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On Information Structures and Nonsequential Stochastic Control

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In many of the present controlled large-scale systems - communication and computer networks, detection networks, manufacturing systems, economic systems, database systems, power systems, etc. - information is decentralized. Furthermore, in the abovementioned systems it may be impossible to order the control actions *a priori*, independently of the set of control laws that determines these actions. Such systems are called *nonsequential*. The theory of nonsequential stochastic controlled systems is at a very early stage of development. In this paper, we first present a survey of existing results on nonsequential systems within the framework of Witsenhausen's intrinsic model; then, we discuss some open problems arising from the research performed so far.

1. Introduction

In centralized stochastic controlled systems, all control actions are taken by one control station where all the information is gathered. The station has perfect recall and can base each action on all the information gathered up to the time the action must be taken. The theoretical foundations for the analysis and optimization of centralized stochastic controlled systems are by now well-developed (see, for example, [12], [18], [26], [27], [28], [33]).

Most of the present large-scale systems such as communication and computer networks, manufacturing systems, economic systems, database systems, power systems, etc., are informationally decentralized. The salient features of these systems are the following: (1) there are several control stations that have access to different information; (2) the stations may communicate among each other by signaling through the system itself or through noisy channels (that are part of the system); (3) the stations have a common objective; (4) the stations have to coordinate their control strategies to optimize that common objective.

The decentralization of information and the possibility of communication among control stations make decentralized decision problems drastically different from centralized stochastic control problems. The difficulties arising in informationally decentralized ststems are clearly pointed out by Witsenhausen in [34] as follows: In informationally decentralized systems "the data available for a certain decision may be insufficient to determine what the control values chosen at earlier decisions were. Worse yet, the data may be insufficient to determine which decisions have been made and which are in the future and could possibly have their data dependent upon the decision under consideration. This is because for any agent (device) which is to implement a decision, the time (and place) of that decision may depend upon the random inputs to the system and on the values decided upon by other stations."

Because of the abovementioned difficulties, the fundamental techniques of analysis and optimization of centralized stochastic controlled systems cannot be used to analyze and optimize the performance of informationally decentralized systems (cf. Section 6). One of the reasons is that in decentralized systems it may be impossible to order the stations' control actions *a priori*, independently of the set of control laws, called the *design* (or *control policy*), that determines the actions. Such systems are called *nonsequential.* In the simplest case, a nonsequential system's actions may be ordered *a priori*, given any design, but the order varies from design to design. In general, for at least one design, the actions' order depends on the system's uncontrolled inputs (the noise variables), i.e. action α_1 may depend on action α_2 under some circumstances while α_2 may depend on α_1 under other circumstances. Examples of systems that exhibit such interdependence are: (1) packet-switched data networks [29] - packet routing, buffering, and reassembling interdependencies; (2) distributed databases [11] - transaction scheduling and locking interdependencies; (3) flexible manufacturing systems [24] - part delivery, buffering and assembly interdependencies; and (4) decentralized detection networks [4, Appendix A], [1, Appendix L] - observation and signaling interdependencies. It has been shown in [4, Appendix A] that nonsequential systems can potentially perform better that sequential systems, i.e. systems where the control actions can be ordered *a priori*, independently of the design. However, nonsequential systems are subject to *deadlock*, i.e. it is possible that for some design two or more actions are mutually dependent, - e.g. action α_1 depends on α_2 and vice versa.

The theory of nonsequential stochastic controlled systems is at a very early stage of development. The performance of these systems crucially depends on what information is available for each control action. Thus, some of the fundamental issues associated with the performance of nonsequential stochastic controlled systems are:

P1. Who should know what and when?
P2. Who should communicate with whom and when?
P3. Given that communication must be limited, either because channels have

limited capacity or because stations have limited memory to store data and limited processing capability, what information must be exchanged in *real-time* among stations so that they can improve the quality of their actions?

P4. What information should be available to each control station so that the system is deadlock-free?

P5. Given that the design of highly concurrent systems is desirable, but concurrency can only increase by increasing the complexity of the system's information gathering sources, what are the fundamental tradeoffs between system concurrency and the complexity of the system's information gathering sources?

P6. How does one optimize the performance of nonsequential stochastic controlled systems?

In this paper, we first present the intrinsic model for stochastic control which is a mathematical model for nonsequential stochastic controlled systems; then, we briefly survey existing results on nonsequential systems within the framework of the intrinsic model; and finally, we discuss some open questions arising from the research performed thus far.

2. Witsenhausen's Intrinsic Model for Stochastic Control

At least five different classes of models have been proposed for modeling nonsequential systems: (1) a quantum mechanical model [9]; (2) discrete event models (e.g.[16], [17], [22], [23], [30], [31]); (3) a game- theoretic model [25], [32]; (4) a hybrid dynamical model [10], [20]; and (5) an intrinsic model [34], [36]. These models provide a statistical, logical, informational, logical/temporal, and informational characterization of nonsequentiality, respectively. Witsenhausen's intrinsic model for stochastic control, [34], [36], provides the framework for the results that will be presented and discussed in this paper.

Consider a generic stochastic controlled system in which the number of control actions (decisions), and the number of primitive random inputs, are both finite (Figure 1). From a game theoretic perspective, the controller's decisions can be viewed as being the decisions of N autonomous, single-decision *agents* (usually, computers or devices) acting on the controller's behalf (cf. [32]). Likewise, the primitive random inputs can be viewed as being a single decision of nature (chance). This perspective entails no loss of generality since realizations of the system's uncertainties can always be selected before any control decisions are made and then forwarded to the system as needed. Denote nature's decision by $\omega := (\omega^0, \omega^1, \dots, \omega^N) \in \Omega$, and the agents' observations and decisions by $y := (y^1, y^2, \dots, y^N) \in Y$ and $u := (u^1, u^2, \dots, u^N) \in U$, respectively. Let nature's decision model the initial uncertainty in the system (ω^0) and all other uncertainties ($\omega^k,\ 1 \leq k \leq N$) affecting the agent's observations. Let the agents' observations be measurable functions of the system's *intrinsic variables*, ω and u (e.g., $y^k = h^k(\omega, u),\ 1 \leq k \leq N$), and constrain each agent's decision policy to be a measurable function of its observation (e.g., $u^k = g^k(y^k),\ 1 \leq k \leq N$). As long as the superscripts on ω, y, and u are not

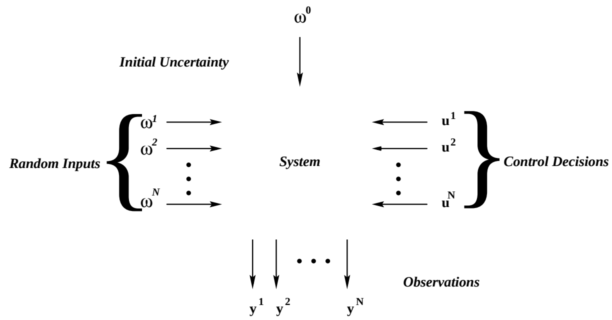

FIGURE 1. A generic stochastic control system

assumed to index time, this setup imposes no *a priori* constraints on the agents' decision order. It follows that nonsequential systems can be represented within this framework.

Witsenhausen's intrisic model for stochastic control, [34],[36], simplifies in a theoretical sense the preceding representation. Witsenhausen adopts a "snapshot" approach that simultaneously relates all of the system's uncertain inputs and control actions to the information that determines the control actions. His crucial observations are:
(1) All agents' decisions are determined by the system's intrinsic variables; that is, $u^k = g^k(y^k) = (g^k \cdot\ h^k)(\omega, u^1, u^2, \ldots, u^N) = \gamma^k(\omega, u^1, u^2, \ldots, u^N)$, where $\gamma^k,\ k = 1, \ldots, N$, are measurable functions of all of the system's intrinsic variables.
(2) The kth agent's observation, $k = 1, 2, \ldots, N$, only affects the kth agent's decision indirectly via the information subfield it induces on the space of intrinsic variables; that is, if $\mathcal{Y}^k$ is the σ-field on Y^k, h^k induces the subfield $[h^k]^{-1}(\mathcal{Y}^k)$ on $\Omega \times U$. Consequently, it is unnecessary to model the observations explicitly.
The measurability constraints on $\gamma^k,\ k = 1, 2, \ldots, N$, replace the observation equations as the sole determinants of the relationship among the uncertain inputs, the control actions and the information that determines the control actions. Thus, within the intrinsic model's framework, the control process can be viewed as a feedback loop that maps information into control actions via the control laws, and control actions into information via the measurability constraints. The principal advantage of Witsenhausen's intrinsic informational characterization of nonsequentiality is that it provides a theoretical framework that is appropriate for the investigation of Problems **P1** – **P6** posed in Section 1.

Formally, Witsenhausen's intrinsic model, [34], [36], has three components: An information structure $\mathcal{I}$, a design constraint set Γ_c, and a description of nature's randomized control policy.

1. The information structure $\mathcal{I} := \{N, (\Omega,\ \mathcal{B}), (U^k, \mathcal{U}^k), \Im^k, k = 1, 2, ..., N\}$

specifies the system's allowable decisions and distinguishable events.

(i) $N \in \mathbf{N}$ is the number of agents in the system excluding nature.

(ii) $(\Omega,\ \mathcal{B})$ is the measurable space from which ω, nature's random action, is selected. (Ω is a set and $\mathcal{B}$ is a σ-algebra of subsets of Ω.)

(iii) $(U^k,\ \mathcal{U}^k)$, k = 1, 2, ..., N, is the measurable space from which u^k, the kth agent's control action is selected. It is assumed that the singletons of U^k belong to $\mathcal{U}^k$, and that the cardinality of U^k, is greater than 1 (see [1]). The measurable product containing the agents' collective actions, $u := (u^1, u^2, ..., u^N)$, is denote by $(U, \mathcal{U}) := \left(\Pi_{i=1}^N U^i; \otimes_{i=1}^N \mathcal{U}^i\right)$

(iv) $\Im^k, k = 1, 2, ..., N$, is the information subfield of the product σ-field $\mathcal{B} \otimes \mathcal{U}$ characterizing the kth agent's set of distinguishable events.

2. The design constraint set Γ_c constrains N-tuples of the agents' control policies $\gamma := (\gamma^1, \gamma^2, ..., \gamma^N), \gamma^k : (\Omega \times U, \Im^k) \to (U^k, \mathcal{U}^k), k = 1, 2, ..., N$, called *designs*, to a nonempty subset of $\Gamma := \Pi_{i=1}^N \Gamma^i$, where $\Gamma^k, k = 1, 2, ..., N$, denotes the set of all $\Im^k/\mathcal{U}^k$- measurable functions.
3. A probability measure P on $(\Omega, \mathcal{B})$ specifies the randomized control policy used by nature.

Note that the intrinsic model does not exclude the possibility of an agent employing a mixed (i.e., randomized) decision policy, or a policy which occasionally dictates that the agent not act. To model the mixed policy, randomizing devices can be included as factors in $(\Omega,\ \mathcal{B},\ \mathrm{P})$, and the effects of the devices' outputs can be specified in $\Im^k,\ k = 1, 2, \ldots, N$. To model the occasional inaction, the agent can be allowed to make decisions that have no effect.

3. The Generic Stochastic Control Problem

We are concerned with the following generic stochastic control problem.

(P) Given an information structure $\mathcal{I}$, a design constraint set Γ_c, a probability measure P on $\mathcal{B}$, and a bounded, nonnegative, $\mathcal{B}\otimes\mathcal{U}$-measurable reward function V, identify a design γ in Γ_c, that achieves $sup_{\gamma\in\Gamma_c} E_\omega^\gamma \left[V(\omega, u_\omega^\gamma)\right]$ exactly, or within $\varepsilon > 0$.

In the above problem, the notation u_ω^γ indicates that $u := (u^1, u^2, ...u^N)$ depends on ω through γ; that is, $u^k = \gamma^k(\omega, u)$ for all $k = 1, 2, \ldots, N$.

Several issues associated with Problem (**P**) arise. Since the problem may not be sequential it need not be deadlock-free (i.e. for some design two or more control actions may be mutually dependent) or well-posed (i.e. some design $\gamma \in \Gamma_c$ may not possess an expected reward $E_\omega^\gamma\left(V(\omega, u_\omega^\gamma)\right)$ so that optimization may not be possible) as demonstrated by the following two examples.

Example 3.1 Consider a system consisting of three agents and nature. Assume

$$\Omega = U^1 = U^2 = U^3 = \{0,1\}, \tag{3.1}$$

$$\mathcal{B} = \mathcal{U}^1 = \mathcal{U}^2 = \mathcal{U}^3 = \{\phi, \{0\}, \{1\}, \{0,1\}\}, \tag{3.2}$$

$$\begin{aligned}\Im^1 = \ & \{\phi, \Omega \times U, \left\{(\omega, u^1, u^2, u^3) : \omega\bar{u}^2 u^3 = 1\right\}, \\ & \left\{(\omega, u^1, u^2, u^3) : \omega\bar{u}^2 u^3 = 0\right\}\},\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\Im^2 &= \left\{\phi, \Omega \times U, \left\{\left(\omega, u^1, u^2, u^3\right) : \omega \bar{u}^3 u^1 = 0\right\},\right. \\
&\quad \left.\left\{\left(\omega, u^1, u^2, u^3\right) : \omega \bar{u}^3 u^1 = 1\right\}\right\}, &(3.4)\\
\Im^3 &= \left\{\phi, \Omega \times U, \left\{\left(\omega, u^1, u^2, u^3\right) : \omega \bar{u}^1 u^2 = 0\right\},\right. \\
&\quad \left.\left\{\left(\omega, u^1, u^2, u^3\right) : \omega \bar{u}^1 u^2 = 1\right\}\right\}, &(3.5)
\end{aligned}$$

where $\bar{u}^i$ denotes the binary complement of $u^i \in \{0,1\}, i = 1,2,3$; that is, $\bar{u}^i = (1+u^i) \bmod 2$, $i = 1,2,3$.

Consider the following design $\gamma := \left(\gamma^1, \gamma^2, \gamma^3\right)$,

$$\gamma^1\left(\omega, u^1, u^2, u^3\right) = \begin{cases} 1, & \text{if } \omega \bar{u}^2 u^3 = 1 \\ 0, & \text{otherwise,} \end{cases} \tag{3.6}$$

$$\gamma^2\left(\omega, u^1, u^2, u^3\right) = \begin{cases} 1, & \text{if } \omega \bar{u}^3 u^1 = 1 \\ 0, & \text{otherwise,} \end{cases} \tag{3.7}$$

$$\gamma^3\left(\omega, u^1, u^2, u^3\right) = \begin{cases} 1, & \text{if } \omega \bar{u}^1 u^2 = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

Then, when $\omega = 1$ occurs, γ^1 depends on u^2 and u^3, γ^2 depends on u^3 and u^1, and γ^3 depends on u^1 and u^2. Consequently, no agent can act and a deadlock occurs. □

EXAMPLE 3.2 Consider a system consisting of two agents and nature. Let

$$\Omega = U^1 = U^2 = \{0,1\}, \tag{3.9}$$

$$\mathcal{B} = \mathcal{U}^1 = \mathcal{U}^2 = \{\phi, \{0\}, \{1\}, \{0,1\}\}, \tag{3.10}$$

$$\Im^1 = \Im^2 = \mathcal{B} \otimes \mathcal{U}^1 \otimes \mathcal{U}^2. \tag{3.11}$$

Consider the following design $\gamma := \left(\gamma^1, \gamma^2\right)$

$$\gamma^1\left(\omega, u^1, u^2\right) = \begin{cases} 0, \text{if } u^2 = 0 \\ 1, \text{if } u^2 = 1, \end{cases} \tag{3.12}$$

$$\gamma^2\left(\omega, u^1, u^2\right) = \begin{cases} 0, \text{if } u^1 = 0 \\ 1, \text{if } u^1 = 1. \end{cases} \tag{3.13}$$

Then, at $\omega = 0$ the closed-loop equations

$$u^1 = \gamma^1\left(\omega, u^1, u^2\right), u^2 = \gamma^2\left(\omega, u^1, u^2\right) \tag{3.14}$$

fail to possess a unique solution

$$u_\omega^\gamma := \left(u_\omega^{\gamma^1}, u_\omega^{\gamma^2}\right) \tag{3.15}$$

as

$$0 = \gamma^1(0,0,0),\ 0 = \gamma^2(0,0,0) \tag{3.16}$$

and

$$1 = \gamma^1(0,1,1),\ 1 = \gamma^2(0,1,1) \tag{3.17}$$

both satisfy (3.14). A similar situation arises at $\omega = 1$ where

$$0 = \gamma^1(1,0,0),\ 0 = \gamma^2(1,0,0) \tag{3.18}$$

and

$$1 = \gamma^1(1,1,1),\ 1 = \gamma^2(1,1,1) \tag{3.19}$$

both satisfy (3.14). In this case the reward $\mathrm{V}(\omega, u_\omega^\gamma)$ induced by any ω under γ is not unique, the expectation $E_\omega^\gamma \left[V\left(\omega, u_\omega^\gamma\right)\right]$ does not exist and Problem (**P**) is not well-posed. □

Therefore, it is important to identify conditions on the information structure to ensure that Problem (**P**) is deadlock-free and well-posed. Since there exist problems of the form (**P**) where some, but not all, nontrivial designs are deadlock-free and possess expected rewards (see [4], Appendix A), two classes of conditions can be considered: (1) conditions based on the problem's *design-independent* properties (that is, properties that hold for all $\gamma \in \Gamma$); and (ii) conditions based on the problem's *design-dependent* properties (that is, properties that may hold only for specific designs $\gamma \in \Gamma$). We examine these conditions separately in the following two sections.

4. Design-Independent Properties

In this section we present the properties the information structure $\mathcal{I}$ must possess to ensure that Problem (**P**) is well-posed and deadlock-free for all designs $\gamma \in \Gamma$.

4.1. Properties DF, S, SM.

To ensure that Problem (**P**) is deadlock-free, it is sufficient to require that its information structure, $\mathcal{I}$ possesses Property DF (deadlock-freeness).

Definition 4.1 ([1], [4]). An information structure $\mathcal{I}$ possesses *Property DF* (deadlock- freeness) if for each $\gamma = (\gamma^1, \gamma^2, ..., \gamma^N) \in \Gamma$ and for every $\omega \in \Omega$, there exists an ordering of γ's N control laws $\gamma^{s_1(\omega)}, \gamma^{s_2(\omega)}, ..., \gamma^{s_N(\omega)}$ such that no control action $u^{s_n(\omega)}, n = 1, 2, ..., N$, depends on itself or the control actions that follow. □

Property DF generalizes the usual notion of causality since it does not assume that the actions' order is fixed independently of the random input $\omega \in \Omega$ and the design $\gamma \in \Gamma$. To ensure that Problem (**P**) is well-posed we must guarantee that every design $\gamma \in \Gamma$ induces a unique outcome u_ω^γ (or a unique reward $V(\omega, u,_\omega^\gamma))$ for every decision ω of nature and that the expected reward $E_\omega^\gamma \left[V(\omega, u,_\omega^\gamma)\right]$ is also defined. These requirements lead to Properties S and SM defined below.

Definition 4.2 ([36]). An information structure $\mathcal{I}$ possesses *Property S* (solvability) when for each design $\gamma \in \Gamma$ and every $\omega \in \Omega$ there exists a unique $u_\omega^\gamma \in U$ satisfying the closed-loop equations.

$$u^k = \gamma^k(\omega, u),\ k = 1, 2, ...N. \tag{4.1}$$ □

When Problem (**P**)'s information structure possesses Property S, every design $\gamma \in \Gamma$ induces a unique closed-loop solution map $\sum^{\gamma} : \Omega \to U$ via the closed-loop solutions $\{u^{\gamma}_{\omega} \in U : \gamma(\omega, u^{\gamma}_{\omega}) = u^{\gamma}_{\omega}\}$. Hence, we have $\sum^{\gamma}(\omega) = u^{\gamma}_{\omega}$ and therefore Problem (**P**)'s reward $V(\omega, u^{\gamma}_{\omega})$ can be uniquely defined as $V(\omega, \sum^{\gamma}(\omega))$ for all $\gamma \in \Gamma$. The expected reward $E^{\gamma}_{\omega}[V(\omega, \sum^{\gamma}(\omega))]$ is also defined when the map $\sum^{\gamma} : \Omega \to U$ is $\mathcal{B}/\mathcal{U}-$ measurable.

DEFINITION 4.3 ([36]). An information structure $\mathcal{I}$ possessing Property S possesses *Property SM* (solvability - measurability) when for each $\gamma \in \Gamma$ the induced map $\sum^{\gamma} : \Omega \to U$ is $\mathcal{B}/\mathcal{U}-$ measurable. □

In general, it is not known whether Property S implies Property SM. However, in two important cases S implies SM.

THEOREM 4.1 ([1], [3]).*Property S implies Property SM when either of the two conditions is satisfied:*

(i) $U^k, k = 1, 2...N$*, are countable sets, or*

(ii) $(\Omega,\ \mathcal{B})$ *and* $(U^k,\ \mathcal{U}^k)$, $\ k = 1, 2, ..N$*, are Souslin measurable spaces.* □

Most measurable spaces of interest are Souslin [19]. For example, countable spaces, standard Borel spaces [15] and Blackwell spaces [14] where all singletons are measurable are Souslin spaces. Furthermore, spaces of the form (X, B(X)), where is X is Borel, or more generally an analytic subset of $\mathcal{R}^n$, and B(X) is the Borel σ-field of X, are Souslin. Consequently, for most cases of interest Properties S and SM are equivalent. Property DF implies property SM; this will become clear from the results of Section 4.2. In general, Property SM does not imply Property DF as the following example shows.

EXAMPLE 4.1 Consider the information structure $\mathcal{I}$ of Example 3.1. This information structure posseses Property S. Since U^k, k = 1,2,3, are finite, by Theorem 4.1, $\mathcal{I}$ possesses Property SM. However, as shown in Example 3.1, $\mathcal{I}$ does not possess Property DF. □

Since Properties DF, S and SM ensure that the generic stochastic control problem (**P**) is deadlock-free and well-posed, it would be desirable to determine properties of the information structure $\mathcal{I}$ that guarantee Properties DF, S and SM. This leads to Properties C and CI discussed below.

4.2. Properties C and CI.

Property DF suggests that deadlocks cannot arise if for each $\omega \in \Omega$ and each design $\gamma \in \Gamma$ the agents can be ordered in a such way that each agent's information depends only on ω and its predecessors' actions. To formalize this observation we adopt the following notation: For all $k = 1, 2 \ldots N$, we define S_k to be the set of all k-agent orderings, that is, all injections of $\{1, 2, ...k\}$ into $\{1, 2, ..., N\}$. For all $j = 0, 1, 2, ...N$, and $k = j, j+1, ...N$, we let $T^k_j : S_k \to S_j$ denote a truncation map that returns the ordering of the first j agents of a k-agent ordering, that is, T^k_j restricts $s \in S_k$ to the domain $\{1, 2, ...j\}$ or to ϕ when j =0. For all $s := (s_1, s_2, ..., s_k) \in S_k$, and $k = 1, 2, ...N$, we define $\mathcal{P}_s$ to be the projection of $\Omega \times U$ onto $(\Omega \times \Pi^k_{i=1} U^{s_i})$, that is

$$\mathcal{P}_s(\omega, u) := (\omega, u^{s_i}, ..., u^{s_k}), \quad \mathcal{P}_\phi(\omega, u) = (\omega) \tag{4.2}$$

Finally, for all $s \in S_k, k = 1, 2, ..., N$, we denote by $\mathcal{F}(T^k_{k-1}(s))$ the σ-field of intrinsic events that is induced by the actions of nature and the first k-1 agents in s. Then, the intuitive notion of causality can be formalized as a constraint on Problem (**P**)'s information structure as follows:

DEFINITION 4.4 ([36]). An information structure $\mathcal{I}$ possesses *Property C* (causality) when there is at least one map $\psi : \Omega \times U \to S_N$ such that for all $s := (s_1, s_2, .., s_k) \in S_k, k = 1, 2, ...N$,

$$\Im^{s_k} \cap \left[T^N_k \cdot \psi\right]^{-1}(s) \subset \mathcal{F}\left(T^k_{k-1}(s)\right). \tag{4.3}$$ □

DEFINITION 4.5 ([4]). An information structure $\mathcal{I}$ possesses *Property CI* (causal implementability) when there exists at least one map $\psi : \Omega \times U \to S_N$ such that for all k=1,2,...N, and $(\omega, u) \in \Omega \times U$,

$$\begin{aligned}
&\Im^{s_k} \cap \left[\mathcal{P}_{T^N_{k-1}(s)}\right]^{-1}\left(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)\right) \subset \\
&\subset \left\{\phi, \left[\mathcal{P}_{T^N_{k-1}(s)}\right]^{-1}\left[\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)\right]\right\}
\end{aligned} \tag{4.4}$$

when $s := (s_1, s_2, ...s_N) = \psi(\omega, u)$. □

We now intuitively interpret each of the above definitions. The function ψ maps every intrinsic outcome $(\omega, u) \in \Omega \times U$ into an N-agent decision (action) order. $\left[T^N_k \cdot \psi\right]^{-1}(s)$ is the set of intrinsic outcomes that are mapped by ψ into decision orders where the order of the first k agents is $s \in S_k$. $\Im^{s_k} \cap \left[T^N_k \cdot \psi\right]^{-1}(s), s \in S_k$, is the σ-field of intrinsic events that agent s_k can distinguish, given that the order of the first k agents, as determined by ψ, is s. Property C ensures that there exists an order function ψ, such that for all possible orderings $s \in S_k, k = 1, 2, ...N$, the events that agent s_k can distinguish, (given that the ordering of the first k agents, as determined by ψ, is s), are events that can be induced by the decisions of nature, and the s_k agent's predecessors in s. Property CI's interpretation proceeds along similar arguments. As before, ψ is a function that maps every intrinsic outcome $(\omega, u) \in \Omega \times U$ into an N-agent ordering.

$$\left[\mathcal{P}_{T^N_{k-1}(s)}\right]^{-1}\left(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)\right) = \left[\mathcal{P}_{T^N_{k-1}(s)}\right]^{-1}(w, u^{s_1}, ..., u^{s_{k-1}}) \tag{4.5}$$

is the cylinder set induced on $\Omega \times U$, when the intrinsic outcome is (ω, u), by the actions of nature and the first k-1 agents in $s := (s_1, s_2, ..., s_N) = \psi(\omega, u)$. The σ-field

$$\Im^{s_k} \cap \left[\mathcal{P}_{T^N_{k-1}(s)}\right]^{-1}\left(\mathcal{P}_{T^N_{k-1}(s)}(\omega, u)\right) \tag{4.6}$$

denotes the trace of the s_kth agent's information field on this cylinder set. Requirement (4.4) constrains the cylinder set (4.5) to be a subset of all events containing (ω, u) in the s_kth agent's information field $\Im^{s_k}$; that is, no event containing (ω, u) may depend on $u^{s_k}, u^{s_{k+1}}, ..., u^{s_N}$. Therefore, Property CI ensures that for all outcomes $(\omega, u) \in \Omega \times U$, there exists an order $s := (s^1, s^2, ..., s^N) = \psi(\omega, u)$ such that for all $k = 1, 2, \ldots N$, the s_kth agent's information at the point (ω, u) depends only on the actions of nature and its predecessors in s.

The significance of Properties C and CI stems from the following results:

THEOREM 4.2 ([36]).*If an information structure $\mathcal{I}$ possesses Property C then:*

(i) $\mathcal{I}$ possesses property SM;

(ii) $\mathcal{I}$ possesess property DF. □

THEOREM 4.3 ([4]).*Let $\mathcal{I}$ be an arbitrary information structure. Then,*

(i) $\mathcal{I}$ possesses property SM if $\mathcal{I}$ posseses property CI, and

(ii) $\mathcal{I}$ possesses property DF if and only if $\mathcal{I}$ possesses property CI. □

Part (i) of Theorems 4.2 and 4.3 ensures that when Problem (**P**) satisfies either Property C or Property CI then it is well-posed. Theorem 4.2 provides a sufficient condition, whereas Theorem 4.3 provides a necessary and sufficient condition for deadlock-free operation of nonsequential systems. The reason for this difference is the following: The requirement expressed by (4.3) in the definition of Property C imposes certain measurability constraints on the order function ψ (Lemma 5, [36]); however, there are order functions ψ for which $\mathcal{I}$ possesses property CI and which do not satisfy the measurability constraints imposed by Property C. Such an order function is given in the following example:

EXAMPLE 4.2 ([4]). Consider a nonsequential information structure $\mathcal{I}$ of the form

$$\begin{aligned}
N &= 3, \\
\Omega &= U^1 = U^2 = U^3 = \{0, 1\}, \\
\mathcal{B} &= \mathcal{U}^1 = \mathcal{U}^2 = \mathcal{U}^3 = \{\phi, \{0\}, \{1\}, \{0, 1\}\}, \\
\Im^1 &= \{\phi, \{(\omega, u) : \omega u^2 = 0\}, \{(\omega, u) : \omega u^2 = 1\}, \Omega \times U\}, \\
\Im^2 &= \{\phi, \{(\omega, u) : \bar{\omega} u^1 = 0\}, \{(\omega, u) : \bar{\omega} u^1 = 1\}, \Omega \times U\}, \\
\Im^3 &= \{\phi, \{(\omega, u) : \omega = 0\}, \{(\omega, u) : \omega = 1\}, \Omega \times U\},
\end{aligned} \tag{4.7}$$

where $\bar{\omega}$ is the binary complement of $\omega \in \{1, 0\}$.

The order function ψ defined by

$$\psi(\omega, u^1, u^2, u^3) = \begin{cases} (1,2,3), & \text{when } \omega = 0 \\ (3,2,1), & \text{when } \omega u^3 = 1 \\ (2,1,3), & \text{otherwise} \end{cases} \tag{4.8}$$

is such that $\mathcal{I}$ possesses property CI but not property C; Eq. (4.3) fails when $k = 1, s = 3 \in S_1$, because $\left[T_1^3 \cdot \psi\right]^{-1}(3) = \{(\omega, u) : \omega u^3 = 1\} \notin \mathcal{F}(\phi) = \mathcal{B} \otimes \{\phi, U\}$. □

Since Property C implies Property DF (Theorem 4.2) and Property DF implies Property CI (Theorem 4.3) the following is clear.

THEOREM 4.4 ([4]). *Property C implies Property CI.* □

Furthermore, since there exist nonsequential information structures $\mathcal{I}$, and order functions ψ such that $\mathcal{I}$ possesses Property CI but not Property C, (Example 4.1), Property CI may not imply Property C, and general proofs that Property CI implies Property C must be constructive. It is not known, in general, whether Property CI implies Property C; the implication however, holds in the following cases.

THEOREM 4.5 ([4], [36]). *Property CI implies Property C whenever $N \leq 2$.* □

THEOREM 4.6 ([4]). *Property CI implies Property C whenever Ω and U^k, $k = 1, 2, ..., N$, are countable sets, and $\mathcal{B}$ contains the singletons of Ω.* □

THEOREM 4.7 ([4]). *All constant order functions ψ such that $\mathcal{I}$ possesses Property CI are order functions such that $\mathcal{I}$ possesses Property C.* □

An information structure $\mathcal{I}$ is said to be *sequential* when Property CI holds for some constant order function ψ. Therefore, by Theorem 4.7, Property CI implies Property C when $\mathcal{I}$ is sequential.

We conclude this section by noting that the motivation and development of Properties DF, S, SM, C and CI was done independently of any properties of the reward function V. Consequently, the results presented in this section apply to nonsequential stochastic controlled systems as well as games. The results of this section imply that a game with a finite number of decisions, chosen from decision spaces that satisfy the constraints imposed by the intrinsic model, has an extensive form, [25], if and only if its information structure possesses Property CI.

5. DESIGN-DEPENDENT PROPERTIES

The real world imposes independent constraints on the information available to a system's agents. Thus, problems that do not satisfy property CI arise in practice. Nevertheless, many of those problems' admissible designs possess expected rewards and deadlock-free implementations. For example, when cast as decision problems, many routing, flow control and concurrency control problems are such that some protocols (designs) are deadlock-free whereas others are not ([11], [29]). Thus, it is important to determine necessary and sufficient conditions for individual designs to possess expected rewards and deadlock-free implementation, and to restrict optimization to the set of all designs that possess the above characteristics.

The above considerations motivate the development of the *design-dependent* analogues of Properties DF, S, SM, C and CI. These analogues are Properties DF*, S*, SM*, C* and CI*, respectively. Their development parallels that of Properties DF, S, SM, C and CI and will not be presented here. The precise definition of Properties DF*, S*, SM*, C* and CI*, the subtleties that arise in their construction from their design-independent analogues, as well as their implications, are presented in detail in [1], [5]. Here we only remark that a design γ's possession of Properties DF* and SM* guarantees that γ possesses a deadlock-free implementation and an expected reward, respectively. Furthermore, γ's possession of Property C* or Property CI* ensures that γ possesses

Property DF* and SM* [5]. Thus, optimization of real-world nonsequential systems may be performed among those designs that possess Properties CI* or C*.

Design-dependent properties provide a finer characterization of a design's closed-loop solvability and deadlock-freeness than design-independent properties. This happens because, as pointed out at the beginning of this section, there are many designs whose deadlock-freeness and closed-loop solvability can not be characterized using any design-independent property. Such a situation is presented in the following example.

EXAMPLE 5.1 ([5]). Consider the nonsequential information structure $\mathcal{I}$ with

$$\begin{aligned} N &= 3, && (5.1)\\ \Omega &= U^1 = U^2 = U^3 = \{0,1\}, && (5.2)\\ \mathcal{B} &= \mathcal{U}^1 = \mathcal{U}^2 = \mathcal{U}^3 = \{\phi, \{0\}, \{1\}, \{0,1\}\}, && (5.3)\\ \Im^1 &= \{\phi, \{\omega, u) : \omega = 0\}, \{(\omega, u) : \omega = 1\}, \Omega \times U\}, && (5.4)\\ \Im^2 &= \big\{\phi, \{(\omega, u) : \max\ (\bar{\omega}\bar{u}^1\bar{u}^3, u^1u^3) = 0\big\}, \\ &\qquad \big\{(\omega, u) : \max\ (\bar{\omega}\bar{u}^1\bar{u}^3, u^1u^3) = 1\big\}, \Omega \times U\big\}, && (5.5)\\ \Im^3 &= \big\{\phi, \big\{(\omega, u) : \omega u^2 = 0\big\}, \big\{(\omega, u) : \omega u^2 = 1\big\}, \Omega \times U\big\}, && (5.6) \end{aligned}$$

where $\bar{\omega}$ and $\bar{u}^i, i = 1, 2, 3$, denote the binary complement of $\omega, u^i \in \{0, 1\}, i = 1, 2, 3$, respectively.

Consider the designs $\gamma := (\gamma^1, \gamma^2, \gamma^3)$ and $\hat{\gamma} := (\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3)$ defined as follows:

$$\gamma^1(\omega, u^1, u^2, u^3) = \begin{cases} 0, & \text{if } \omega = 1 \\ 1, & \text{otherwise,} \end{cases} \tag{5.7}$$

$$\gamma^2(\omega, u^1, u^2, u^3) = \begin{cases} 1, & \text{if } \max\ (\bar{\omega}\bar{u}^1\bar{u}^3, u^1u^3) = 1 \\ 0, & \text{otherwise,} \end{cases} \tag{5.8}$$

$$\gamma^3(\omega, u^1, u^2, u^3) = \begin{cases} 1, & \text{if } \omega u^2 = 1 \\ 0, & \text{otherwise,} \end{cases} \tag{5.9}$$

$$\hat{\gamma}^1(\omega, u^1, u^2, u^3) = \begin{cases} 1, & \text{if } \omega = 1 \\ 0, & \text{otherwise,} \end{cases} \tag{5.10}$$

$$\hat{\gamma}^2(\omega, u^1, u^2, u^3) = \gamma^2(\omega, u^1, u^2, u^3), \tag{5.11}$$

$$\hat{\gamma}^3(\omega, u^1, u^2, u^3) = \gamma^3(\omega, u^1, u^2, u^3). \tag{5.12}$$

For k =1,2,3, γ^k and $\hat{\gamma}^k$ both induce the same information subfield $\widetilde{\Im}^k$, that is,

$$\widetilde{\Im}^k := [\gamma^k]^{-1}(\mathcal{U}^k) = [\hat{\gamma}^k]^{-1}(\mathcal{U}^k). \tag{5.13}$$

The *graph* G^γ of γ, defined by

$$G^\gamma := \{(\omega, u) : u = \gamma(\omega, u)\}, \tag{5.14}$$

is found to be

$$G^\gamma = \{(0, 1, 0, 0), (1, 0, 0, 0)\}. \tag{5.15}$$

The graph of $\hat{\gamma}$ is

$$G^{\hat{\gamma}} = \{(0,0,1,0), (1,1,0,0), (1,1,1,1)\}. \tag{5.16}$$

Equation (5.16) implies that $\hat{\gamma}$ does not possess Property S*[1]. Furthermore, it is known ([5], Theorem 5) that an information structure $\mathcal{I}$ possesses Property S if and only if all $\gamma \in \Gamma$ possess Property S*. Consequently, no information structure (including $\mathcal{I}$) that can be associated with $\hat{\gamma}$ can possess Property S; that is, no information structure

$$\bar{\mathcal{I}} := \{(\Omega, \mathcal{B}),\ (U^k, \mathcal{U}^k),\ \bar{\Im}^k;\ k = 1,2,3\}, \tag{5.17}$$

such that

$$[\hat{\gamma}^k]^{-1}(\mathcal{U}^k) = \widetilde{\Im}^k \subset \bar{\Im}^k,\ k = 1,2,3, \tag{5.18}$$

can possess Property S.

On the other hand, Eq. (5.15) implies that the design γ possesses Property S*. Furthermore, it can be shown that the order function $\psi : G^\gamma \to S_N$ defined by

$$\psi(\omega, u^1, u^2, u^3) = \begin{cases} (1,3,2), & \text{if } (\omega, u^1, u^2, u^3) = (0,1,0,0) \\ (1,2,3), & \text{if } (\omega, u^1, u^2, u^3) = (1,0,0,0) \end{cases} \tag{5.19}$$

is such that γ possesses property C*[2]. Consequently, by the results of [5] (Thm. 1), γ possesses properties DF* and SM*.

However, since $\hat{\gamma}$ does not possess Property S* and (5.13) holds, γ cannot be associated with any information structure possessing Property S, let alone properties SM, CI or C. □

6. Optimization

In centralized stochastic control the dynamic programming algorithm (see for example [21], [26]) is an approach to optimization. Dynamic programming provides a recursive decomposition of the optimization problem and is based on the following fact ([8], [26]): If the future control laws are fixed, then, given the information available at the current step and the action taken at the present

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1 A design γ possesses Property S* when for every $\omega \in \Omega$ there exists a unique $u \in U$ satisfying the system of equations $u^k = \gamma^k(\omega, u),\ k = 1,2,...N$.

2 A design γ possesses property C* when $\mathcal{P}_\phi(G^\gamma) = \Omega$, and there exists at least one map $\psi : G^\gamma \to S_N$ such that for all $s := (s^1, s^2, ...s^k) \in S_k$ and k =1,2,...,N,

$$\begin{aligned} &\mathcal{J}^{\gamma^{s_k}} \cap \left[\mathcal{P}_{T^k_{k-1}(s)}\right]^{-1} \left(\mathcal{P}_{T^k_{k-1}(s)}\left([T^N_k \cdot \psi]^{-1}(s)\right)\right) \\ &\subset \mathcal{F}(T^k_{k-1}(s)) \cap \left[\mathcal{P}_{T^k_{k-1}(s)}\right]^{-1}\left(\mathcal{P}_{T^k_{k-1}(s)}(G^\gamma)\right), \end{aligned}$$

where $\mathcal{J}^{\gamma^j}$ is the jth agent's infrormation partition induced by γ, $\mathcal{J}^{\gamma^j} := \left\{[\gamma^j]^{-1}(u^j) :\ u^j \in U^j\right\}$.

step, the total conditional expected cost is independent of the past and present control laws.

Decentralized sequential stochastic control problems are also amenable (in theory) to recursive decomposition. Witsenhausen's dynamic programming algorithm, [35], in terms of unconditional distributions provides such a decomposition, but is often computationally forbidding.

Optimization of nonsequential stochastic controlled systems that are deadlock-free and well-posed is complicated by the fact that dependencies among control actions are dynamic; that is, the order of actions is not fixed in advance but depends on the problem's random inputs (the nature's choice) and the design. Such dependencies, at first glance,would preclude recursive decomposition of nonsequential controlled systems. Nevertheless, it is possible, within the framework of Witsenhausen's intrinsic model, to reduce unconstrained nonsequential problems to equivalent sequential problems. Such a reduction can be achieved when Property CI and a mild measurability condition are satisfied [2].

To convert a N-agent nonsequential stochastic control problem into an equivalent sequential one (within the framework of Witsenhausen's intrinsic model) the dynamic dependencies among control actions must be eliminated. To eliminate these dependencies Andersland [2] proceeds as follows. Starting with the original N-agent intrinsic model, (cf. Section 2), he introduces one additional agent and considers the following new $(N+1)$-agent intrinsic model:

1. The information structure is

$$\hat{\mathcal{I}} := \left\{(\Omega, \mathcal{B}), (\hat{U}^k, \widehat{\mathcal{U}}^k), \hat{\Im}^k, 1 \leq k \leq N+1\right\},$$

   where

   (i) $(\Omega, \mathcal{B})$, are the same as in the original N-agent intrinsic model,

   (ii) $(\hat{U}^k,\ \widehat{\mathcal{U}}^k) = \begin{cases} (U,\ \mathcal{U}) & \text{for } k=1 \\ (U^{k-1},\ \mathcal{U}^{k-1}) & \text{for } k=2,3,...,N+1, \end{cases}$

   $U^k, \mathcal{U}^k,\ k=1,2,\ldots,N$ and $U = \Pi_{i=1}^{N}, U^i, \mathcal{U} = \otimes_{i=1}^{N} \mathcal{U}^i$ are the same as in the original N-agent intrinsic model,

   (iii) $\hat{\Im}^k = \begin{cases} \mathcal{B} \otimes \{\phi, U\} \otimes \{\phi, U\} & \text{for } k=1 \\ \Im^{k-1} \otimes \{\phi, U\} & \text{for } k=2,3,...,N+1, \end{cases}$

   and $\Im^k, k=1,2,\ldots,N$, are the same as in the original N-agent intrinisic model.
2. The set of admissible control laws is $\hat{\Gamma} := \Pi_{i=1}^{N+1}\hat{\Gamma}^i$, where $\hat{\Gamma}^k, k=1,2,...,N+$ 1, denotes the set of all $\hat{\Im}^k/\widehat{\mathcal{U}}^k$ measurable functions.
3. The probability measure P on $(\Omega,\ \mathcal{B})$ is the same as in the original intrinsic model.

The proposed new model has the following features:

(F1) Agents $2,3,\ldots,N+1$, are the same as agents $1,2,3,\ldots,N$, in the original model.

(F2) Agent 1, (the additional agent), always acts first and its action in effect simulates the actions $(u^1, u^2, \ldots, u^N)$ of the agents of the original intrinsic model via a $\mathcal{B}/\mathcal{U}$-measurable function, say θ, that is, $\hat{u}^1 := \hat{u}^\theta_\omega = \theta(\omega)$.
(F3) The subfields $\hat{\Im}^1$ and $\hat{\Im}^k$, $k = 2, 3, \ldots, N+1$, are degenerate; thus, all $\hat{\gamma} := (\hat{\gamma}^1, \hat{\gamma}^2, ..., \hat{\gamma}^{N+1}) \in \hat{\Gamma}$ can be viewed as pairs (θ, γ), where θ is as above and γ is in Γ (cf. Section 2). Therefore, we write $\hat{\gamma} := (\theta, \gamma) = (\theta, \gamma^1, \gamma^2, \ldots, \gamma^N)$.
(F4) The actions $\hat{u}^k, k = 2, 3, \ldots, N+1$, of the N agents of the original model are decoupled in the new model because (by the definition of $\hat{\Im}^k, k = 1, 2, \ldots, N+1$) each $\hat{u}^k, k = 2, 3, \ldots, N+1$, is selected as a function of ω and $\hat{u}^1 (\hat{u}^1 = \hat{u}^\theta_\omega)$. Hence, we can write ${}^1u^{\theta,\gamma}_\omega := (\hat{u}^2, \hat{u}^3, \ldots, \hat{u}^{N+1}) = \gamma(\omega, \hat{u}^\theta_\omega)$.

Within the framework of the above $(N+1)$-agent intrinsic model Andersland formulates the following control problem:

($\hat{\mathbf{P}}$) Given the information structure $\hat{\mathcal{I}}$, the design set $\hat{\Gamma}$ and a real nonnegative, bounded, $\mathcal{B} \otimes \mathcal{U}$- measurable payoff function V, identify a design (θ, γ) in $\hat{\Gamma}$ that achieves

$$\sup_{(\theta,\gamma)\in\hat{\Gamma}} E\left[V\left(\omega,\ {}^1u^{\theta,\gamma}_\omega\right) 1_{\{\hat{u}=u\}}\left(\hat{u}^\theta_\omega,\ {}^1u^{\theta,\gamma}_\omega\right)\right]$$

exactly or within $\varepsilon > 0$.

In the above problem, $1_{\{\hat{u}=u\}}$ denotes the indicator function of the set

$$\{(\hat{u}, u) \in U \times U : \ \hat{u} = u\}.$$

Problem ($\hat{\mathbf{P}}$) has the following features:
(F5) Its expected payoff depends on $\gamma \in \Gamma$ as well as on (the simulation) θ.
(F6) Problem ($\hat{\mathbf{P}}$) is sequential. This is true because according to (F2) agent 1 always acts first, and according to (F4) the actions $\hat{u}^k, k = 2, 3, \ldots, N+1$, are decoupled.
(F7) The reward function for Problem ($\hat{\mathbf{P}}$) is such that the expected payoff can be maximized only when the action of (the simulation) θ agrees with the actions of the agents of the original (N-agent) intrinsic model. Consequently, ε-optimal designs for Problem ($\hat{\mathbf{P}}$) determine, via a simple correspondence, ε-optimal designs for Problem ($\mathbf{P}$).

The observations made in (F6) and (F7) have been formalized by Andersland in [2] as follows:
THEOREM 6.1 ([2]). *Problem* ($\hat{\mathbf{P}}$) *is a sequential* $(N+1)$*-agent problem of the form* ($\mathbf{P}$) *when* $\mathcal{U}^k, k = 1, 2, \ldots, N$, *are countably generable.* □
THEOREM 6.2 ([2]). *When the information structure* $\mathcal{I}$ *possesses property CI and* $\mathcal{U}^k, k = 1, 2, \ldots, N$, *are countably generable, the following is true: Whenever the expected payoff of* $(\theta, \gamma) \in \hat{\Gamma}$ *is within* $\varepsilon > 0$ *of optimal for Problem* ($\hat{\mathbf{P}}$), *the payoff of* $\gamma \in \Gamma$ *is within* ε *of optimal for Problem* ($\mathbf{P}$). □

The above theorems have the following implications:

(i) Most nonsequential problems of the form (**P**) can be reduced to sequential problems of the form ($\hat{\mathbf{P}}$) because most measurable spaces $(X, \mathcal{X})$ (standard Borel [19], Souslin [19]) have countably generable $\mathcal{X}$ that contains the singletons of X.
(ii) To determine an ε-optimal design $\gamma^\varepsilon := (\gamma^{1,\varepsilon}, \gamma^{2,\varepsilon}, \ldots, \gamma^{N,\varepsilon})$ for the nonsequential Problem (**P**) it is sufficient to determine an ε-optimal design $\hat{\gamma}^\varepsilon := (\hat{\gamma}^{1,\hat{\varepsilon}}, \hat{\gamma}^{2,\varepsilon}, \ldots, \hat{\gamma}^{N+1,\varepsilon})$ for the sequential Problem ($\hat{\mathbf{P}}$) and to set

$$\gamma^\varepsilon = (\hat{\gamma}^{2,\varepsilon}, \ldots, \hat{\gamma}^{N+1,\varepsilon}).$$

7. Concurrency – Some Open Problems

The results of Sections 4-6 address Problems (**P1**) - (**P4**) and (**P6**), posed in Section l, by determining properties of the information structure that guarantee deadlock-freeness and well-posedness of nonsequential stochastic controlled systems, and by providing an approach to the optimization of these systems. To the best of our knowledge, within the framework of the intrinsic model, concurrency of nonsequential systems has not been investigated so far. In this section we discuss issues of concurrency in nonsequential systems. We present a few ideas that lead to open problems the solution of which, we believe, will shed light into the role of the system's information gathering sources on the system's parallelism.

Nonsequential systems that are deadlock–free and well–posed can exhibit a high degree of concurrency (parallelism). The degree of concurrency that is actually achieved in a nonsequential system depends on the physical structure of the information sources that provide the data required for each operation, e.g. the system's sensors or signaling network. Thus, to understand issues of concurrency it is necessary to incorporate into the intrinsic model the observation functions

$$h^k : (\Omega \times U, \mathcal{B} \otimes \mathcal{U}) \to (Y^k, \mathcal{Y}^k), \qquad k = 1, 2, \ldots, N, \tag{7.1}$$

that induce the subfields $\Im^k = [h^k]^{-1}(\mathcal{Y}^k)$ [3]. The motivation for incorporating this additional detail into the intrinsic model comes from the following fact: Identical sources that provide identical outputs given the outcome of one set of operations may behave differently given the outcomes of a subset of these operations. This is demonstrated by the following example.

Example 7.1 ([7]). Consider the nonsequential system with

$$\begin{aligned}
N &= 2, & (7.2)\\
\Omega &= U^1 = U^2 = \{0, 1\}, & (7.3)\\
\mathcal{B} &= \mathcal{U}^1 = \mathcal{U}^2 = \{\phi, \{0\}, \{1\}, \{0, 1\}\}, & (7.4)\\
Y^1 &= Y^2 = \{L_1, L_2\}, & (7.5)\\
\mathcal{Y}^1 &= \mathcal{Y}^2 = \{\phi, \{L_1\}, \{L_2\}, \{L_1, L_2\}\}, & (7.6)
\end{aligned}$$

[3] In this situation, as pointed out in Section 2, a design $\gamma \in \Gamma$ is of the form $\gamma := (\gamma^1, \gamma^2, \ldots, \gamma^N) = (g^1 \cdot h^1, g^2 \cdot h^2, \ldots, g^N \cdot h^N)$ where $g^k : (Y^k, \mathcal{Y}^k) \to (U^k, \mathcal{U}^k)$, $k = 1, 2, \ldots, N$.

and the following two pairs of observation functions:

$$h^1(\omega, u^1, u^2) \quad = \quad \begin{cases} L_1, & \text{when } \omega = 0, \text{ (and any } u^1, u^2), \text{ or} \\ & \qquad u^2 = 0 \text{ (and any } \omega, u^1) \\ L_2, & \text{when } \omega = 1, \ u^2 = 1, \text{ (and any } u^1), \end{cases} \tag{7.7}$$

$$h^2(\omega, u^1, u^2) \quad = \quad \begin{cases} L_1, & \text{when } \omega = 1, \text{ (and any } u^1, u^2), \text{ or} \\ & \qquad u^1 = 1 \text{ (and any } \omega, u^2) \\ L_2, & \text{when } \omega = 0, \ u^1 = 0, \text{ (and any } u^2), \end{cases} \tag{7.8}$$

and

$$\hat{h}^1(\omega, u^1, u^2) \quad = \quad \begin{cases} L_1, & \text{when } \omega = 0, \ u^2 = 1, \text{ (and any } u^1) \\ L_1, & \text{when } \omega = 0, \ u^2 = 0, \text{ (and any } u^1) \\ L_1, & \text{when } \omega = 1, \ u^2 = 0, \text{ (and any } u^1) \\ L_2, & \text{when } \omega = 1, \ u^2 = 1, \text{ (and any } u^1), \end{cases} \tag{7.9}$$

$$\hat{h}^2(\omega, u^1, u^2) \quad = \quad \begin{cases} L_1, & \text{when } \omega = 1, \text{ (and any } u^1, u^2), \text{ or} \\ & \qquad u^1 = 1 \text{ (and any } \omega, u^2) \\ L_2, & \text{when } \omega = 0, \ u^1 = 0, \text{ (and any } u^2), \end{cases} \tag{7.10}$$

respectively. Both pairs (h^1, h^2) and $(\hat{h}^1, \hat{h}^2)$ induce identical information subfields $\Im^1 = [h^1]^{-1}(\mathcal{Y}^1) = [\hat{h}^1]^{-1}(\mathcal{Y}^1)$, $\Im^2 = [h^2]^{-1}(\mathcal{Y}^2) = [\hat{h}^2]^{-1}(\mathcal{Y}^2)$, that satisfy property CI. However, only the system with observation functions (h^1, h^2) is deadlock-free. When $\omega = 0$, $\hat{h}^1$ does not give an output until u^2 is known and $\hat{h}^2$ does not give an output until u^1 is known; thus, u^1 depends on u^2 and vice versa, and the system deadlocks. In this example (h^1, h^2) and $(\hat{h}^1, \hat{h}^2)$ are different physical realizations of the same function. The major difference between these physical realizations is that when $\omega = 0$ $\hat{h}^1$ unnecessarily delays reporting L_1 even though $\omega = 0$ implies L_1 independently of u^2. This delay in reporting the observation results in a dramatic change in the system's behavior. □

Example 7.1 suggests that any design γ satisfying Property CI* can have a deadlock free implementation if and only if for any $(\omega, u) \in G^\gamma$ and any ordering $u^{s_1(\omega)}$, $u^{s_2(\omega)}$, , ..., $u^{s_N(\omega)}$ consistent with Property CI*, for all $k = 1, 2, \ldots, N$, given $(\omega,\ u^{s_1(\omega)},\ u^{s_2(\omega)}, ,\ \ldots,\ u^{s_{k-1}(\omega)}), h^{s_k}$ provides an output. The implementation of such information maps requires simultaneous monitoring of several subsets of $\Omega \times U$. As the number of subsets of $\Omega \times U$ requiring simultaneous monitoring increases the complexity of implementation of these information sources increases. In the case of Example 7.1, implementation of h^1 (respectively h^2) requires simultaneous monitoring of ω and (ω, u^2) (respectively ω and (ω, u^1)). On the other hand, implementation of $\hat{h}^1$ (respectively $\hat{h}^2$) requires only monitoring of (ω, u^2) (respectively ω and (ω, u^1)). Hence, implementation of h^1, that is, writing a program to implement h^1, is more complex than implementation of $\hat{h}^1$ in the sense that it requires more input ports and processors.

The above example and discussion suggest that: (**Qi**) the relationship between the physical realization of a system's observation functions and its deadlock–free operation should be formalized; and (**Qii**) a characterization of the tradeoff between the system's concurrency and the complexity of the physical realization of the system's observation functions should be developed.

The precise relationship between deadlock-freeness and the physical realization of a system's observation functions will be developed elsewhere ([7]). To find an answer to the issues raised in (**Qii**) we can proceed in several steps: First it will be important to find a simple characterization of Ψ, the set of all functions ψ from $\Omega \times U$ to S_N that satisfy Property CI, in terms of a function from $\Omega \times U$ to partial orders on the set A of N agents. For each $(\omega, u) \in \Omega \times U$ as ψ runs through Ψ one obtains a set of total orders $\psi(\omega, u)$ on A. Let $\phi(\omega, u)$ be the strongest partial order on A compatible with all these total orders. Can Ψ be recovered from ϕ? That is, does the set Φ of total orders generated by ϕ contain Ψ? Under what conditions is $\Phi = \Psi$? This is the *causality problem* posed by Witsenhausen in [34]. The solution to this problem, when the spaces $(\Omega, \mathcal{B})$, $(U^k, \mathcal{U}^k)$, $k = 1, 2, \ldots, N$, are *discrete* ([13]) will be presented elsewhere ([6]). For more general spaces, Witsenhausen's causality problem remains unsolved. The advantage of having ϕ is that we can use it to generate all total order functions $\psi : \Omega \times U \to S_N$ that satisfy Property CI, as well as all functions $\hat{\phi}_i$, from $\Omega \times U$ to partial orders on A such that for every $(\omega, u) \in \Omega \times U$ each $\hat{\phi}_i(\omega, u)$ is compatible with $\psi(\omega, u)$, $\psi \in \Psi$, and each $\hat{\phi}_i(\omega, u)$ is a weaker partial order than $\phi(\omega, u)$. Each of the functions ψ and $\hat{\phi}_i$ in turn suggests a physical realization of the system's observation functions that result in a deadlock-free operation of the system. Furthermore, associated with each of the above characterizations of the system's observation functions is the complexity of implementation (that is, the number of input ports and processors required to realize the program implementing the observation function), and the system's speed of response (concurrency). Thus, we can begin to understand the tradeoffs between the system's concurrency and the complexity of the system's information sources. Two open problems whose solution could increase our understanding of the abovementioned tradeoffs are: (i) a characterization of the complexity of implementation of *maximally concurrent* systems, i.e. systems where the information sources report the information without any delay; and (ii) a characterization of the complexity of implementation of *minimally concurrent* systems, i.e. systems where the information sources induce deadlock-free information structures and report information with maximum delay. Within the intrinsic model's framework, the precise formulation of the problem of reconciling the conflicting goals of maximizing concurrency and minimizing the required resources (parallel processes) remains open.

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