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Vectors and Matrices

1. Vectors

A vector is an object with magnitude and direction (velocity, force, acceleration, etc). A scalar is an object with just magnitude (temperature, pressure, age, length). A vector, denoted $\vec{v}$ or $\mathbf{v}$, has no initial point and is often best thought of as at the origin. A vector may be written in component form $\vec{v} = \langle v_1, v_2, v_3 \rangle$ (in three dimensions) but it is important to remember that the components are dependent on the coordinate system in use. The vector as a whole is a real object; the components are a tool. One special vector to keep in mind is the zero (or null) vector $\vec{0}$ whose components are all zero (in every coordinate system).

The **length**, **norm**, or **magnitude** of a vector is given by

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

and sometimes denoted $|\vec{v}|$.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ be two vectors and c be a scalar.

1) The **vector sum**, **direct sum**, or just **sum** of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle.$$

2) The **scalar multiple** of c and $\vec{a}$ is the vector

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle.$$

3) The **negative** of $\vec{a}$ is the vector

$$-\vec{a} = (-1)\vec{a} = \langle -a_1, -a_2, -a_3 \rangle.$$

4) The **vector difference** or just **difference** of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle.$$

5) The **direct product** of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \otimes \vec{b} = \langle a_1 b_1, a_2 b_2, a_3 b_3 \rangle.$$

Some properties to keep in mind about vectors are (given vectors $\vec{a}, \vec{b}$, and $\vec{c}$ and scalars c and d):

1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (Commutative property)
2) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Associative property)
3) $\vec{a} + \vec{0} = \vec{a}$ (Additive identity property)
4) $\vec{a} + (-\vec{a}) = \vec{0}$ (Additive inverse property)
5) $c(d\vec{a}) = (cd)\vec{a}$ (Associativity with Scalars)
6) $(c + d)\vec{a} = c\vec{a} + d\vec{a}$ (Distributive property)

7) $c(\vec{a} + \vec{b}) = c\vec{a} + d\vec{b}$ (Distributive property)
8) $1(\vec{a}) = \vec{a}$ (Multiplicative identity property)
9) $0(\vec{a}) = \vec{0}$

Given a vector $\vec{v}$ and a scalar c,

$$||c\vec{v}|| = |c|\,||\vec{v}||.$$

Given a vector $\vec{v}$ the **unit vector**, or **normal vector**, in the direction of $\vec{v}$, denoted $\hat{v}$ is given by

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}.$$

The vector $\hat{v}$ is parallel to $\vec{v}$ meaning that $\hat{v} = t\vec{v}$ where t is a scalar with $t = 1/||\vec{v}||$.

There are three standard unit vectors in Cartesian coordinates: $\hat{i} = \langle 1, 0, 0 \rangle, \hat{j} = \langle 0, 1, 0 \rangle$, and $\hat{k} = \langle 0, 0, 1 \rangle$. Thus the vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$ in Cartesian coordinates can be written $\vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$.

Two vectors are equal if and only if (abbreviated iff) they have the same magnitude and the same direction; in other words, each of their respective components is equal.

Two nonzero vectors $\vec{u}$ and $\vec{v}$ are **parallel** if there exists a scalar c such that $\vec{u} = c\vec{v}$.

2. The Dot Product

Another operation on vectors is the dot product. Sometimes it is called the inner product or scalar product (since it produces a scalar). The dot product of $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

It is very important to notice that the result of the dot product is *not* a vector. If there are only two dimensions the third term is not present and so forth for other dimensions.

Some properties of the dot product (given vectors $\vec{u}, \vec{v}$ and $\vec{w}$ and scalar c) are:
1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (Commutative property)
2) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (Distributive property)
3) $c(\vec{u} \cdot \vec{v}) = c\vec{u} \cdot \vec{v} = \vec{u} \cdot c\vec{v}$
4) $0 \cdot \vec{v} = 0$
5) $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$

Given a triangle with sides of lengths a, b, and c and with angle θ opposite side c
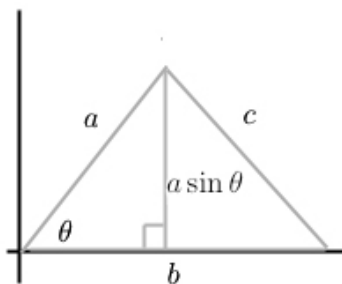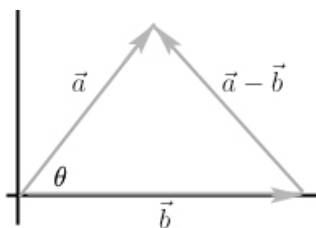

the Pythagorean theorem gives

$$c^2 = (a \sin \theta)^2 + (b - a \cos \theta)^2 = a^2 \sin^2 \theta + a^2 \cos^2 \theta + b^2 - 2ab \cos \theta$$
$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$

Duplicating the diagram with vectors gives

$$||\vec{a} - \vec{b}||^2 = ||\vec{a}||^2 + ||\vec{b}||^2 - 2||\vec{a}|| \, ||\vec{b}|| \cos \theta$$

Using properties of the dot product, the left side can be rewritten

$$||\vec{a} - \vec{b}||^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = (\vec{a} - \vec{b}) \cdot \vec{a} - (\vec{a} - \vec{b}) \cdot \vec{b}$$
$$= \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = ||\vec{a}||^2 + ||\vec{b}||^2 - 2\vec{a} \cdot \vec{b}.$$
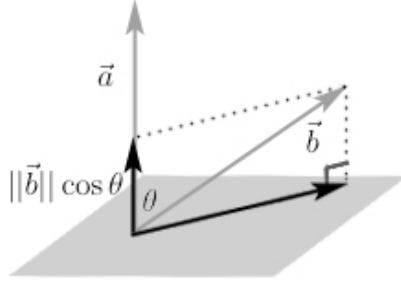
Comparison with the formula above gives

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \, ||\vec{b}|| \cos \theta$$

as an alternate form of the dot product. This can also be used to find the angle θ between two vectors $\vec{a}$ and $\vec{b}$.

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| \, ||\vec{b}||}$$

Two vectors $\vec{a}$ and $\vec{b}$ are then **orthogonal** or **perpendicular** if $\vec{a} \cdot \vec{b} = 0$.

Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors separated by an angle θ.

The vector $\vec{b}$ can be split into two components: one that is parallel to $\vec{a}$ and one that is orthogonal to it. The component of $\vec{b}$ parallel to $\vec{a}$ is called the **projection of $\vec{b}$ onto $\vec{a}$** or the **vector component of $\vec{b}$ along $\vec{a}$** and is denoted $\text{proj}_{\vec{a}}\vec{b}$. Notice that

$$||\text{proj}_{\vec{a}}\vec{b}|| = ||\vec{b}|| \cos\theta$$

and the direction is along the unit vector $\hat{a}$. That gives

$$\text{proj}_{\vec{a}}\vec{b} = \left(||\vec{b}|| \cos\theta\right)\hat{a} = \left(||\vec{b}|| \cos\theta\right)\frac{\vec{a}}{||\vec{a}||} = \left(\frac{||\vec{a}||\,||\vec{b}||\cos\theta}{||\vec{a}||}\right)\frac{\vec{a}}{||\vec{a}||} = \left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\right)\vec{a} = (\hat{a}\cdot\vec{b})\hat{a}.$$

Observe also that

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a} = t\vec{a}$$

where t is a scalar and $t = (\vec{a}\cdot\vec{b})/||\vec{a}||^2$. Thus $\text{proj}_{\vec{a}}\vec{b}$ is *always* parallel to $\vec{a}$.

The component of $\vec{b}$ orthogonal to $\vec{a}$ is then just

$$\text{proj}_{\vec{a}}^{\perp}\vec{b} = \vec{b} - \text{proj}_{\vec{a}}\vec{b} = \vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\right)\vec{a}.$$

Be aware that there is no standard notation for the orthogonal component of one vector with respect to another and $\text{proj}_{\vec{a}}^{\perp}\vec{b}$ was used for illustration.

The projection of $\vec{b}$ along $\vec{a}$ and the perpendicular projection *decompose* $\vec{b}$ which means that $\vec{b} = \text{proj}_{\vec{a}}\vec{b} + \text{proj}_{\vec{a}}^{\perp}\vec{b}$, which is trivially true by the construction of $\text{proj}_{\vec{a}}^{\perp}\vec{b}$. Also $\text{proj}_{\vec{a}}\vec{b}$ and $\text{proj}_{\vec{a}}^{\perp}\vec{b}$ are orthogonal, shown as follows

$$\text{proj}_{\vec{a}}\vec{b} \cdot \text{proj}_{\vec{a}}^{\perp}\vec{b} = \left[\left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\right)\vec{a}\right]\cdot\left[\vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\right)\vec{a}\right]$$

$$= \left[\left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\right)\vec{a}\cdot\vec{b}\right] - \left[\left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\right)\vec{a}\right]\cdot\left[\left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\right)\vec{a}\right]$$

$$= \frac{(\vec{a}\cdot\vec{b})^2}{||\vec{a}||^2} - \left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||^2}\right)^2(\vec{a}\cdot\vec{a}) = \frac{(\vec{a}\cdot\vec{b})^2}{||\vec{a}||^2} - \frac{(\vec{a}\cdot\vec{b})^2}{||\vec{a}||^4}||\vec{a}||^2 = 0.$$

There are two edge cases of projection:

1) If $\vec{a} = \vec{b}$ then the component of $\vec{b}$ along $\vec{a}$ is $\vec{a}$ itself. Restated it says that iff $\vec{a} = \vec{b}$ then

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2}\right)\vec{a} = \left(\frac{\vec{a} \cdot \vec{a}}{||\vec{a}||^2}\right)\vec{a} = \left(\frac{||\vec{a}||^2}{||\vec{a}||^2}\right)\vec{a} = \vec{a}$$

and similarly $\text{proj}_{\vec{a}}^{\perp}\vec{b} = 0$.

2) If $\vec{a} \cdot \vec{b} = 0$ (they are orthogonal or perpendicular) then the component of $\vec{b}$ along $\vec{a}$ is 0 and $\text{proj}_{\vec{a}}^{\perp}\vec{b} = \vec{a}$.

Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors separated by an angle $\pi - \theta$. Instead of placing them tail-to-tail, the head of $\vec{b}$ is placed at the tail of $\vec{a}$ since vectors can be freely translated throughout space. This makes an apparent separation angle between the two as θ.
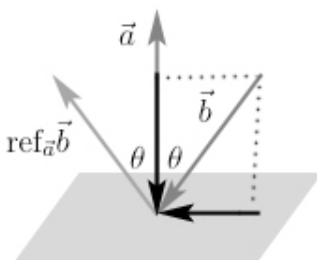
$\vec{b}$ can be broken up into two components. One parallel to $\vec{a}$ and one orthogonal. They trivially add up to make $\vec{b}$.

$$\vec{b} = \text{proj}_{\vec{a}}\vec{b} + \text{proj}_{\vec{a}}^{\perp}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2}\right)\vec{a} + \left[\vec{b} - \left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2}\right)\vec{a}\right]$$

The vector in the same plane as $\vec{a}$ and $\vec{b}$ (any two nonzero vectors form a plane) with angle θ with respect to $\vec{a}$ (instead of $\pi - \theta$) is called the **reflection of $\vec{b}$ across $\vec{a}$**, sometimes denoted $\text{ref}_{\vec{a}}\vec{b}$. This vector has the same orthogonal component to $\vec{a}$ as $\vec{b}$ does, but that its parallel component is flipped. That means

$$\text{ref}_{\vec{a}}\vec{b} = (-1)\left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2}\right)\vec{a} + \left[\vec{b} - \left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2}\right)\vec{a}\right] = \vec{b} - 2\left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2}\right)\vec{a} = \vec{b} - 2(\hat{a} \cdot \vec{b})\hat{a}.$$

Sometimes reflection is used to mean mirroring vector $\vec{b}$ along $\vec{a}$ instead of the use above where it is based off a ray reflecting off a surface. In such a context the ray produced will be the negative of the one derived above.

3. THE CROSS PRODUCT

The cross product between two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is defined as

$$\vec{u} \times \vec{v} = \big\langle (u_2 v_3 - u_3 v_2), (u_3 v_1 - u_1 v_3), (u_1 v_2 - u_2 v_1) \big\rangle.$$

Unlike the dot product the cross product produces a vector and not a scalar. It can be remembered using a trick with determinants (which will be reviewed later):

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\hat{\imath} + (u_3 v_1 - u_1 v_3)\hat{\jmath} + (u_1 v_2 - u_2 v_1)\hat{z}.$$

The cross product is not defined in 2D and if one wishes to use it with two 2D vectors, set them to have a third component equal to zero and proceed. Thus given $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$,

$$\vec{u} \times \vec{v} = \langle u_1, u_2, 0 \rangle \times \langle v_1, v_2, 0 \rangle = \langle 0, 0, u_1 v_2 - u_2 v_1 \rangle.$$

Some properties of the cross product (given vectors $\vec{u}, \vec{v}$ and $\vec{w}$ and scalar c) are:

1) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (Anti-commutative property)
2) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ (Distributive property)
3) $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4) $0 \times \vec{v} = \vec{0}$
5) $\vec{v} \times \vec{v} = \vec{0}$
6) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
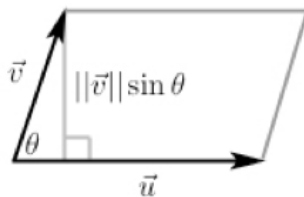7) $\vec{u} \times \vec{v} \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

The cross product $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. That is $(\vec{u} \times \vec{v}) \cdot \vec{u} = (\vec{u} \times \vec{v}) \cdot \vec{v} = 0$.

Also, $\vec{u} \times \vec{v} = 0$ iff $\vec{u} = c\vec{v}$ for some scalar c, meaning that they are parallel.

Notice that

$$\begin{aligned} ||\vec{u} \times \vec{v}|| &= \sqrt{(u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2} \\ &= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2} \\ &= \sqrt{||\vec{u}||^2 ||\vec{v}||^2 - (\vec{u} \cdot \vec{v})^2} \\ &= ||\vec{u}||\, ||\vec{v}|| \sqrt{1 - \frac{(\vec{u} \cdot \vec{v})^2}{||\vec{u}||^2 ||\vec{v}||^2}} \\ &= ||\vec{u}||\, ||\vec{v}|| \sqrt{1 - \cos^2 \theta} \\ ||\vec{u} \times \vec{v}|| &= ||\vec{u}||\, ||\vec{v}|| \sin \theta. \end{aligned}$$
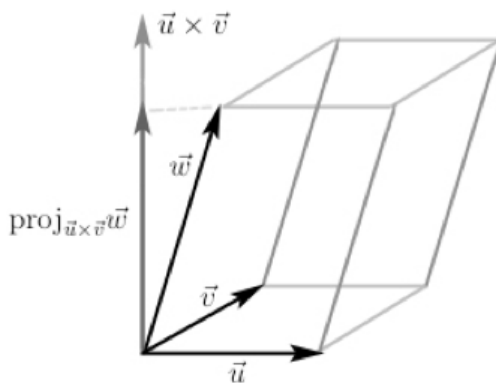
Let $\vec{u}$ and $\vec{v}$ be the two sides of a parallelogram.

Then the area of the parallelogram is

$$\text{Area} = (\text{base})(\text{height}) = ||\vec{u}|| \, ||\vec{v}|| \, \sin\theta = ||\vec{u} \times \vec{v}||.$$

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be the three sides of a parallelepiped.

Then the volume of the parallelepiped is

$$\text{Volume} = (\text{Area of base})(\text{height}) = ||\vec{u}\times\vec{v}|| \, ||\text{proj}_{\vec{u}\times\vec{v}}\vec{w}|| = ||\vec{u}\times\vec{v}|| \left| \frac{\vec{w} \cdot (\vec{u} \times \vec{v})}{||\vec{u} \times \vec{v}||} \right| = |\vec{w}\cdot(\vec{u}\times\vec{v})|.$$

And note that $\vec{w}\cdot(\vec{u}\times\vec{v}) = \vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = -\vec{w}\cdot(\vec{u}\times\vec{v}) = -\vec{u}\cdot(\vec{v}\times\vec{w}) = -\vec{v}\cdot(\vec{w}\times\vec{u})$. Since the volume has an absolute value sign, it does not matter which vector is chosen as which.

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ then

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

is the determinant (explained later); $\vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product**.

4. BASIS

The set of all real numbers is often denoted $\mathbb{R}$. The set of all points in 3D space is written $\mathbb{R}^3$ (the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$) and any point in it can be written as (x, y, z) with $x, y, z \in \mathbb{R}$; however, the components x, y, and z are dependent on the given **basis**.

Given a **vector space** V (the definition of which is not important here) a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is said to **span** V iff for all vectors $\vec{a} \in V$ there exists $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that $\vec{a} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$.

A **basis** for a vector space V is a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ that span V and are **linearly independent**, that is to say that no vector in the set is a linear combination of the others.

A basis is an **orthogonal basis** iff all basis elements are mutually orthogonal. That is, given a basis $\{\vec{v}_n\}$ for V, one has that $\vec{v}_i \cdot \vec{v}_j = 0$ when $i \neq j$. A basis is said to be a **normal basis** if each element has unit length (magnitude). A basis is said to be an **orthonormal basis** when it is both normal and orthogonal.

Some common orthonormal bases (plural of basis) for $\mathbb{R}^3$ are:

1) *Cartesian Coordinates* with basis vectors $\hat{x}, \hat{y}$, and $\hat{z}$.
2) *Cylindrical Coordinates* with basis vectors $\hat{r}$ ($\hat{s}$ or $\hat{\rho}$), $\hat{\theta}$, and $\hat{z}$.
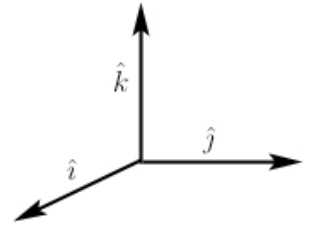 $\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$
   Where θ is the counter-clockwise angle from the x-axis in the XY-plane.
3) *Spherical Coordinates* with basis $\hat{r}, \hat{\theta}$, and $\hat{\phi}$.
   $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$
   $\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$
   $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$
   Sometimes θ and ϕ are reversed so be sure to always check an author's choice.

An orthogonal basis (each vector is orthogonal to the others) with basis vectors $\hat{i}, \hat{j}$, and $\hat{k}$ is said to be **right-handed** (counter-clockwise) iff

$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j}$$

and **left-handed** (clockwise) iff

$$\hat{j} \times \hat{i} = \hat{k}$$
$$\hat{k} \times \hat{j} = \hat{i}$$
$$\hat{i} \times \hat{k} = \hat{j}.$$

OpenGL is right-handed. Direct3D is left-handed.

The vector $\vec{v} = \alpha\hat{r} + \beta\hat{\theta} + \gamma\hat{\phi}$ would be written (α, β, γ) and the vector $\vec{u} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}$ would also be written (α, β, γ). Thus it is extremely important to know what basis is being used. Remember that a vector with its magnitude (length) and direction (sense) is the real object and that the components are just a basis-dependent representation.

Given a basis $\vec{v}_1, \vec{v}_2,$ and $\vec{v}_3$ (in the 3-dimension case) and a vector $\vec{u}$ the **direction angles** are defined as the angles between the vector and each coordinate axis. They are usually written $\alpha, \beta,$ and γ where $\alpha =$ (the angle between $\vec{u}$ and coordinate axis $\vec{v}_1$). β and γ are the angles for the other two coordinate axes. The **direction cosines** are defined as the cosines of the direction angles. That is

$$\cos\alpha = \hat{u} \cdot \hat{v}_1$$
$$\cos\beta = \hat{u} \cdot \hat{v}_2$$
$$\cos\gamma = \hat{u} \cdot \hat{v}_3.$$

5. Lines, Rays, and Planes

A **line** parallel to the vector $\vec{u} = \langle u_x, u_y, u_z \rangle$ and passing through the point (x_0, y_0, z_0) is given by the **parametric equations**

$$x = x_0 + u_x t$$
$$y = y_0 + u_y t$$
$$z = z_0 + u_z t$$

which is equivalent to

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}$$

if $u_x, u_y,$ and u_z are all nonzero.

A **ray** is simply a vector with a specified initial point. It can also be thought of as a line with $t \geq 0$ (it has one endpoint, a line segment has two).

A **plane** containing the point (x_0, y_0, z_0) and having a normal vector (it is orthogonal to the plane) $\vec{n} = \langle A, B, C \rangle$ is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or by letting $D = -(Ax_0 + By_0 + Cz_0)$,

$$Ax + By + Cz + D = 0.$$

The angle θ between two planes with normal vectors $\vec{n}_0$ and $\vec{n}_2$ is given by

$$\cos\theta = \frac{|\vec{n}_0 \cdot \vec{n}_1|}{||\vec{n}_0|| \, ||\vec{n}_1||}.$$

This is a direct application of the angle between two vectors. The numerator has absolute value since one of the planes could be "upside-down" and the angle between two planes must be between 0 and $\pi/2$ inclusively.

The angle between a vector $\vec{v}$ and a plane with normal $\vec{n}$ is just $\cos{(\pi/2 - \theta)} = |\hat{v} \cdot \hat{n}|$.

Let $\vec{n}$ be the normal of a plane containing the point P and let Q be a point not in the plane.
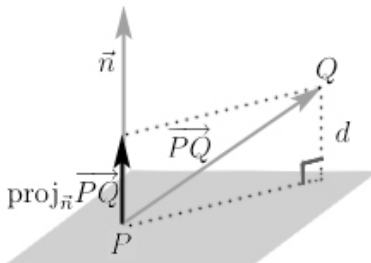
The distance d between the point Q and the plane is then

$$d = ||\text{proj}_{\vec{n}}\overrightarrow{PQ}|| = \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{||\vec{n}||}.$$

Let $\vec{u}$ be the direction vector of a line containing the point P and let Q be a point not on the line. By similar arguments as above the distance d between the point Q and the line is

$$d = ||\overrightarrow{PQ}|| \sin\theta = \frac{||\overrightarrow{PQ} \times \vec{u}||}{||\vec{u}||}$$

where θ is the angle between $\overrightarrow{PQ}$ and $\vec{u}$.

6. Matrices

A matrix, denoted $\hat{M}$ along with many other notations, is a rectangular array of numbers for instance

$$\hat{M} = \begin{pmatrix} 3 & -8 & 16 \\ 6 & 1 & -9 \end{pmatrix}.$$

This particular matrix is of order 2×3, always listing the number of rows first. A matrix with equal number of rows and columns is called a **square matrix**.

The **transpose** of a matrix, denoted $\hat{M}^T$ is a matrix produced by interchanging the rows and columns. The transpose of the matrix above is

$$\hat{M}^T = \begin{pmatrix} 3 & 6 \\ -8 & 1 \\ 16 & -9 \end{pmatrix}.$$

The transpose takes an element at row i and column j, a_{ij}, and moves it row j and column i, a_{ji}.

A vector is simply a matrix with only one column. Vectors are often written as matrices with one row but this is merely for convenience and is incorrect. Thus a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is the same as

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

7. Matrix Multiplication

The multiplication of two matrices is analogous to the dot product. Given vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ we can think of their dot product as multiplying the transpose of one by the other. That is

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}^T \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (u_1 v_1 + u_2 v_2 + u_3 v_3).$$

Think of taking the dot product with a row and a column. In the product $\hat{A}\hat{B}$, the element at row i and column j will be the "dot product" of row i in matrix $\hat{A}$ with column j in matrix $\hat{B}$. An example should help to illustrate the point.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$$

The element at row 1 and column 1 of the product will be row 1 in the left matrix dot column 1 in the right matrix.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 1(5) + 2(7) & ? \\ ? & ? \end{pmatrix}$$

The element at row 1 and column 2 of the product will be row 1 in the left matrix dot column 2 in the right matrix.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 19 & 1(6) + 2(4) \\ ? & ? \end{pmatrix}$$

The element at row 2 and column 1 of the product will be row 2 in the left matrix dot column 1 in the right matrix.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 19 & 14 \\ 3(5) + 1(7) & ? \end{pmatrix}$$

The element at row 2 and column 2 of the product will be row 2 in the left matrix dot column 2 in the right matrix.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 19 & 14 \\ 22 & 3(6) + 1(4) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 19 & 14 \\ 22 & 22 \end{pmatrix}$$

In general matrix multiplication is not commutative.

$$\begin{pmatrix} 5 & 6 \\ 7 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 23 & 16 \\ 19 & 18 \end{pmatrix}$$

Matrix multiplication is only possible if the first matrix's number of columns is equal to the second matrix's number of rows.

The identity matrix is the matrix $\hat{I}$ such that for any square matrix $\hat{M}$ it follows that $\hat{M}\hat{I} = \hat{I}\hat{M} = \hat{M}$. The identity matrix must be the same size as $\hat{M}$ and has ones on the main diagonal and zeros elsewhere. For a square matrix of size three one has

$$\begin{pmatrix} 15 & 8 & 1 \\ 0 & 3 & -9 \\ -7 & -4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 8 & 1 \\ 0 & 3 & -9 \\ -7 & -4 & 2 \end{pmatrix}$$

and thus for order three

$$\hat{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The **determinant** of a square 1×1 matrix $\hat{M}$, denoted $|\hat{M}|$ or $\det(\hat{M})$, is defined as the value of the matrix's one element. That is, if $\hat{M} = [a]$, then $|\hat{M}| = a$. The determinant of a matrix is a scalar.

The **minor** (i, j) of a matrix $\hat{M}$, denoted $\hat{M}_{i,j}$, is the determinant of the matrix made by removing row i and column j from $\hat{M}$.

For example: $\hat{M}_{1,2}$ of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{vmatrix} \cancel{a} & \cancel{b} \\ c & \cancel{d} \end{vmatrix} = |c| = c$

The **cofactor** (i, j) of a matrix $\hat{M}$, denoted $\hat{C}_{i,j}$, is $\hat{C}_{i,j} = (-1)^{i+j}\hat{M}_{i,j}$.

For example: $\hat{C}_{1,2}$ of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\hat{C}_{1,2} = (-1)^{1+2}\hat{M}_{i,j} = -c$.

The **determinant** of any square matrix is obtained by taking any row (or column) and summing the product of each element of that row (or column) with its cofactor.

The **determinant** of a square 2×2 matrix $\hat{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is then by (using the first row)

$$|\hat{M}| = a\left(\hat{C}_{1,1}\right) + b\left(\hat{C}_{1,2}\right) = a(-1)^{1+1}\hat{M}_{1,1} + b(-1)^{1+2}\hat{M}_{1,2} = ad - bc.$$

It is worth nothing that choosing to use the second row or either of the columns would still result in $ad - bc$, the difference of the diagonals.

The **determinant** of a 3×3 matrix $\hat{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is then (using the first row)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(-1)^{1+1}\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{vmatrix} + b(-1)^{1+2}\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{vmatrix} + c(-1)^{1+3}\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - hf) - b(di - gf) + c(dh - ge)$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - ahf - bdi + bgf + cdh - cge$$

This result would have been obtained using any row or column. A trick to taking determinants is to think of minors instead of cofactors. Work along any row or column summing the product of each element with its minor and using signs as follows.

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

8. Inverse of a Matrix

If a matrix is not square the notion of an inverse is vacuous. For any square matrix $\hat{M}$ there exists an inverse $\hat{M}^{-1}$ such that $\hat{M}\hat{M}^{-1} = \hat{M}^{-1}\hat{M} = \hat{I}$ if and only if $|\hat{M}| \neq 0$ (an invertible matrix is also called non-singular).

There are two common ways to find the inverse of a matrix. The first is called Cramer's rule and the second is Gauss-Jordan elimination.

The **cofactor matrix** of a matrix $\hat{M}$, denoted $\hat{C}_{i,j}$ without substituting numbers in for i and j, is the matrix where every element $a_{i,j}$ is the cofactor $\hat{C}_{i,j}$ of $\hat{M}$.

The **adjugate** or **classical adjoint** of a matrix $\hat{M}$, denoted $\mathrm{Adj}(\hat{M})$ is the transpose of the cofactor matrix.

Cramer's rule states that given a square matrix $\hat{M}$ with $|\hat{M}| \neq 0$

$$\hat{M}^{-1} = \frac{1}{|\hat{M}|} \text{Adj}(\hat{M}).$$

As an example take the matrix $\hat{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The cofactors are $\hat{C}_{1,1} = d$, $\hat{C}_{1,2} = -c$, $\hat{C}_{2,1} = -b$, and $\hat{C}_{2,2} = a$. Thus the cofactor matrix is $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ and the adjugate is merely the transpose of that, hence

$$\text{Adj}(\hat{M}) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Thus Cramer's rule states that

$$\hat{M}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

which when either left- or right-multiplied by $\hat{M}$ does indeed produce $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the identity matrix (of order 2).

9. Gauss-Jordan Elimination for the Inverse of a Matrix

There are three **elementary row operations** that can be performed on a matrix.

1) **Row swap.** A row within the matrix can be swapped with any other row.
2) **Row multiplication.** Each element in a row can be multiplied by a non-zero constant.
3) **Row addition.** A row can be replaced by that row added to any other.

Notice that row i could be multiplied through by a constant k, then row i could be added to row j and then row i could be multiplied by $1/k$. This is often quoted as a single step (although a composition of others) simply as *a constant multiple of any row may be added to any other.*

The inverse of a matrix can be found by cleverly applying these elementary operations to a matrix $\hat{M}$ *augmented* with the identity matrix (that means placing them side-by-side to make a larger matrix). Remember that a matrix must be square to have an inverse in the first place so it makes sense to be able to place it along side the identity matrix. This may sound confusing but an simple example should be illustrious. Given the matrix

$$\hat{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the inverse will be found as follows.

Write the matrix augmented with the corresponding identity matrix.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right]$$

Elementary row operations must be performed in order to make the matrix on the "left" look like the identity matrix. Thus where the a is should be a 1. So the first row is multiplied by $1/a$.

$$\begin{bmatrix} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

Where the c is should be a 0, so add $-c$ times the first row to the second row.

$$\begin{bmatrix} 1 & b/a & 1/a & 0 \\ c+(-c) & d+(-c)b/a & 0+(-c)1/a & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & b/a & 1/a & 0 \\ 0 & d-bc/a & -c/a & 1 \end{bmatrix}$$

The entry "$d - bc/a$" should be a 1 so we multiply the second row by $1/(d - bc/a)$.

$$\begin{bmatrix} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c/a}{d-bc/a} & \frac{1}{d-bc/a} \end{bmatrix}$$

$$\begin{bmatrix} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

All that remains is to change the "b/a" to 0. So add the second row times $(-b/a)$ to the first.

$$\begin{bmatrix} 1 & 0 & 1/a+(-b/a)\frac{-c}{ad-bc} & 0+(-b/a)\frac{a}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1/a+\frac{cb/a}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{ad-bc}{a(ad-bc)}+\frac{cb}{a(ad-bc)} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

After using elementary row operations on the augmented matrix, the left side (which was $\hat{M}$) is now the identity matrix and the right side (which was the identity matrix) is now $\hat{M}^{-1}$. This matrix is exactly the inverse of $\hat{M}$ found using Cramer's rule.

Gauss-Jordan Elimination is done (to the augmented matrix of $\hat{M}$ and the identity matrix) as

1) Use row swaps to make the first row have a nonzero leading coefficient.
2) Divide the first row by its leading coefficient to make the leading coefficient 1 or swap in a row that already has it.
3) Add multiples of the first row to all other rows to make each row have a zero in the first column.
4) Repeat the process with the second row and so forth.

Here is a simple example to show it once more. Here the inverse of

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 4 & 1 & 2 \end{pmatrix}$$

will be found.

The determinant is 15 (which is not zero) so the matrix has an inverse and so one proceeds as follows.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array} \right]$$

Swap the first and second rows to put a 1 in place (this is easier than doing division to get it).

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array} \right]$$

Add (-2) times the first row to the second.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -3 & 1 & -2 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array} \right]$$

Add (-4) times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -3 & 1 & -2 & 0 \\ 0 & 1 & -6 & 0 & -4 & 1 \end{array} \right]$$

Swap the second and third rows (or do a division if preferred) to put a 1 as the leading coefficient in the second row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -6 & 0 & -4 & 1 \\ 0 & 3 & -3 & 1 & -2 & 0 \end{array} \right]$$

Add (-3) times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -6 & 0 & -4 & 1 \\ 0 & 0 & 15 & 1 & 10 & -3 \end{array} \right]$$

Divide the third row by 15.

$$
\left[\begin{array}{ccc|ccc}
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & -6 & 0 & -4 & 1 \\
0 & 0 & 1 & 1/15 & 2/3 & -1/5
\end{array}\right]
$$

Add (-2) times the third row to the first.

$$
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & -2/15 & -1/3 & 2/5 \\
0 & 1 & -6 & 0 & -4 & 1 \\
0 & 0 & 1 & 1/15 & 2/3 & -1/5
\end{array}\right]
$$

Add (6) times the third row to the second.

$$
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & -2/15 & -1/3 & 2/5 \\
0 & 1 & 0 & 2/5 & 0 & -1/5 \\
0 & 0 & 1 & 1/15 & 2/3 & -1/5
\end{array}\right]
$$

Thus the matrix on the right is the inverse of the matrix above.

10. System of Linear Equations

Given two equations $ax + by = c$ and $dx + ey = f$ one can write this as

$$
\begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}.
$$

This is because equality of matrices is equality of all components, so multiplying the above out results in the original two equations. One can then apply

$$
\begin{pmatrix} a & b \\ d & e \end{pmatrix}^{-1}
$$

to both sides to find x and y.

11. Wrap-up

So how much should you take away from all of this? You should be able to take the dot product, cross product, or direct product of vectors; you should be able to find the distance between points and lines or points and planes; you should be able to find the reflection of one vector off another; you should be able to multiply any two matrices, find the determinant of one, or the inverse; and if asked you should be able to prove any of the simple identities such as $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$.

It may be in your favor to practice some of the above skills. The mental overhead of computer graphics can be greatly lowered if working with vectors and matrices is second nature!

Transformations and rotations will be covered in later notes.

The vector part of these notes was written using as a guide:
Calculus Early Transcendental Functions 3$^{\text{rd}}$ Ed. by Larson, Hostetler, and Edwards ©2003.