# EECS 487: Transformations

Ari Grant & Sugih Jamin

October 2009

## 1. Affine Transformations

**Definition.** The *n-dimensional Euclidean space* $\mathbb{E}^n$ is the set of all points in $\mathbb{R}^n$ with the included metric (measure of distance) $d(\vec{x}, \vec{y}) = \sqrt{(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})}$.

**Definition.** An *affine combination* of points $\vec{p}_0, \vec{p}_1, \ldots, \vec{p}_n \in \mathbb{E}^n$ is any point $\vec{p} \in \mathbb{E}^n$ such that there exists $a_0, a_1, \ldots a_n \in \mathbb{R}$ with

$$\vec{p} = \sum_{i=0}^{n} a_i \vec{p}_i \quad \text{and} \quad \sum_{i=0}^{n} a_i = 1.$$

**Definition.** An affine combination is a *convex affine combination* if it is also true that $a_i > 0$ for all $i$.

**Definition.** A transformation $\hat{\Lambda} : \mathbb{E}^n \to \mathbb{E}^n$ is a *linear transformation* if

$$\hat{\Lambda}(a\vec{u} + b\vec{v}) = a\Lambda(\vec{u}) + b\Lambda(\vec{v}).$$

**Claim.** All matrices are *linear transformations* when multiplied with a vector.

*Proof.* This fact follows trivially from the fact that matrix multiplication follows the above property. □

**Claim.** Any transformation $\hat{\Phi}$ that is of the form $\hat{\Phi}(\vec{p}) = \hat{A} \cdot \vec{p} + \vec{t}$ preserves affine combination if $\hat{A}$ is a square matrix and $\vec{t}$ is a constant vector.

*Proof.* It is necessary to show that

$$\hat{\Phi}\left(\sum_{i=0}^{n} a_i \vec{p}_i\right) = \sum_{i=0}^{n} a_i \hat{\Phi}(\vec{p}_i).$$

The required manipulation follows.

$$\hat{\Phi}\left(\sum_{i=0}^{n} a_i \vec{p}_i\right) = \hat{A} \cdot \left(\sum_{i=0}^{n} a_i \vec{p}_i\right) + \vec{t} = \left[\sum_{i=0}^{n} \hat{A} \cdot (a_i \vec{p}_i)\right] + \vec{t}$$

Note that $\sum_{i=0}^{n} a_i = 1$. So multiply $\vec{t}$ by 1 but write it as the sum of the affine combination coefficients.

$$= \left[\sum_{i=0}^{n} a_i \hat{A} \cdot \vec{p}_i\right] + \vec{t}\left[\sum_{i=0}^{n} a_i\right] = \left[\sum_{i=0}^{n} a_i \hat{A} \cdot \vec{p}_i\right] + \left[\sum_{i=0}^{n} a_i \vec{t}\right]$$

$$= \sum_{i=0}^{n} \left(a_i \hat{A} \cdot \vec{p}_i + a_i \vec{t}\right) = \sum_{i=0}^{n} a_i \left(\hat{A} \cdot \vec{p}_i + \vec{t}\right) = \sum_{i=0}^{n} a_i \hat{\Phi}(\vec{p}_i)$$

$\square$

**Definition.** A transformation $\hat{\Phi} : \mathbb{E}^n \to \mathbb{E}^n$ is an *affine transformation* if

$$\hat{\Phi}(\vec{p}) = \sum_{i=0}^{n} a_i \Phi(\vec{p}_i)$$

where $\vec{p} = \sum_{i=0}^{n} a_i \vec{p}_i$ is an affine combination.

Note then that any transformation $\hat{\Phi}(\vec{p}) = \hat{A} \cdot \vec{p} + \vec{t}$ is an *affine transformation* if $\hat{A}$ is a square matrix and $\vec{t}$ is a constant vector.

**Remark.** This is a very important fact. It means that given a set of vertices $\{v_i\}$ defining a shape or object $\mathbb{O}$, one can transform the individual vertices instead of the entire object, as long as the transformation is *affine.*

## 2. How Vectors (Vertices) Transform

For everything that follows, we can assume that work is done in $\mathbb{E}^3$, the standard 3-dimensional Euclidean space. Also, no assumptions will be made that basis vectors are orthogonal or normalized.

**Theorem 2.1.** *An affine transformation is fully specified by its action on the basis vectors.*

*Proof.* Let basis $A$ be the set of vectors $\{\vec{x}_A, \vec{y}_A, \vec{z}_A\}$ and let basis $B$ be the set $\{\vec{x}_B, \vec{y}_B, \vec{z}_B\}$. Any vector $\vec{v}$ can be written as a linear combination of the basis. That is, there exist $\alpha_A, \beta_A, \gamma_A, \alpha_B, \beta_B, \gamma_B \in \mathbb{R}$ such that $\vec{v} = \alpha_A \vec{x}_A + \beta_A \vec{y}_A + \gamma_A \vec{z}_A = \alpha_B \vec{x}_B + \beta_B \vec{y}_B + \gamma_B \vec{z}_B$. Thus when writing coordinates it is important to note which basis is being used. The convention will be to specify the basis with a subscript, thus

$$\vec{v} = \begin{pmatrix} \alpha_A \\ \beta_A \\ \gamma_A \end{pmatrix}_A = \begin{pmatrix} \alpha_B \\ \beta_B \\ \gamma_B \end{pmatrix}_B .$$

One assumption must now be made. It will be assumed that the basis vectors are constant throughout all space, thus $\vec{x}_A$ is always parallel to $\vec{x}_A$ no matter where one is. This is not the case for instance in spherical coordinates where the basis vectors $\vec{r}, \vec{\theta}$, and $\vec{\phi}$ point in different directions depending on what point in space is being used.

Note that this assumption is equivalent to there being affine maps from the standard $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$ to the basis $A$ and to the basis $B$.

This assumption means that there exist $a, b, c, d, e, f, g, h, i \in \mathbb{R}$ such that

$$\begin{aligned}
\vec{x}_B &= a\vec{x}_A + b\vec{y}_A + c\vec{z}_A \\
\vec{y}_B &= d\vec{x}_A + e\vec{y}_A + f\vec{z}_A \\
\vec{z}_B &= g\vec{x}_A + h\vec{y}_A + i\vec{z}_A.
\end{aligned}$$

Let us examine the vector above, $\vec{v} = \alpha_A\vec{x}_A + \beta_A\vec{y}_A + \gamma_A\vec{z}_A = \alpha_B\vec{x}_B + \beta_B\vec{y}_B + \gamma_B\vec{z}_B$.

$$\begin{aligned}
\vec{v} &= \begin{pmatrix} \alpha_B \\ \beta_B \\ \gamma_B \end{pmatrix}_B = \alpha_B\vec{x}_B + \beta_B\vec{y}_B + \gamma_B\vec{z}_B \\
&= \alpha_B(a\vec{x}_A + b\vec{y}_A + c\vec{z}_A) + \beta_B(d\vec{x}_A + e\vec{y}_A + f\vec{z}_A) + \gamma_B(g\vec{x}_A + h\vec{y}_A + i\vec{z}_A) \\
&= (\alpha_B a + \beta_B d + \gamma_B g)\vec{x}_A + (\alpha_B b + \beta_B e + \gamma_B h)\vec{y}_A + (\alpha_B c + \beta_B f + \gamma_B i)\vec{z}_A \\
&= \begin{pmatrix} \alpha_B a + \beta_B d + \gamma_B g \\ \alpha_B b + \beta_B e + \gamma_B h \\ \alpha_B c + \beta_B f + \gamma_B i \end{pmatrix}_A = \left[\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} \alpha_B \\ \beta_B \\ \gamma_B \end{pmatrix}\right]_A
\end{aligned}$$

Notice that the matrix that transforms the components of $\vec{v}$ from the basis $A$ to the basis $B$ can be rewritten as follows

$$\begin{pmatrix} \alpha_A \\ \beta_A \\ \gamma_A \end{pmatrix}_A = \left[\begin{pmatrix} | & | & | \\ (\vec{x}_B)_A & (\vec{y}_B)_A & (\vec{z}_B)_A \\ | & | & | \end{pmatrix} \begin{pmatrix} \alpha_B \\ \beta_B \\ \gamma_B \end{pmatrix}\right]_A.$$

Thus given the coordinates of a vector in basis $B$, the coordinates in basis $A$ can be found by multiplying $(\vec{v})_B$ by a matrix whose columns are the basis vectors of basis $B$ expressed in terms of basis $A$.

$$(\vec{v})_A = \begin{pmatrix} | & | & | \\ (\vec{x}_B)_A & (\vec{y}_B)_A & (\vec{z}_B)_A \\ | & | & | \end{pmatrix} (\vec{v})_B$$

$\square$

**Corollary 2.2.** *Applying a transformation $\hat{M}$ to the coordinates of a set of vectors $\{\vec{v}_i\}$ is the same as instead applying $\hat{M}^T$ to the basis $\{\vec{e}_i\}$ while maintaining the vector coordinates.*

*Proof.* Let $\vec{v} \in \mathbb{E}^3$, $E = \{\vec{a}, \vec{b}, \vec{c}\}$ be a basis, and $a_E, b_E, c_E \in \mathbb{R}$ be the coordinates. That is,

$$\vec{v} = a_E\vec{a} + b_E\vec{b} + c_E\vec{v} = \begin{pmatrix} a_E \\ b_E \\ c_E \end{pmatrix}_E = (a_E \;\; b_E \;\; c_E) \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}.$$

Now apply the transformation matrix

$$\hat{M} = \begin{pmatrix} d & e & f \\ g & h & i \\ j & k & l \end{pmatrix}$$

to the coordinate vector $(a_E, b_E, c_E)^T$.

$$\begin{aligned} \hat{M} \begin{pmatrix} a_E \\ b_E \\ c_E \end{pmatrix} &= \begin{pmatrix} d & e & f \\ g & h & i \\ j & k & l \end{pmatrix} \begin{pmatrix} a_E \\ b_E \\ c_E \end{pmatrix} \\ &= \begin{pmatrix} da_E + eb_E + fc_E \\ ga_E + hb_E + ic_E \\ ja_E + kb_E + lc_E \end{pmatrix}_E \end{aligned}$$

This is the coordinate vector of the transformed vector. If instead the transpose of the matrix is applied to the basis vector one has $\vec{v}\,' = a_E\hat{M}^T\vec{a} + b_E\hat{M}^T\vec{b} + c_E^T\hat{\vec{c}}$.

$$\begin{aligned} \vec{v}\,' &= \begin{pmatrix} d & e & f \\ g & h & i \\ j & k & l \end{pmatrix}^T \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} \\ &= \begin{pmatrix} d & g & j \\ e & h & k \\ f & i & l \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} \\ &= \begin{pmatrix} d\vec{a} + g\vec{b} + j\vec{c} \\ e\vec{a} + h\vec{b} + k\vec{c} \\ f\vec{a} + i\vec{b} + l\vec{c} \end{pmatrix} \end{aligned}$$

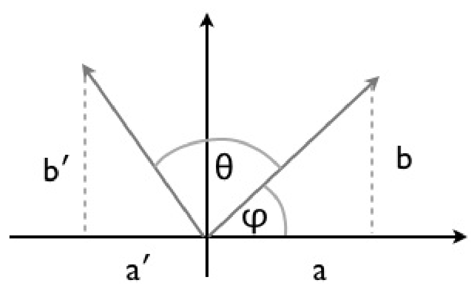
Applying the coordinate vector to get the transformed vector gives

$$
\begin{aligned}
\vec{v}\,' &= (a_E \;\; b_E \;\; c_E) \begin{pmatrix} d\vec{a} + g\vec{b} + j\vec{c} \\ e\vec{a} + h\vec{b} + k\vec{c} \\ f\vec{a} + i\vec{b} + l\vec{c} \end{pmatrix} \\
&= a_E(d\vec{a} + g\vec{b} + j\vec{c}) + b_E(e\vec{a} + h\vec{b} + k\vec{c}) + c_E(f\vec{a} + i\vec{b} + l\vec{c}) \\
&= (a_E d + b_E e + c_E f)\vec{a} + (a_E g + b_E h + c_E i)\vec{b} + (a_E j + b_E k + c_E l)\vec{c} \\
&= \begin{pmatrix} da_E + eb_E + fc_E \\ ga_E + hb_E + ic_E \\ ja_E + kb_E + lc_E \end{pmatrix}_E
\end{aligned}
$$

□

**Remark.** This has a profound importance. Given the components of a set of vectors in some basis. The set can be transformed by multiplying the basis vectors by a matrix *or* by multiplying the components (as a vector) by the transpose of that matrix! Thus if asked how to rotate a vector by $\theta$ is it the same as multiplying the transpose of that transformation (rotation by $-\theta$).

**Example.** In $\mathbb{E}^n$ rotate the vector $\vec{v} = (a, b)^T$ counterclockwise by the angle $\theta$.




The vector $\vec{v}$ has length $\sqrt{a^2 + b^2}$ and an angle $\varphi = \arctan(b/a)$ above the $x-$axis. The new vector has the same length but has a polar angle $\theta + \phi$. Thus

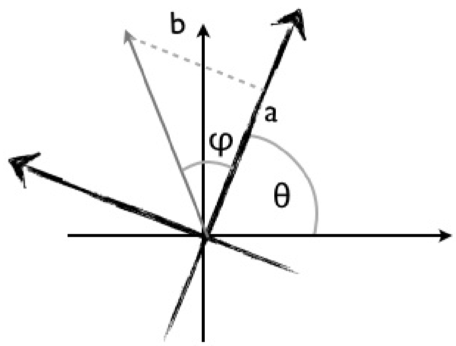
$$
\begin{aligned}
a' &= \sqrt{a^2 + b^2}\cos(\theta + \phi) = \sqrt{a^2 + b^2}\,(\cos\theta\cos\phi - \sin\theta\sin\phi) \\
&= \sqrt{a^2 + b^2}\left(\cos\theta\frac{a}{\sqrt{a^2 + b^2}} - \sin\theta\frac{b}{\sqrt{a^2 + b^2}}\right) = a\cos\theta - b\sin\theta \\
b' &= \sqrt{a^2 + b^2}\sin(\theta + \phi) = \sqrt{a^2 + b^2}\,(\sin\theta\cos\phi + \cos\theta\sin\phi) \\
&= \sqrt{a^2 + b^2}\left(\sin\theta\frac{a}{\sqrt{a^2 + b^2}} + \cos\theta\frac{b}{\sqrt{a^2 + b^2}}\right) = a\sin\theta + b\cos\theta
\end{aligned}
$$

Thus it follows that

$$
\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.
$$

This is the matrix that rotates a vector $(a, b)^T$ by an angle $\theta$ in a plane formed by two vectors $\vec{x}$ and $\vec{y}$ around the axis $\vec{x} \times \vec{y}$.

Now instead of working with the components, try rotating the coordinate system in a way that $\vec{v}$ still has the same components but its direction matches what is expected.




It may appear as if the same transformation has been applied to the basis as to the vector in the last part, but that is not true! The new $x-$axis is given by $\vec{x}\,' = \cos\theta\vec{x} + \sin\theta\vec{y}$ and the new $y-$axis is $\vec{y}\,' = -\sin\theta\vec{x} + \cos\theta\vec{y}$. Thus we have

$$\begin{pmatrix} \vec{x}\,' \\ \vec{y}\,' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}.$$

Hence the axes were indeed transformed by the transpose matrix used for the vector.