma(\omega, u)$ it suffices to show that, $L_N^\gamma(\omega, r) = L_N^\gamma(\omega, \overline{r})$ for all $\overline{r} \in U$, or, equivalently, that

$$\mathcal{P}_{T_{k-1}^N(s)}(L_N^\gamma(\omega,r)) = \mathcal{P}_{T_{k-1}^N(s)}(L_N^\gamma(\omega,\overline{r})) \tag{B.11}$$

when $k = N + 1$. When $k = 1$, $T_{k-1}^N(s) = \emptyset$, and

$$\begin{aligned} \mathcal{P}_\emptyset(L_N^\gamma(\omega,r)) &= \mathcal{P}_\emptyset(\omega, \gamma(L_{N-1}^\gamma(\omega,r))) \\ &= (\omega) \\ &= \mathcal{P}_\emptyset(\omega, \gamma(L_{N-1}^\gamma(\omega,\overline{r}))) \\ &= \mathcal{P}_\emptyset(L_N^\gamma(\omega,\overline{r})). \end{aligned} \tag{B.12}$$

For $k > 1$, suppose that (B.11) holds. Then, just as (B.6) and (B.7) imply (B.5), (B.6) and (B.11) imply that

$$\gamma^{s_k}(L_N^\gamma(\omega,r)) = \gamma^{s_k}(L_N^\gamma(\omega,\overline{r})); \tag{B.13}$$

accordingly,

$$\begin{aligned} \mathcal{P}_{T_k^N(s)}(L_N^\gamma(\omega,r)) &= (\mathcal{P}_{T_{k-1}^N(s)}(L_N^\gamma(\omega,r)), \gamma^{s_k}(L_N^\gamma(\omega,r))) \\ &= (\mathcal{P}_{T_{k-1}^N(s)}(L_N^\gamma(\omega,\overline{r})), \gamma^{s_k}(L_N^\gamma(\omega,\overline{r}))) \\ &= \mathcal{P}_{T_k^N(s)}(L_N^\gamma(\omega,\overline{r})). \end{aligned} \tag{B.14}$$

It follows, by induction, that (B.11) holds for all $k = 1, 2, \ldots, N+1$; hence, $L_N^\gamma(\omega, r) = L_N^\gamma(\omega, \overline{r})$ for all $\overline{r} \in U$, and consequently, the unique solution u_ω^γ to the closed-loop equation $u = \gamma(\omega, u)$ is $\pi_U(L_N^\gamma(\omega, r))$, where $r \in U$ is the (arbitrary) "seed" that starts the recursive solution process.

Measurability. Fix $r \in U$ and let π_U and π_Ω denote, respectively, the canonical projections of $\Omega \times U$ onto U and Ω. To establish the $\mathcal{B}/\mathcal{U}$-measurability of the induced closed-loop solution map $\Sigma^\gamma : \Omega \to U$, it suffices to show that the u-section of $\pi_U \circ L_N^\gamma$, $\pi_U \circ L_N^\gamma|_r$, is $\mathcal{B}/\mathcal{U}$-measurable—because, for fixed r,

$$\Sigma^\gamma(\omega) = (\pi_U \circ L_N^\gamma|_r)(\omega) := (\pi_U \circ L_N^\gamma)(\omega, r). \tag{B.15}$$

To begin, note that (B.1) implies that

$$L^\gamma(\omega, r) = (\pi_\Omega(\omega, u), \gamma(\omega, r)). \tag{B.16}$$

By definition, π_Ω and π_U are, respectively, $\mathcal{B}\otimes\mathcal{U}/\mathcal{B}$- and $\mathcal{B}\otimes\mathcal{U}/\mathcal{U}$-measurable. Likewise, $\gamma^k, k=1,2,\ldots,N$, is $\mathcal{J}^k/\mathcal{U}^k$-measurable, accordingly, $\gamma := (\gamma^1,\gamma^2,\ldots,\gamma^N)$ is $\mathcal{B}\otimes\mathcal{U}/\mathcal{U}$-measurable (since $\mathcal{J}^k \subset \mathcal{B}\otimes\mathcal{U}$ for all k). It follows that L^γ, and by composition ([7, Thm. 13.1]), L_k^γ ((B.2)) and $\pi_U \circ L_N^\gamma$, are, respectively, $\mathcal{B}\otimes\mathcal{U}/\mathcal{B}\otimes\mathcal{U}$-, $\mathcal{B}\otimes\mathcal{U}/\mathcal{B}\otimes\mathcal{U}$-, and $\mathcal{B}\otimes\mathcal{U}/\mathcal{U}$-measurable. But all u-sections of $\mathcal{B}\otimes\mathcal{U}/\mathcal{U}$-measurable functions are $\mathcal{B}/\mathcal{U}$-measurable ([7, Thm. 18.1]); consequently, $\Sigma^\gamma = \pi_U \circ L_N^\gamma|_r$ is $\mathcal{B}/\mathcal{U}$-measurable.

Proof of (ii).

Sufficiency. Fix $\gamma \in \Gamma$, and suppose that ψ is an order function such that $\mathcal{I}$ possesses property CI. To prove that γ possesses property DF, it suffices to show that for each $\omega \in \Omega$, the agents can be ordered, such that no agent's decision depends on itself or the decisions of its successors.

Fix $\omega \in \Omega$. By (i), the closed-loop equation $u = \gamma(\omega, u)$ possesses a unique solution $u_\omega^\gamma \in U$. Let

$$s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u_\omega^\gamma). \tag{B.17}$$

Since property CI holds with order function ψ, for all $k = 1, 2, \ldots N$,

$$\mathcal{J}^{s_k} \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u_\omega^\gamma)) \subset \{\emptyset, [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u_\omega^\gamma))\}. \tag{B.18}$$

But (B.18) implies that, at the point $(\omega, u_\omega^\gamma) \in \Omega \times U$, all $\mathcal{J}^{s_k}/\mathcal{U}^{s_k}$-measurable functions, including γ^{s_k}, do not depend on components $(s_k+1), (s_{k+1}+1), \ldots$, and (s_N+1) of $(\omega, u_\omega^\gamma)$; consequently, for all $k = 1, 2, \ldots, N$, the s_kth agent's decision does not depend on the decisions of agents $s_k, s_{k+1}, \ldots$, and s_N. This proves sufficiency.

Necessity. Suppose that $\mathcal{I}$ does not possess property CI for any order function ψ. Then there exists at least one outcome in $\Omega \times U$, say (ω^*, u^*), such that for all N-agent orderings $s := (s_1, s_2, \ldots, s_N) \in S_N$,

$$\mathcal{J}^{s_k} \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega^*, u^*)) \subset \{\emptyset, [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega^*, u^*))\} \tag{B.19}$$

fails for at least one $k \in \{1, 2, \ldots, N\}$. To prove necessity, it suffices to construct a design $\gamma \in \Gamma$ that does not possess property DF.

For all $s \in S_N$, and $k = 1, 2, \ldots, N$, let

$$\mathcal{L}_s^k := \left\{A \in \mathcal{J}^{s_k} : (\omega^*, u^*) \in A, A \bigcap C_s^k(\omega^*, u^*) \notin \{\emptyset, C_s^k(\omega^*, u^*)\}\right\}, \tag{B.20}$$

where

$$C_s^k(\omega^*, u^*) := [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega^*, u^*)). \tag{B.21}$$

When (B.19) holds, $\mathcal{L}_s^k = \emptyset$. When (B.19) fails, $\mathcal{L}_s^k$ contains those events in $\mathcal{J}^{s_k}$ that contain (ω^*, u^*) and depend on the decisions of agents that have yet to act under the decision order s—i.e., those events containing (ω^*, u^*) that, under the decision order s, cannot be distinguished without precognition.

For all $s \in S_N$, and $k = 1, 2, \ldots, N$, set

$$A_s^{s_k} = U^{s_k} \text{ when } \mathcal{L}_s^k = \emptyset, \tag{B.22}$$

set

$$A_s^{s_k} = A, \ A \in \mathcal{L}_s^k, \ A \neq \emptyset, \text{ when } \mathcal{L}_s^k \neq \emptyset, \tag{B.23}$$

let $r^k \neq u^{*k}$ be an arbitrary reference element in U^k (such an r^k exists since card $(U^k) > 1$), and let

$$\gamma^k(\omega, u) := \begin{cases} u^{*k} & (\omega, u) \in \bigcap_{s' \in S_N} A^k_{s'} \\ r^k & \text{else.} \end{cases} \tag{B.24}$$

Since $\text{card}(S_N) = N!$, and $A^k_s \in \mathcal{J}^k$ for all $s \in S_N, \bigcap_{s' \in S_N} A^k_{s'}$ is $\mathcal{J}^k$-measurable; accordingly, γ^k is a $\mathcal{J}^k/\mathcal{U}^k$-measurable function for all $k = 1, 2, \ldots, N$, and consequently, $\gamma := (\gamma^1, \gamma^2, \ldots, \gamma^N)$ is an element of Γ.

The design γ, however, is not deadlock-free. Consider the outcome (ω^*, u^*), fix $s \in S_N$, and let k^* denote a k for which (B.19) fails. By construction (ω^*, u^*) satisfies the closed-loop equation (i.e., $u^* = \gamma(\omega^*, u^*)$); moreover, $\mathcal{L}^{k^*}_s \neq \emptyset$. It follows from (B.23) that $A^{s_{k^*}}_s \in \mathcal{L}^{k^*}_s$, and $A^{s_{k^*}}_s \neq \emptyset$; accordingly,

$$\begin{aligned} [\gamma^{s_{k^*}}]^{-1}(u^{*s_{k^*}}) \bigcap C^{k^*}_s(\omega^*, u^*) &= \left(\bigcap_{s' \in S_N} A^{s_{k^*}}_{s'} \right) \bigcap C^{k^*}_s(\omega^*, u^*) \\ &\notin \{\emptyset, C^{k^*}_s(\omega^*, u^*)\}. \end{aligned} \tag{B.25}$$

However, (B.25) implies that, at the point $(\omega^*, u^*) \in \Omega \times U$, agent s_{k^*}'s decision depends on the decision of agents that have yet to act under s. Since the same argument applies for all $s \in S_N, \gamma$ does not possess property DF. This proves necessity. □

Appendix C: Proof of Corollary 1. Although this corollary is an immediate consequence of Theorems 2(ii) and 1(ii) (property C $\Rightarrow$ property DF $\Rightarrow$ property CI), it is instructive to prove it directly.

Suppose that ψ is an order function such that $\mathcal{I}$ possesses property C. It suffices to show that ψ is also an order function such that $\mathcal{I}$ possesses property CI—i.e., that (4.1) of property C (with $s = T^N_k(\psi(\omega, u)) \in S_k$), implies (3.3) of property CI (with $s = \psi(\omega, u) \in S_N$), for all $(\omega, u) \in \Omega \times U$ and $k = 1, 2, \ldots, N$.

Fix $(\omega, u) \in \Omega \times U$ and $k \in \{1, 2, \ldots, N\}$, and let

$$s := (s_1, s_2, \ldots, s_N) = \psi(\omega, u). \tag{C.1}$$

Since $T^N_k(s) \in S_k$, and $T^N_{k-1} = T^k_{k-1} \circ T^N_k$, (4.1) of property C implies that

$$\mathcal{J}^{s_k} \bigcap [T^N_k \circ \psi]^{-1}(T^N_k(s)) \subset \mathcal{F}(T^N_{k-1}(s)). \tag{C.2}$$

Restricting both sides of (C.2) to

$$[\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) \tag{C.3}$$

yields the desired result—(3.3) of property CI—if

$$\begin{aligned} &[T^N_k \circ \psi]^{-1}(T^N_k(s)) \bigcap [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) \\ &\quad = [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) \end{aligned} \tag{C.4}$$

and

$$\begin{aligned} &\mathcal{F}(T^N_{k-1}(s)) \bigcap [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)) \\ &\quad = \{\emptyset, [\mathcal{P}_{T^N_{k-1}(s)}]^{-1}(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u))\}. \end{aligned} \tag{C.5}$$

Equation (C.5) follows from the definition of $\mathcal{F}(T_{k-1}^N(s))$,

$$\mathcal{F}(T_{k-1}^N(s)) := [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i}\right)\right), \tag{C.6}$$

and the fact that inverse images preserve intersections—i.e.,

$$\begin{aligned} [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i}\right)\right) \bigcap \ & [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u)) \\ &= \{\emptyset, [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u))\}. \end{aligned} \tag{C.7}$$

Equation (C.4) follows from the observation that

$$[T_k^N \circ \psi]^{-1}(T_k^N(s)) \in \mathcal{F}(T_{k-1}^N(s)) \tag{C.8}$$

(to see this substitute $\Omega \times U \in \mathcal{J}^{s_k}$ for $\mathcal{J}^{s_k}$, and $\in$ for $\subset$, in (C.2)), (C.5), and the fact that

$$[T_k^N \circ \psi]^{-1}(T_k^N(s)) \bigcap [\mathcal{P}_{T_{k-1}^N(s)}]^{-1}(\mathcal{P}_{T_{k-1}^N(s)}(\omega, u)) \neq \emptyset \tag{C.9}$$

since both sets contain (ω, u). $\square$

Appendix D: Proof of Theorem 3. Suppose that $\mathcal{I}$ is sequential. Then there exists a *constant* order function ψ such that $\mathcal{I}$ possesses property CI. It suffices to show that ψ is also an order function such that $\mathcal{I}$ possesses property C—i.e., that for all $k = 1, 2, \ldots, N$, the fact that (3.3) of property CI holds for all $(\omega, u) \in \Omega \times U$ with $s = s^* \in S_N$ constant, implies that (4.1) of property C holds for all $s \in S_k$.

Fix $k \in \{1, 2, \ldots, N\}$ and let

$$s^* := (s_1^*, s_2^*, \ldots, s_N^*) \tag{D.1}$$

denote the constant order induced by ψ. Since

$$[T_k^N \circ \psi]^{-1}(s) = \begin{cases} \Omega \times U & \text{when} \quad s = T_k^N(s^*) \\ \emptyset & \qquad\quad \text{else} \end{cases} \tag{D.2}$$

for all $s \in S_k$, and since $T_{k-1}^N = T_{k-1}^k \circ T_k^N$, to prove that (4.1) of property C holds for all $s \in S_k$, it suffices to show that

$$\mathcal{J}^{s_k^*} \subset \mathcal{F}(T_{k-1}^N(s^*)). \tag{D.3}$$

By definition, $\mathcal{J}^{s_k^*}$ is a subfield of

$$\mathcal{B} \otimes \mathcal{U} = [\mathcal{P}_{T_N^N(s^*)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{N} \mathcal{U}^{s_i^*}\right)\right). \tag{D.4}$$

Since (3.3) holds for all $(\omega, u) \in \Omega \times U$ when $s = s^*$, all events in $\mathcal{J}^{s_k^*}$ must be of the form

$$[\mathcal{P}_{T_N^N(s^*)}]^{-1}\left(A \times \left(\prod_{i=k}^{N} U^{s_i^*}\right)\right), \tag{D.5}$$

where $A \subset \Omega \times \Pi_{i=1}^{k-1} U^{s_i^*}$; accordingly, $\mathcal{J}^{s_k^*}$ is also a subfield of

$$\mathcal{C}_{s^*} := \sigma\left([\mathcal{P}_{T_N^N(s^*)}]^{-1}\left(A \times \left(\prod_{i=k}^{N} U^{s_i^*} \right)\right) : A \subset \Omega \times \prod_{i=1}^{k-1} U^{s_i^*} \right)$$

$$= \sigma\left([\mathcal{P}_{T_{k-1}^N(s^*)}]^{-1}(A) : A \subset \Omega \times \prod_{i=1}^{k-1} U^{s_i^*} \right) \tag{D.6}$$

—the cylindrical extension of the power set of $\Omega \times \prod_{i=1}^{k-1} U^{s_i^*}$ to $\Omega \times U$. However,

$$(\mathcal{B} \otimes \mathcal{U}) \bigcap \mathcal{C}_{s^*} = [\mathcal{P}_{T_{k-1}^N(s^*)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i^*} \right)\right)$$

$$:= \mathcal{F}(T_{k-1}^N(s^*)); \tag{D.7}$$

consequently, $\mathcal{J}^{s_k^*} \subset \mathcal{F}(T_{k-1}^N(s^*))$. □

Appendix E: Proof of Theorem 4. Suppose that ψ is an order function such that $\mathcal{I}$ possesses property CI. It suffices to construct an order function $\hat{\psi}$ such that $\mathcal{I}$ possesses property C.

To simplify property C's verification, it is convenient to construct $\hat{\psi}$ recursively. The recursion has N steps, the kth of which, $k = 1, 2, \ldots, N$, corresponds to the construction of a function

$$f_k : \Omega \times U \to S_k \tag{E.1}$$

with the following properties:

(1) For all $j \in \{1, 2, \ldots, k-1\}, T_j^k \circ f_k = f_j$, and

(2) For all $s := (s_1, s_2, \ldots s_k) \in S_k, \mathcal{J}^{s_k} \bigcap [f_k]^{-1}(s) \subset \mathcal{F}(T_{k-1}^k(s))$.

Property (1) suffices to ensure that $f_k = [T_k^N \circ f_N]$; consequently, property (2) suffices to ensure that $\hat{\psi} = f_N$ is an order function such that $\mathcal{I}$ possesses property C (see Definition 3 in §4).

For all $(\omega, u) \in \Omega \times U$,

$$s := (s_1, s_2, \ldots, s_{k-1}) \in S_{k-1}, \tag{E.2}$$

and $k = 1, 2, \ldots, N$: let

$$C_s(\omega, u) := [\mathcal{P}_s]^{-1}(\mathcal{P}_s(\omega, u)) \tag{E.3}$$

denote the cylinder set induced on $\Omega \times U$ by $(\omega, u^{s_1}, \ldots, u^{s_{k-1}})$; let

$$\langle\langle s \rangle\rangle := (\langle\langle s \rangle\rangle_1, \langle\langle s \rangle\rangle_2, \ldots, \langle\langle s \rangle\rangle_N) \in S_N \tag{E.4}$$

denote the unique element in S_N for which $T_{k-1}^N(\langle\langle s \rangle\rangle) = s$, and $\langle\langle s \rangle\rangle_k < \langle\langle s \rangle\rangle_{k+1} < \cdots < \langle\langle s \rangle\rangle_N$; and let

$$s, \langle\langle s \rangle\rangle_j := (s_1, s_2, \ldots, s_{k-1}, \langle\langle s \rangle\rangle_j) \tag{E.5}$$

$j = k, k+1, \ldots, N$, denote the concatenation of $\langle\langle s \rangle\rangle_j$ to s. Then the recursive construction of $\hat{\psi}$, given ψ, can be described as follows:

1. For all $j = 1, 2, \ldots, N$, let

$$f_1(\omega, u) = j \tag{E.6}$$

when

$$(\omega, u) \in C_\emptyset([T_1^N \circ \psi]^{-1}(j)) \backslash \left(\bigcup_{i=1}^{j-1} C_\emptyset([T_1^N \circ \psi]^{-1}(i)) \right).\text{[10]} \tag{E.7}$$

$$\vdots$$

k. For all $s \in S_{k-1}$, and $j = k, k+1, \ldots, N$, let

$$f_k(\omega, u) = s, \langle\langle s \rangle\rangle_j \tag{E.8}$$

when

$$\begin{aligned}(\omega, u) \in [f_{k-1}]^{-1}(s) \bigcap & \Bigg(C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s \rangle\rangle_j)) \\ & \backslash \left(\bigcup_{i=k}^{j-1} C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s \rangle\rangle_i)) \right) \Bigg).\end{aligned} \tag{E.9}$$

$$\vdots$$

N. For all $s \in S_{N-1}$, let

$$f_N(\omega, u) = s, \langle\langle s \rangle\rangle_N \tag{E.10}$$

when

$$(\omega, u) \in [f_{N-1}]^{-1}(s) \bigcap C_s([T_{N-1}^N \circ \psi]^{-1}(s, \langle\langle s \rangle\rangle_N)). \tag{E.11}$$

To verify that the preceding constructions give rise to legitimate functions it suffices to check, for all $k = 1, 2, \ldots, N$, that $\{[f_k]^{-1}(s) : s \in S_k\}$ partitions $\Omega \times U$. The following facts will be used without comment:

- Unions and intersections are distributive.
- Inverse and direct images preserve unions and inclusions.
- $\{[T_k^N \circ \psi]^{-1}(s) : s \in S_k\}$ partitions $\Omega \times U$ for all $k = 1, 2, \ldots, N$; moreover, since

$$[T_{k-1}^k]^{-1}(s) = \bigcup_{i=k}^{N} (s, \langle\langle s \rangle\rangle_i), \tag{E.12}$$

for all $s \in S_{k-1}, k = 1, 2, \ldots, N$,

$$\begin{aligned} [T_{k-1}^N \circ \psi]^{-1}(s) &= [T_{k-1}^k \circ T_k^N \circ \psi]^{-1}(s) \\ &= [T_k^N \circ \psi]^{-1}([T_{k-1}^k]^{-1}(s)) \\ &= [T_k^N \circ \psi]^{-1}\left(\bigcup_{i=k}^{N} (s, \langle\langle s \rangle\rangle_i) \right) \\ &= \bigcup_{i=k}^{N} [T_k^N \circ \psi]^{-1}(s, \langle\langle s \rangle\rangle_i). \end{aligned} \tag{E.13}$$

[10] For sets $A, B \in X, A \backslash B := \{x \in A : x \notin B\}$.

- When A, B, C, D, E are sets,

$A \cup (B \backslash A) = A \cup B,\ A \cup B \cup (C \backslash (A \cup B)) = A \cup B \cup C$, and so on, and

$$A \cap (B \backslash A) = \emptyset, C \cap (E \backslash (A \cup B \cup C \cup D)) = \emptyset, \text{and so on.} \tag{E.14}$$

When $k = 1$,

$$\begin{aligned}
[f_1]^{-1}(S_1) &:= [f_1]^{-1}\left(\bigcup_{j=1}^{N}\{j\}\right) \\
&= \bigcup_{j=1}^{N}[f_1]^{-1}(j) \\
&:= \bigcup_{j=1}^{N}\left(C_\emptyset([T_1^N \circ \psi]^{-1}(j)) \backslash \left(\bigcup_{i=1}^{j-1} C_\emptyset([T_1^N \circ \psi]^{-1}(i))\right)\right) \\
&= \bigcup_{j=1}^{N} C_\emptyset([T_1^N \circ \psi]^{-1}(j)) \qquad \text{(E.15)} \\
&= C_\emptyset\left(\bigcup_{j=1}^{N}[T_1^N \circ \psi]^{-1}(j)\right) \\
&= C_\emptyset(\Omega \times U) \\
&\supset \Omega \times U.
\end{aligned}$$

Moreover, (E.6) and (E.7) imply that for all $m, n \in \{1, 2, \ldots, N\}, m < n$,

$$\begin{aligned}
[f_1]^{-1}(m) \bigcap [f_1]^{-1}(n) &:= \left(C_\emptyset([T_1^N \circ \psi]^{-1}(m)) \backslash \left(\bigcup_{i=1}^{m-1} C_\emptyset([T_1^N \circ \psi]^{-1}(i))\right)\right) \\
&\qquad \bigcap \left(C_\emptyset([T_1^N \circ \psi]^{-1}(n)) \backslash \left(\bigcup_{i=1}^{n-1} C_\emptyset([T_1^N \circ \psi]^{-1}(i))\right)\right) \\
&\qquad \subset C_\emptyset([T_1^N \circ \psi]^{-1}(m)) \qquad \text{(E.16)} \\
&\qquad \bigcap \left(C_\emptyset([T_1^N \circ \psi]^{-1}(n)) \backslash \left(\bigcup_{i=1}^{n-1} C_\emptyset([T_1^N \circ \psi]^{-1}(i))\right)\right) \\
&\qquad = \emptyset.
\end{aligned}$$

It follows that $\{[f_1]^{-1}(s) : s \in S_1\}$ partitions $\Omega \times U$.

For $k > 1$, suppose that $\{[f_{k-1}]^{-1}(s) : s \in S_{k-1}\}$ partitions $\Omega \times U$. Then

$$\begin{aligned}
[f_k]^{-1}(S_k) &:= [f_k]^{-1}\left(\bigcup_{s' \in S_k} s'\right) \\
&:= [f_k]^{-1}\left(\bigcup_{s \in S_{k-1}} \bigcup_{j=k}^{N}(s, \langle\langle s \rangle\rangle_j)\right) \\
&= \bigcup_{s \in S_{k-1}} \bigcup_{j=k}^{N}[f_k]^{-1}(s, \langle\langle s \rangle\rangle_j)
\end{aligned}$$

$$
\begin{aligned}
&:= \bigcup_{s\in S_{k-1}} \bigcup_{j=k}^{N} \Bigg([f_{k-1}]^{-1}(s) \bigcap \Bigg(C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_j)) \\
&\qquad \setminus \Bigg(\bigcup_{i=k}^{j-1} C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_i)) \Bigg)\Bigg)\Bigg) \\
&= \bigcup_{s\in S_{k-1}} \Bigg([f_{k-1}]^{-1}(s) \bigcap \Bigg(\bigcup_{j=k}^{N} \Bigg(C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_j)) \\
&\qquad \setminus \Bigg(\bigcup_{i=k}^{j-1} C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_i)) \Bigg)\Bigg)\Bigg)\Bigg) \qquad \text{(E.17)} \\
&= \bigcup_{s\in S_{k-1}} \Bigg([f_{k-1}]^{-1}(s) \bigcap \Bigg(\bigcup_{j=k}^{N} C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_j)) \Bigg)\Bigg) \\
&= \bigcup_{s\in S_{k-1}} \Bigg([f_{k-1}]^{-1}(s) \bigcap C_s \Bigg(\bigcup_{j=k}^{N} [T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_j) \Bigg)\Bigg) \\
&= \bigcup_{s\in S_{k-1}} \Bigg([f_{k-1}]^{-1}(s) \bigcap C_s([T_{k-1}^N \circ \psi]^{-1}(s)) \Bigg) \\
&= \Bigg(\bigcup_{s\in S_{k-1}} [f_{k-1}]^{-1}(s) \Bigg) \bigcap \Bigg(\bigcup_{s\in S_{k-1}} C_s([T_{k-1}^N \circ \psi]^{-1}(s)) \Bigg) \\
&= (\Omega \times U) \bigcap \Bigg(\bigcup_{s\in S_{k-1}} C_s([T_{k-1}^N \circ \psi]^{-1}(s)) \Bigg) \\
&= \bigcup_{s\in S_{k-1}} C_s([T_{k-1}^N \circ \psi]^{-1}(s)) \\
&\supset \bigcup_{s\in S_{k-1}} [T_{k-1}^N \circ \psi]^{-1}(s) \\
&= \Omega \times U.
\end{aligned}
$$

Moreover, for all $s, \overline{s} \in S_k$ such that $s \neq \overline{s}$, when $T_{k-1}^k(s) \neq T_{k-1}^k(\overline{s})$, (E.8) and (E.9) and the induction hypothesis imply that

$$
\begin{aligned}
[f_k]^{-1}(s) \bigcap [f_k]^{-1}(\overline{s}) &\subset [f_{k-1}]^{-1}(T_{k-1}^k(s)) \bigcap [f_{k-1}]^{-1}(T_{k-1}^k(\overline{s})) \\
&= \emptyset, \qquad \text{(E.18)}
\end{aligned}
$$

and when $T_{k-1}^k(s) = T_{k-1}^k(\overline{s})$ (implying that $s_k \neq \overline{s}_k$), (E.8) and (E.9) and the induction hypothesis imply that for some $m < n$ (say $s_k = \langle\langle s\rangle\rangle_m, \overline{s}_k = \langle\langle \overline{s}\rangle\rangle_n$)

$$
\begin{aligned}
[f_k]^{-1}(s) \bigcap [f_k]^{-1}(\overline{s}) \subset\; & \Bigg(C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_m)) \\
& \setminus \Bigg(\textstyle\bigcup_{i=k}^{m-1} C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_i)) \Bigg)\Bigg) \\
& \bigcap \Bigg(C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_n))
\end{aligned}
$$

$$
\begin{aligned}
&\setminus \left(\bigcup_{i=k}^{n-1} C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_i))\right)\Bigg)\\
&\subset C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_m))\\
&\bigcap \Bigg(C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_n))\\
&\setminus \left(\bigcup_{i=k}^{n-1} C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s\rangle\rangle_i))\right)\Bigg)\\
= \quad &\emptyset.
\end{aligned}
\tag{E.19}
$$

Consequently, $\{[f_k]^{-1}(s) : s \in S_k\}$ partitions $\Omega \times U$. It follows, by induction, that for all $k = 1, 2, \ldots, N$, $\{[f_k]^{-1}(s) : s \in S_k\}$ partitions $\Omega \times U$.

Having established, for all $k = 1, 2, \ldots, N$, that f_k is a legitimate function, it remains to show that f_k satisfies properties (1) and (2) (cf. the discussion following (E.1)). To verify property (1) it suffices to prove, for all $k = 1, 2, \ldots, N$, that

$$T_{k-1}^k \circ f_k = f_{k-1}, \tag{E.20}$$

or, equivalently, that

$$[T_{k-1}^k \circ f_k]^{-1}(s) = [f_{k-1}]^{-1}(s) \tag{E.21}$$

for all $s \in S_{k-1}$. Fix $k \in \{1, 2, \ldots, N\}$. By (E.8) and (E.9)

$$[f_k]^{-1}(s, \langle\langle s\rangle\rangle_j) \subset [f_{k-1}]^{-1}(s) \tag{E.22}$$

for all $s \in S_{k-1}$ and $j = k, k+1, \ldots, N$; consequently, since $\{[f_{k-1}]^{-1}(s) : s \in S_{k-1}\}$ partitions $\Omega \times U$, for all $s, \overline{s} \in S_{k-1}$ such that $s \neq \overline{s}$, and for arbitrary $j \in \{k, k+1, \ldots, N\}$,

$$[f_k]^{-1}(s, \langle\langle s\rangle\rangle_j) \bigcap [f_{k-1}]^{-1}(\overline{s}) = \emptyset. \tag{E.23}$$

However, $\{[f_k]^{-1}(s) : s \in S_k\}$ also partitions $\Omega \times U$; accordingly, (E.23) implies that for all $s \in S_{k-1}$,

$$
\begin{aligned}
[T_{k-1}^k \circ f_k]^{-1}(s) &= [f_k]^{-1} \circ [T_{k-1}^k]^{-1}(s)\\
&= [f_k]^{-1}\left(\bigcup_{i=k}^{N}(s, \langle\langle s\rangle\rangle_i)\right)\\
&= \bigcup_{i=k}^{N}[f_k]^{-1}(s, \langle\langle s\rangle\rangle_i)\\
&= \left(\bigcup_{\overline{s}\in S_{k-1}} \bigcup_{i=k}^{N}[f_k]^{-1}(\overline{s}, \langle\langle \overline{s}\rangle\rangle_i)\right) \bigcap [f_{k-1}]^{-1}(s)\\
&= \left(\bigcup_{s'\in S_k} [f_k]^{-1}(s')\right) \bigcap [f_{k-1}]^{-1}(s)\\
&= (\Omega \times U) \bigcap [f_{k-1}]^{-1}(s)\\
&= [f_{k-1}]^{-1}(s)
\end{aligned}
\tag{E.24}
$$

—i.e., (E.21), and, consequently, property (1) hold.

To verify property (2) (see the discussion following (E.1)) it is necessary to establish the following lemma.

LEMMA E1. *Suppose that Ω, and $U^k, k = 1, 2, \ldots, N$, are countable sets, and suppose that $\mathcal{B}$ contains the singletons of Ω. Then if ψ is an order function such that $\mathcal{I}$ possesses property* CI, *for all $s \in S_k, k = 1, 2, \ldots, N$,*

$$\mathcal{J}^{s_k} \bigcap C_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s)) \subset \mathcal{F}(T^k_{k-1}(s)). \tag{E.25}$$

Proof. By assumption, the σ-fields $\mathcal{B}$ and $\mathcal{U}^k, k = 1, 2, \ldots, N$, contain, respectively, the singletons of the countable sets Ω and $U^k, k = 1, 2, \ldots, N$ ($\mathcal{U}^k$ contains the singletons of U^k due to §2.2, 1(c)). Accordingly, for all $s := (s_1, s_2, \ldots, s_k) \in S_k, k = 1, 2, \ldots, N$, the product field $\mathcal{B} \otimes (\bigotimes_{i=1}^k \mathcal{U}^{s_i})$ contains the singletons of the countable set $\Omega \times (\Pi_{i=1}^k U^{s_i})$, implying that $\mathcal{B} \otimes (\bigotimes_{i=1}^k \mathcal{U}^{s_1})$ is the power set of $\Omega \times (\Pi_{i=1}^k U^{s_i})$. It follows, for all $s \in S_k, k = 1, 2, \ldots, N$, that

$$\begin{aligned} \mathcal{F}(T^k_{k-1}(s)) \;&:=\; [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}\left(\mathcal{B} \otimes \left(\bigotimes_{i=1}^{k-1} \mathcal{U}^{s_i}\right)\right) \\ &=\; \sigma\left([\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(A) : A \subset \Omega \times \prod_{i=1}^{k-1} U^{s_i}\right) \end{aligned} \tag{E.26}$$

—i.e., it follows that $\mathcal{F}(T^k_{k-1}(s))$ is the cylindrical extension of the power set of $\Omega \times \Pi_{i=1}^{k-1} U^{s_i}$ to $\Omega \times U$.

Fix $k \in \{1, 2, \ldots, N\}$ and $s \in S_k$. Since property CI holds with order function ψ, (E.3), (3.3), and (E.26) imply that for all $(\omega, u) \in [T^N_k \circ \psi]^{-1}(s)$ and $A \in \mathcal{J}^{s_k}$,

$$\begin{aligned} A \bigcap C_{T^k_{k-1}(s)}(\omega, u) \;&=\; A \bigcap [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}(\omega, u)) \\ &\in\; \{\emptyset, [\mathcal{P}_{T^k_{k-1}(s)}]^{-1}(\mathcal{P}_{T^k_{k-1}(s)}(\omega, u))\} \\ &\subset\; \mathcal{F}(T^k_{k-1}(s)). \end{aligned} \tag{E.27}$$

Since $[T^N_k \circ \psi]^{-1}(s) \in \Omega \times U$ is a countable set, and since inverse and direct images preserve unions, it follows that

$$\begin{aligned} A \bigcap C_{T^k_{k-1}(s)}([T^N_k \circ \psi]^{-1}(s)) \;&=\; A \bigcap C_{T^k_{k-1}(s)}\left(\bigcup_{(\omega,u) \in [T^N_k \circ \psi]^{-1}(s)} (\omega, u)\right) \\ &=\; \bigcup_{(\omega,u) \in [T^N_k \circ \psi]^{-1}(s)} (A \bigcap C_{T^k_{k-1}(s)}(\omega, u)) \\ &\in\; \mathcal{F}(T^k_{k-1}(s)). \end{aligned} \tag{E.28}$$

This proves the lemma since (E.28) holds for all $A \in \mathcal{J}^{s_k}$, and consequently, implies (E.25). □

Given Lemma E1, by induction, all f_k can be shown to possess property (2). For $k = 1$, fix $j \in \{1, 2, \ldots, N\}$. By Lemma E1, for all $A \in \mathcal{J}^j$,

$$A \bigcap C_\emptyset([T^N_1 \circ \psi]^{-1}(j)) \in \mathcal{F}(\emptyset). \tag{E.29}$$

Likewise, since $\Omega \times U \in \mathcal{J}^i$ for all i, for all $i = 1, 2, \ldots, N$,

$$C_\emptyset([T^N_1 \circ \psi]^{-1}(i)) \in \mathcal{F}(\emptyset); \tag{E.30}$$

accordingly,

$$\bigcup_{i=1}^{j-1} C_\emptyset([T_1^N \circ \psi]^{-1}(i)) \in \mathcal{F}(\emptyset). \tag{E.31}$$

It follows, from (E.29) and (E.31), that

$$A \bigcap [f_1]^{-1}(j) \ := \ A \bigcap \left(C_\emptyset([T_1^N \circ \psi]^{-1}(j)) \right. \tag{E.32}$$

$$\left. \setminus \left(\bigcup_{i=1}^{j-1} C_\phi([T_1^N \circ \psi]^{-1}(i)) \right) \right)$$

$$\in \ \mathcal{F}(\emptyset). \tag{E.33}$$

Since (E.32) holds for all $A \in \mathcal{J}^j$, f_1 satisfies property (2)—i.e., for all $j \in S_1$,

$$\mathcal{J}^j \bigcap [f_1]^{-1}(j) \subset \mathcal{F}(\emptyset). \tag{E.34}$$

For $k > 1$, suppose that f_{k-1} satisfies property (2)—i.e., suppose that, for all $s \in S_{k-1}$,

$$\mathcal{J}^{s_{k-1}} \bigcap [f_{k-1}]^{-1}(s) \subset \mathcal{F}(T_{k-2}^{k-1}(s)). \tag{E.35}$$

Then, since $\Omega \times U \in \mathcal{J}^i$ for all $i = 1, 2, \ldots, N$, for all $s \in S_{k-1}$,

$$[f_{k-1}]^{-1}(s) \subset \mathcal{F}(T_{k-2}^{k-1}(s)) \subset \mathcal{F}(s). \tag{E.36}$$

Fix $s \in S_{k-1}$ and $j \in \{k, k+1, \ldots, N\}$. By Lemma E1, for all $A \in \mathcal{J}^{\langle\langle s \rangle\rangle_j}$,

$$A \bigcap \left(C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s \rangle\rangle_j)) \right) \in \mathcal{F}(s). \tag{E.37}$$

Likewise, since $\Omega \times U \in \mathcal{J}^i$ for all i, for all $i = k, k+1, \ldots, N$,

$$C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s \rangle\rangle_i)) \in \mathcal{F}(s); \tag{E.38}$$

accordingly,

$$\bigcup_{i=k}^{j-1} C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s \rangle\rangle_i)) \in \mathcal{F}(s). \tag{E.39}$$

It follows, from (E.35), (E.36), and (E.38), that

$$A \bigcap [f_k]^{-1}(s, \langle\langle s \rangle\rangle_j) \ := \ A \bigcap [f_{k-1}]^{-1}(s) \bigcap \left(C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s \rangle\rangle_j)) \right.$$

$$\left. \setminus \left(\bigcup_{i=k}^{j-1} C_s([T_k^N \circ \psi]^{-1}(s, \langle\langle s \rangle\rangle_i)) \right) \right) \tag{E.40}$$

$$\in \ \mathcal{F}(s).$$

Since (E.39) holds for all $A \in \mathcal{J}^{\langle\langle s \rangle\rangle_j}$, f_k satisfies property (2)—i.e., for all $s \in S_k$

$$\mathcal{J}^{s_k} \bigcap [f_k]^{-1}(s) \subset \mathcal{F}(T_{k-1}^k(s)). \tag{E.41}$$

It follows, by induction, that f_k satisfies property (2) for all $k = 1, 2, \ldots, N$; consequently, since all f_k's also satisfy property (1), $\hat{\psi} = f_N$ is an order function such that $\mathcal{I}$ possesses property C (see the discussion following (E.1)). This proves the theorem. $\square$

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