Progressive Multigrid Eigensolvers for Multiscale Spectral Segmentation

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Overview

Goal: Efficient multiscale spectral clustering

Solution: Multigrid eigensolver on multiscale graph

- Pyramid with cross-scale constraints
- Scale-dependent cues active at each pyramid level
- Multigrid:
  - Process coarse-to-fine sub-pyramids
  - Many early iterations on small sub-pyramids
  - Fewer later iterations on full pyramid
  - Work with intermediate basis instead of eigenvectors
- Parallelizable for *arbitrary* graph structure (unlike [2])

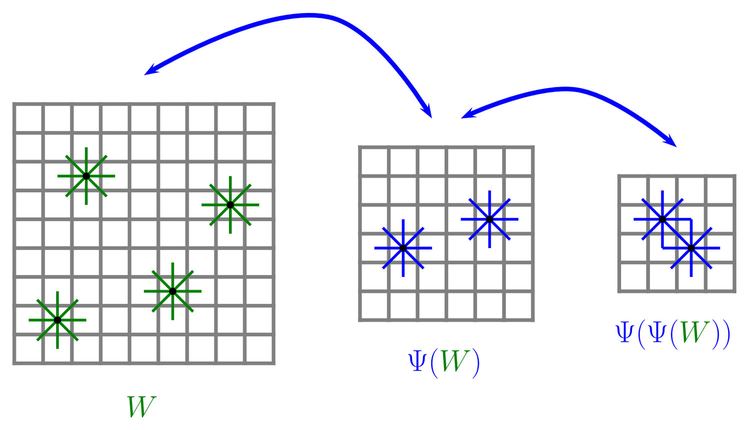
Technical Approach:

- Angular Embedding [9] (generalization of Normalized Cuts)
- Constrained spectral clustering [10]
- Constraint-based coarse-to-fine interpolation
- Randomized methods for matrix approximation [5]
- Self-check for convergence

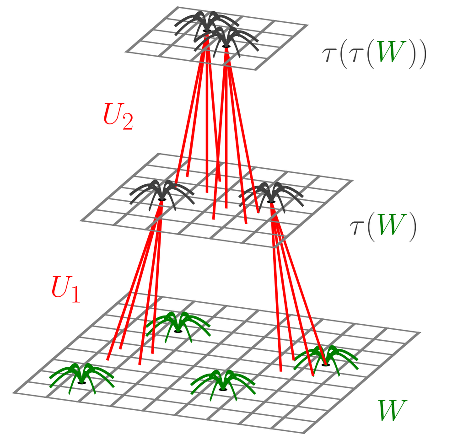
Image Segmentation Results:

- Order of magnitude speedup (from multigrid alone; parallelization may further improve)
- Automatic inter-scale edge alignment

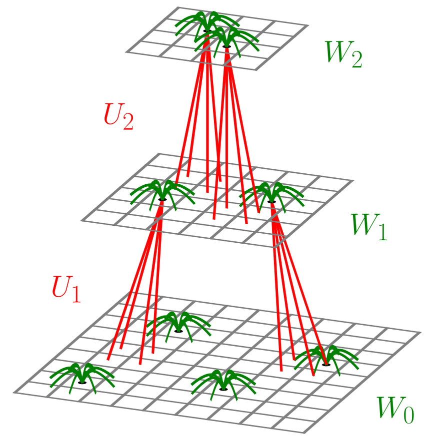
System Comparison



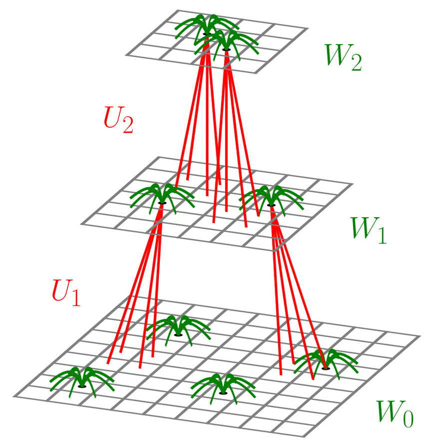
Multigrid [3, 6]



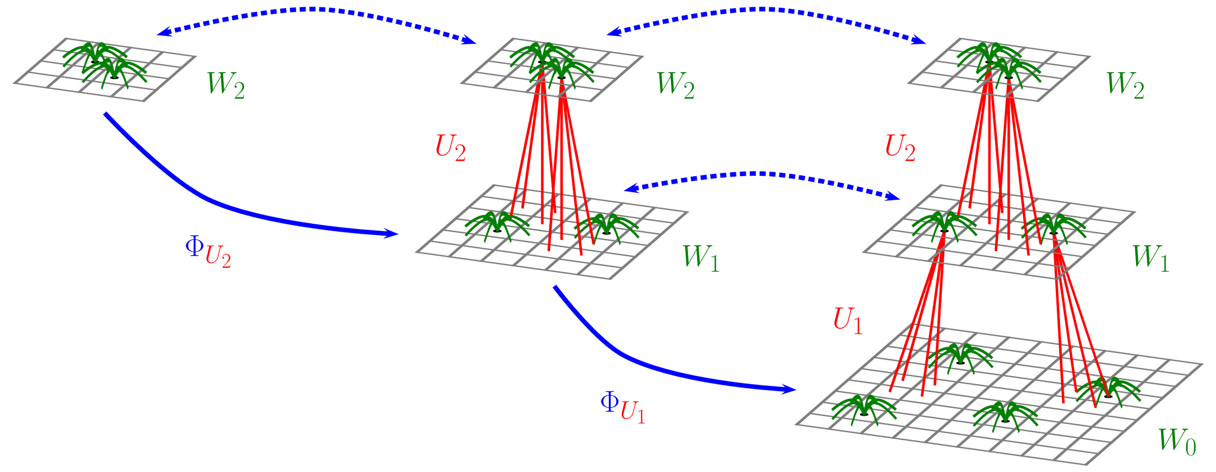
Multirange [4]



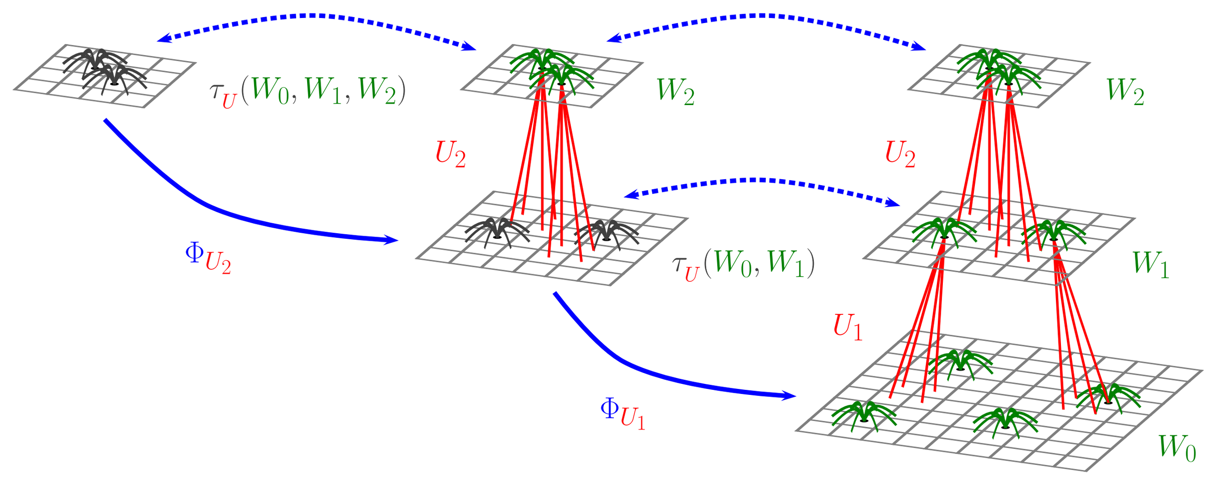
Multiscale [4, 8]



Multiscale



Progressive Multigrid Multiscale



Transformed Progressive Multigrid Multiscale

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Multiscale Constrained Angular Embedding

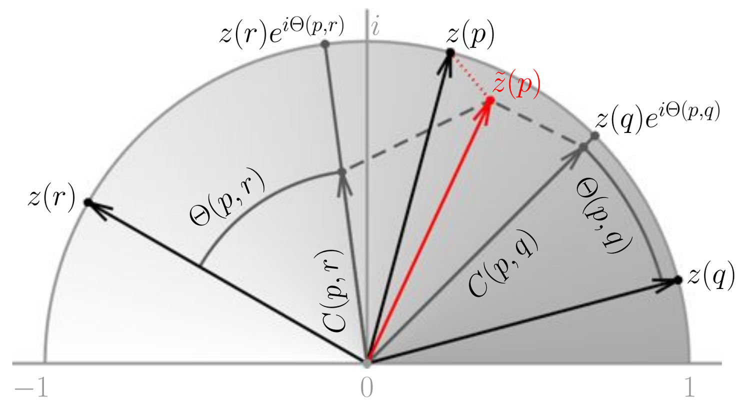
Let (C, Θ, U) define a constrained Angular Embedding (AE) problem by specifying relationships between n graph nodes:

Θ: $n \times n$ pairwise relative ordering matrix

C: $n \times n$ pairwise confidence matrix

U: $n \times u$ matrix of u linear constraints

AE recovers global ordering $\theta(p)$ by embedding: $p \to z = e^{i\theta(p)}$



Minimize: $\varepsilon = \sum_p \frac{\sum_q C(p,q)}{\sum_{p,q} C(p,q)} \cdot |z(p) - \tilde{z}(p)|^2$ (s.t. $U^* z = 0$)

Relax to generalized eigenproblem $QPQz = \lambda z$ where:

$$P = D^{-1}W$$

$$Q = I - D^{-1}U(U^T D^{-1} U)^{-1} U^T$$

with: $D = \mathrm{diag}(C\mathbf{1})$ and $W = C \bullet \exp(i\Theta)$

Multiscale:

- Upgrade C, Θ, U to arrays $\mathbf{C}, \mathbf{\Theta}, \mathbf{U}$, indexed by level s
- Pairwise relationships restricted to be within-level:
  - n_s is the # of nodes in level s
  - $\tilde{n}_s = \sum_{\hat{s} \geq s} n_{\hat{s}}$ is the cumulative # of nodes in levels s and coarser
  - $\mathbf{C}_s$, $\mathbf{\Theta}_s$ are $n_s \times n_s$ matrices
- Constraints organized into incremental sets:
  - u_s is the # of additional constraints at level s
  - $\tilde{u}_s = \sum_{\hat{s} \geq s} u_{\hat{s}}$ is the cumulative # of active constrains at level s
  - $\mathbf{U}_s$ is an $\tilde{n}_s \times u_s$ matrix
  - $\mathbf{U}_s$ involves only nodes appearing at levels s and coarser
- Extract problem on sub-pyramid:
  - $C_s \leftarrow \mathrm{Diag}(\mathbf{C}_{s_{\max}}, \ldots, \mathbf{C}_s)$
  - $\Theta_s \leftarrow \mathrm{Diag}(\mathbf{\Theta}_{s_{\max}}, \ldots, \mathbf{\Theta}_s)$
  - $U_s \leftarrow [\mathbf{U}_{s_{\max}}; \ldots; \mathbf{U}_s]$
  - Compute D_s, W_s from C_s, Θ_s
  - Define $M_s = Q_s P_s Q_s$
- Leading eigenvectors of M_0 solve the full multiscale problem

Multigrid: solve M_s to assist solution of M_{s-1}

Eigensolver using Randomized Matrix Approximation

Fixed Rank Problem:

Given: $n \times n$ sparse matrix M

Find: $n \times l$ dense matrix A, where $l = 2m \ll n$

Such that: range of A approximates range of M

Randomized Subspace Iteration [5]:

Draw $n \times l$ Gaussian matrix Ω

$Y \leftarrow (MM^*)^q M\Omega$ *increase q until convergence*†

$A \leftarrow \text{QR-ORTHONORMALIZE}(Y)$

Coarse-to-Fine Interpolation:

Find A_1 approximating M_1 via subspace iteration

Write A approximating M_0 as: $A = [A_1;\ A_0]$

Look at constraint: $U^* A = 0$

Rewrite as: $[U_{[n_1]};\ U_{[n_0]}]^* [A_1;\ A_0] = 0$

Least-squares interpolate: $\widetilde{A}_0 = U_{[n_0]}(U_{[n_0]}^* U_{[n_0]})^{-1}(-U_{[n_1]}^* A_1)$

Use $\widetilde{A} = [A_1;\ \widetilde{A}_0]$ as starting guess in subspace iteration for A

Eigensolver:

$B \leftarrow A^* M A$ *$l \times l$ matrix*

$(V, \Lambda) \leftarrow \text{EIGS}(B, m)$ *small eigenproblem*

$Z \leftarrow AV$ *m leading eigenvectors*

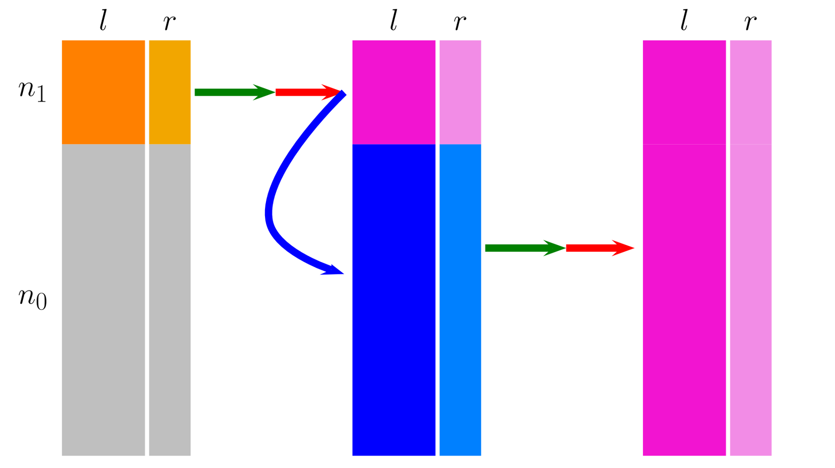
Convergence Check:

Evolve two independent bases $\hat{A}$ and $\check{A}$ of sizes l and r

Check whether the l space contains the r vectors: $E \leftarrow \check{A} - \hat{A}\hat{A}^*\check{A}$

Return reconstruction error: $\tau = \max_{j=0,\ldots,r-1} \left\| E_{[0:(n-1),\ j]} \right\|$

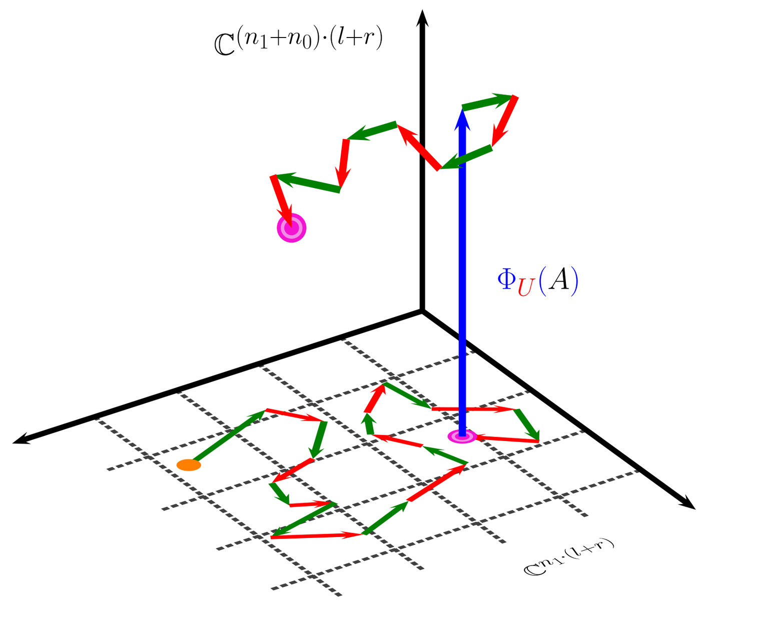
† Avoid explicit computation of M. See paper for details.



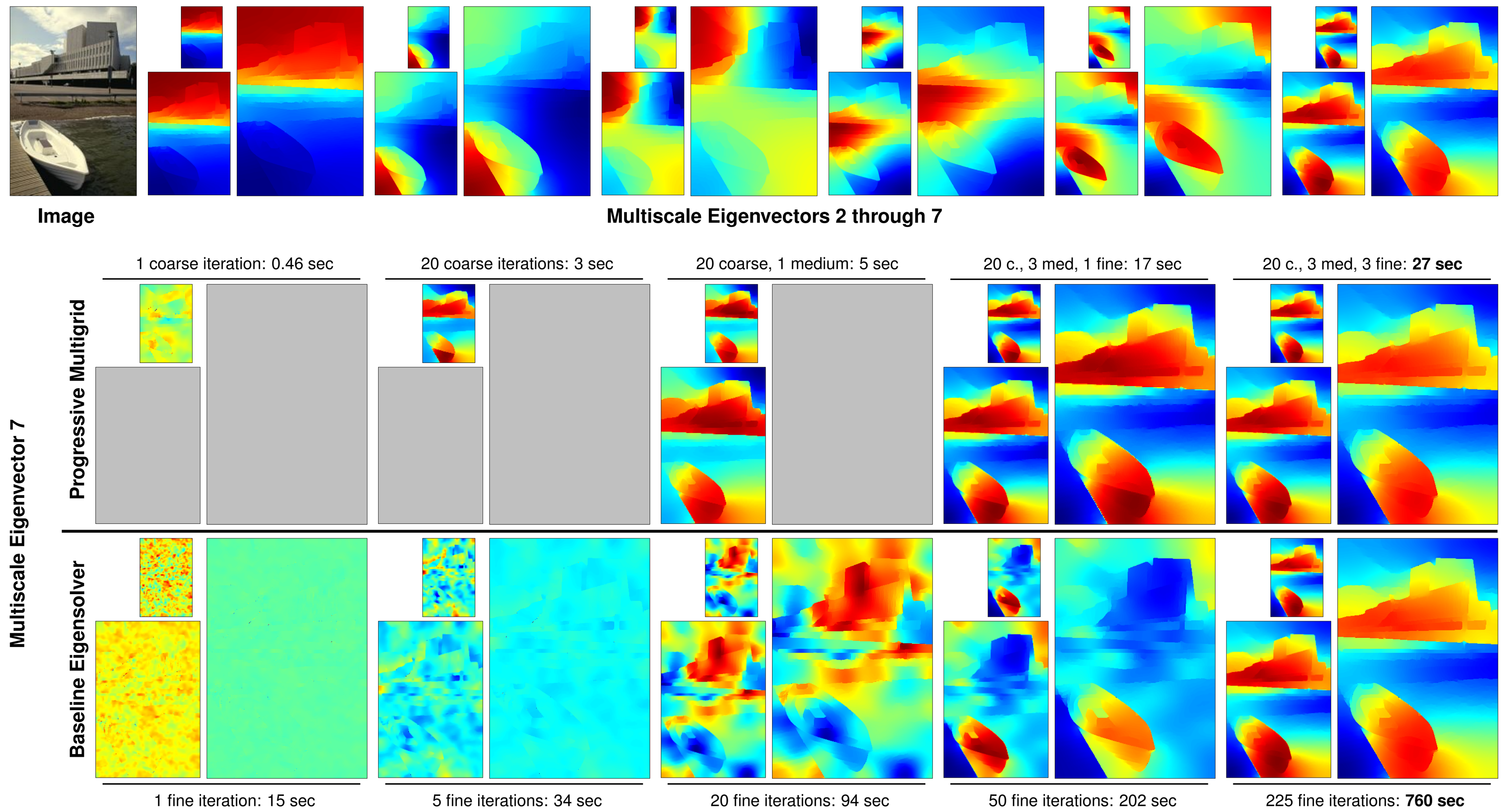
Diffuse: $A = D^{-\frac{1}{2}} W D^{-\frac{1}{2}} A$

Project: $A = (I - D^{-\frac{1}{2}} U (U^* D^{-1} U)^{-1} U^* D^{-\frac{1}{2}}) A$

Interpolate: $A = [A;\ D_{[n_0]}^{\frac{1}{2}} U_{[n_0]} (U_{[n_0]}^* U_{[n_0]})^{-1} (-U_{[n_1]}^* D_{[n_1]}^{-\frac{1}{2}} A)]$

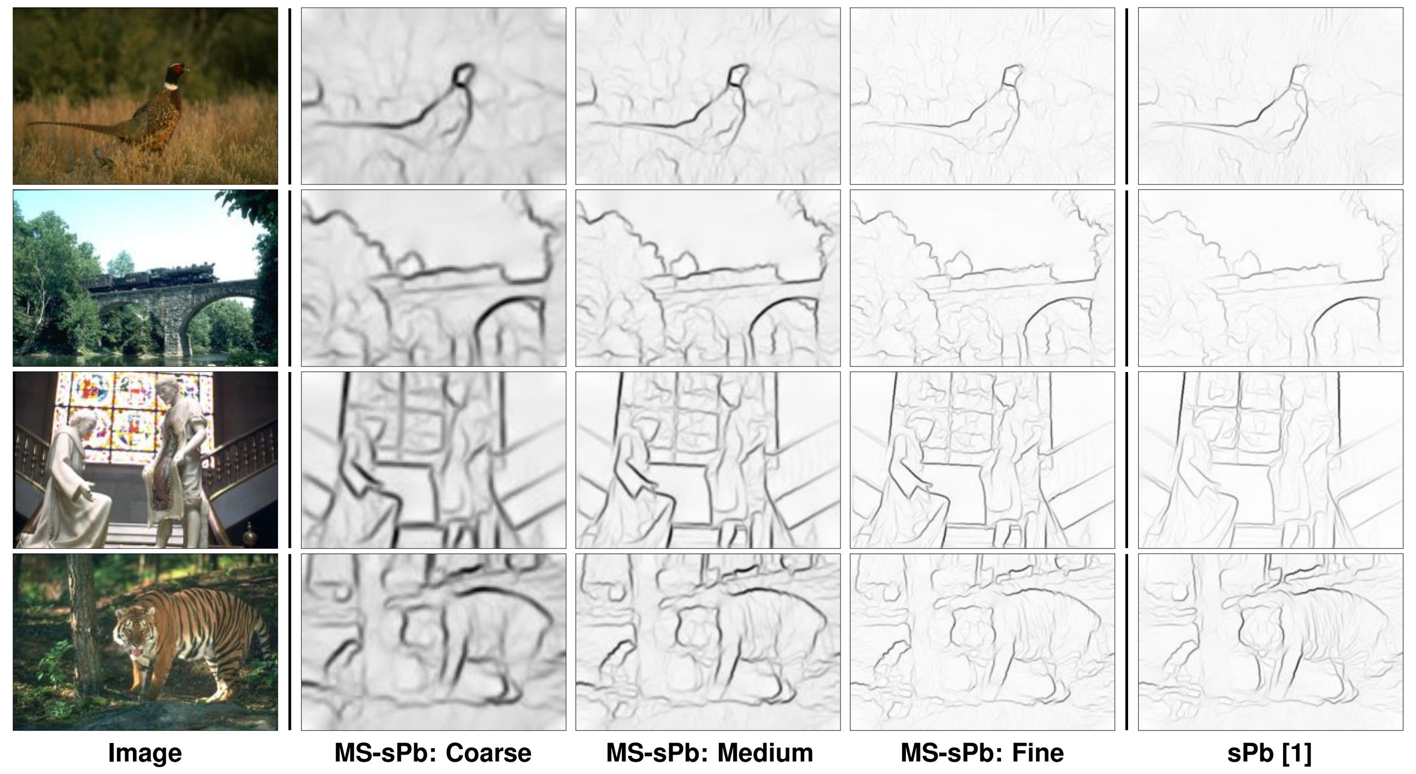


Eigenvector Convergence Comparison



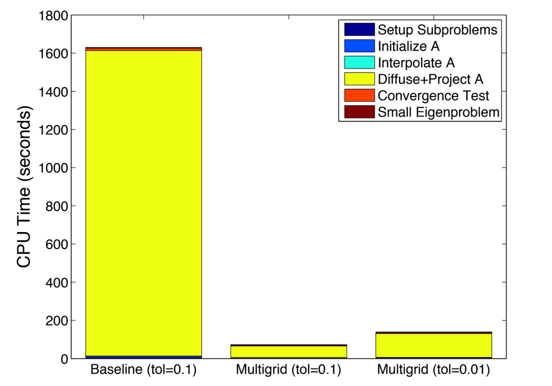
Our solver processes coarse-to-fine sub-pyramids, converging far faster (27 sec vs 760 sec) than the baseline solver, which starts work on the finest pyramid.

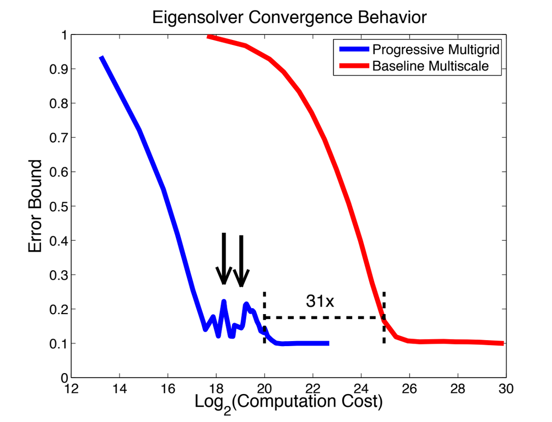
Multiscale Spectral Pb



Our eigenvectors live on an image pyramid and produce consistent coarse-to-fine boundaries across scale-space.

Runtime





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