Optimal Diversity Allocation in Multiuser Communication Systems—Part I: System Model

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***Abstract*—A class of multiuser multicarrier communication systems is introduced to study the influence of resource allocation on the performance of multiuser communication systems operating over fading channels. This class of systems includes both systems that employ exclusive allocation schemes, where users are allotted time-bandwidth slots without interference from other users, and systems that employ shared allocation schemes, where users are allotted time-bandwidth slots that are also employed by other users. The optimal weighting factors used in the combining of the received signals from the slots of a single user for the conventional receiver is derived, and the performance of systems in the class is characterized. For each of a number of popular multiuser architectures, it is shown that there exists a system in the class with nearly identical performance. Based on these relations, it is concluded that a class of systems has been introduced that allows the study of the merits of different types of time-bandwidth allocation under a single framework.**

***Index Terms*— DS-CDMA, FH-CDMA, multicarrier CDMA, multipath fading channels, multiuser communications.**

I. Introduction

THIS two-part paper is motivated by the desire to find a single framework that encompasses a number of multiuser wireless communication system architectures; such a framework would conceivably allow an equitable comparison of the included architectures.

Flexible universal communications is the ultimate goal of modern communication systems. One of the critical components of such a system is a wireless communications component that supports multiple users in a given physical area over the multipath fading channel. However, a universally agreed upon method of resource allocation for these users remains undecided, as evidenced by the competing methodologies being proffered in the cellular telephone market: exclusive time-bandwidth allocation in the form of time-division multiple access for the Global System for Mobile Communications and maximum shared time-bandwidth allocation in the form of direct-sequence code-division multiple access (DS/CDMA) for the IS-95 system.

Because of the critical nature of this problem, optimal system specification for wireless systems has been considered in the research community. However, when comparing various allocation schemes, comparisons have generally been done qualitatively by comparing attributes of the signals for each of the types of resource allocation to argue their suitability for a given wireless environment [1], [2]. Quantitative comparisons have been made between frequency-hopped CDMA (FH/CDMA) and DS/CDMA (e.g., [3]–[5]), but the considered systems have not included exclusive resource allocation schemes and lack the single system hardware architecture as considered here.

The proposed class of communication systems is developed from reasonable assumptions on multiuser communication systems operating over frequency-selective fading channels with independently faded users. Choosing a system in the class is a decision on which type of time-bandwidth allocation is preferable. To make the relation of systems in the class to commonly implemented systems more concrete, the mathematical relation between systems in the class and representative systems of each type of time-bandwidth allocation is demonstrated in this paper. The representative of systems employing exclusive time-bandwidth allocation is an FH/CDMA system with users employing orthogonal hopping patterns. The representative of the maximum shared resource allocation systems is DS/CDMA. It will also be demonstrated that hybrid systems, systems that are a combination of the tenets of DS and FH/CDMA, fit under the class as well, as do the multicarrier (MC) DS/CDMA (MC/DS/CDMA) systems proposed in [6] and [7].

In Part II [8], optimization over the class of systems is presented for a fixed number of users per unit bandwidth. In a system where the only interference not due to system users is additive white Gaussian noise (AWGN), it will be observed that either an exclusive allocation scheme or a maximum shared allocation scheme is optimal for all signal-to-noise ratios (SNR's) and system user densities. It will then be demonstrated that systems that employ exclusive allocation schemes are preferable to maximum resource sharing schemes for any reasonable SNR and user density in a single-cell environment. Next, optimization is performed in the presence of partial-band interference, where it is observed that the presence of partial-band interference enlarges the set of channel

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conditions where the maximum resource sharing scheme is optimal, especially when the probability of a particular time-bandwidth slot experiencing interference is high. We conclude that by stripping away many of the implementation issues and characterizing the performance of a large number of systems under a single framework with equitable receiver assumptions, not only can we optimize over a large number of systems but also make a comparison that is reasonably fair.

The organization of Part I of this paper is as follows. Section II develops the proposed class of systems by developing a coded MC system from the characteristics of the frequency-selective fading channel. Section III presents the conventional receiver for systems in the class and the characterization of the performance of the conventional receiver. Section IV demonstrates that the proposed class contains systems that are mathematically equivalent to representative systems of various types of time-bandwidth allocation. Finally, Section V presents the conclusions and continuing work.

II. Development of the Class of MC Systems

A. Fading Channel Assumptions

In this work, a zero-mean frequency-selective time-nonselective fading channel is assumed that fits the Gaussian wide-sense stationary uncorrelated scattering model presented in [9]. It will be assumed that the receiver is unable to resolve specific paths of the channel, thus making the channel response $h_k(\tau)$ of user k a zero-mean Gaussian random process with autocorrelation function $E[h_k(\tau_i)h_k^*(\tau_j)] = \phi_c(\tau_i)\delta(\tau_i - \tau_j)$, where $\phi_c(\tau)$, the multipath intensity profile, is the expected power of $h_k(\tau)$, $\delta(x)$ is the Dirac delta function, and x^* denotes the complex conjugate of x. It will be assumed that $\phi_c(\tau)$ is the same for all users and is nonzero if and only if $\tau \in [0, \tau_{\max})$.

The communications link in a cellular system from the users to the base station is generally considered the limiting link [10] and will be the link that is addressed here. For the reverse link, the fading processes of different users can be modeled as independent. Although coherent communication presents a challenging implementation on the reverse link due to the need for phase acquisition, coherent communication is not only plausible [10, p. 86] but preferable [11]–[13].

B. MC Modulation and Coding Framework

For a signal bandwidth sufficiently smaller than the coherence frequency $(\Delta f)_c \triangleq (1/\tau_{\max})$ [14, p. 708] of the channel, the Fourier transform of $h_k(\tau)$ can be modeled as approximately constant across the signaling band, and thus only gain equalization is required. Define a subchannel as a communications channel with bandwidth B over which the frequency response of the channel can be modeled as approximately constant. The bandwidth B and the pulse shape $p(t)$ employed fix a maximum rate $1/T_s$ at which symbols can be sent on each subchannel. The aggregate of the large number of subchannels in the total system bandwidth W defines an MC system. Thus, the MC system breaks the available time-frequency plane into a number of slots, as shown on Fig. 1.

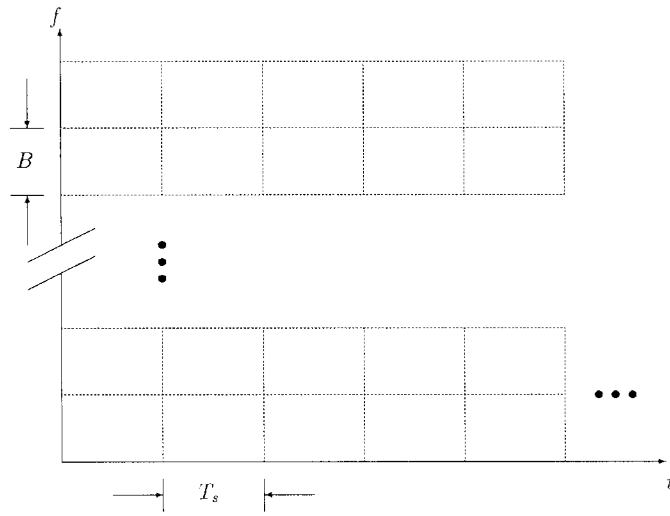

Fig. 1. Time and frequency slot arrangement from the MC approach.

Uncoded systems employing narrow-band signals do not perform well in multipath fading, and thus the need for coding in MC systems has been widely recognized [15]–[17]. Since $B \ll (\Delta f)_c$, adjacent slots in frequency will exhibit correlated fading. Thus, if the code symbols of a codeword are placed across adjacent slots, the effective channel that the encoder–decoder pair operates over will exhibit memory. Although channels with memory have capacity at least as large as their memoryless counterparts [18], most error control codes are designed for channels with independent fading on the channel symbols of a given codeword. Thus, it will be assumed that each user interleaves the transmitted symbols to a depth which achieves independent fading on the code symbols of a given codeword. The ratio of the total bandwidth W of the system and frequency coherence $(\Delta f)_c$ of the channel yields the maximum amount of diversity N, which is the maximum length for the perfectly interleaved codeword of a given user.[1]

Since the possibility of shared slots will be allowed, some form of simple user separation is necessary. Thus, it will be assumed that each user employs repetition coding of rate $1/L$ followed by randomly generated but known binary scrambling. In other words, a data bit of user k will be replicated L times, and the lth replica multiplied by $a_{k,l}$, a binary random variable that is equally likely to be $+1$ or -1. Each of the L resulting symbols is transmitted on an independent slot; thus, each user achieves Lth-order diversity to mitigate the multipath fading. This yields an equivalent lowpass transmitted waveform of $s_k(t) = \Sigma_{l=0}^{L-1} s_{k,l}(t) = \Sigma_{l=0}^{L-1} b_k(t)\sqrt{2P_c}a_{k,l}e^{j\omega_l t}$, where $b_k(t) = \Sigma_{i=-\infty}^{\infty} b_k^i p(t - iT_s)$ for a synchronous system, $b_k^i \in \{-1, +1\}$ corresponds to the ith data bit of user k, ω_l is the frequency of the lth subchannel, P_c is the power of each user per subchannel, and $p(t)$ is an arbitrary unit-energy pulse shape. Thus, the transmitter is sending the waveform $s_{k,l}(t) = b_k(t)\sqrt{2P_c}a_{k,l}$ on the subchannel corresponding to frequency ω_l. By the definition of the subchannel bandwidth, each

[1] For clarity of exposition, the channel is assumed to be time-nonselective. For a channel exhibiting time-selectivity, the maximum amount of diversity will be a function of the coherence time of the channel and the allowable delay as well.

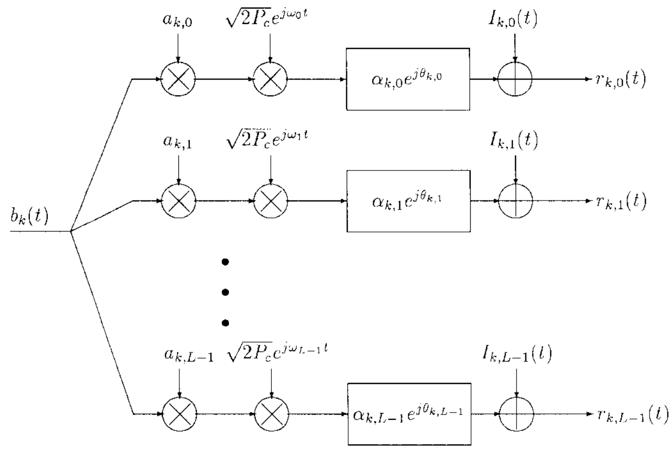

Fig. 2. Equivalent lowpass transmitter and channel for user k.

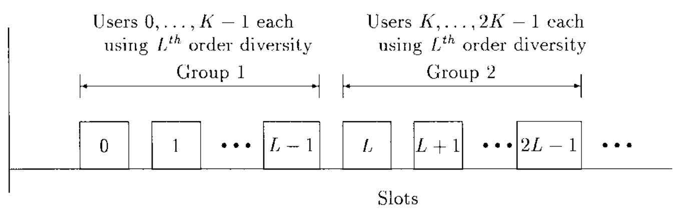

Fig. 3. Distribution of users over independently faded slots. Note that the horizontal axis is *not* frequency.

subchannel is faded nonselectively, and thus the equivalent lowpass transmitter and channel can be represented, as shown in Fig. 2. The fading of the lth subchannel of user k is given by $\alpha_{k,l}e^{j\theta_{k,l}}$, where $\alpha_{k,l}$ is Rayleigh distributed, and $\theta_{k,l}$ is uniformly distributed. The AWGN and multiple-access interference that user k sees on subchannel l is contained in $I_{k,l}(t)$, and $r_{k,l}(t)$ is the received signal on the lth subchannel of user k.

The L independent slots a given user occupies will be denoted a group. In a multiuser system, however, it is not clear that the L slots of a single group should be dedicated exclusively to one user. For this reason, each group will contain $K \geq 1$ users, *each signaling over all L slots in the group*. Fig. 3 displays a picture of the division of users across slots. The proposed class of MC systems consists of all systems described above such that $L \in \{1, \cdots, N\}$ and $K \in \{1, \cdots, \infty\}$.

As noted above, it will be assumed that the system is using random scrambling sequences. Although better performance can be obtained in a multiuser environment where the users are equally faded by using sequences that are orthogonal to those of other users [19], the work here will be concerned with the mobile-to-base link where the independent Rayleigh fading between users randomly changes the phase of the spreading sequences on each slot. However, the systems introduced here enjoy the same eigenfunction interpretation that is exploited directly by the simple gain equalization in [19].

C. Qualitative Interpretation

The total number of users per N slots is given as the product of the number of users per group and the number of

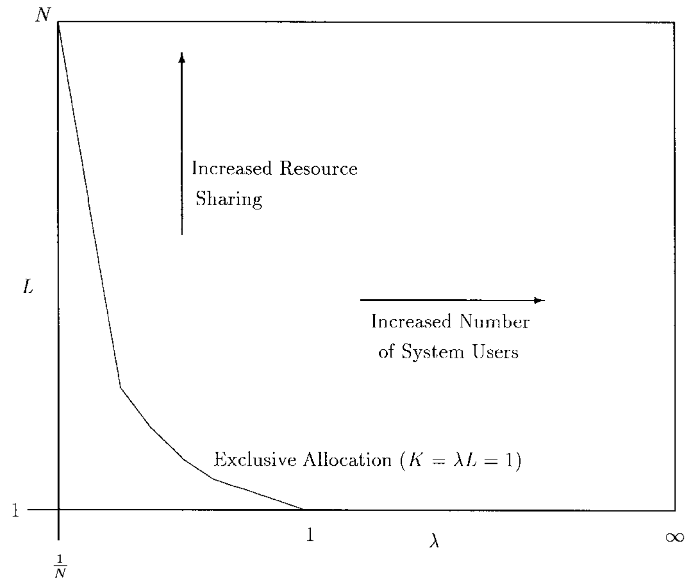

Fig. 4. Interpretation of the class of systems.

groups as $K(N/L) = \lambda N$, where the user density is defined as $\lambda \triangleq (K/L)$ users/slot. The total number of users in the system is proportional to the user density λ, yielding the interpretation shown in Fig. 4. The curve in the lower left-hand corner indicates the exclusive allocation systems that occur when $K = \lambda L = 1$. Moving upward in the plane increases the diversity that each user is employing, but since the system user density is fixed, this implies that each slot is shared with an increasing number of users. Moving right on the plane keeps the diversity that each user employs constant, but the system user density increases, thereby increasing the number of users in the system but reducing the performance of each user.

III. Performance

Although joint decoding of other users' signals can provide improved performance for systems in this MC framework [20], attention is restricted here to the conventional receiver. The conventional receiver will be defined as implementing the optimal decision on the matched filter output of a given user without knowledge of the other users' data bits, matched filter outputs, timing, or spreading waveforms. It will be established that maximum-ratio combining of the matched filter outputs results in the optimal conventional receiver for both synchronous and asynchronous systems. Following the derivation of the optimal combining factors, the performance of the conventional receiver for systems within the class is characterized. Throughout this paper, perfect fading estimation, timing estimation, and carrier acquisition will be assumed.

A. Receiver

Recall that perfect interleaving and independent fading between users is assumed; in other words, $\{\alpha_{k,l}, k = 0, \cdots, K-1, l = 0, \cdots, L-1\}$ is a set of mutually independent Rayleigh random variables, and $\Theta \triangleq \{\theta_{k,l} : k = 1, \cdots, K-1, l = 0, \cdots, L-1\}$ is a set of mutually independent random variables

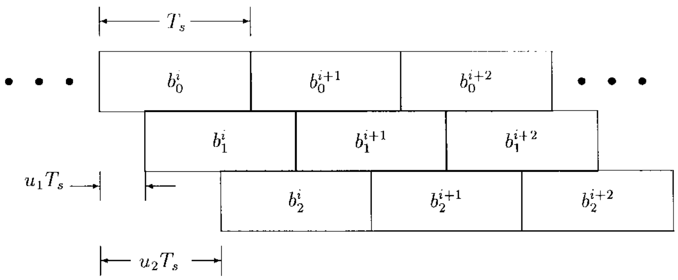

Fig. 5. Asynchronous system bit timings.

that are uniformly distributed on $[0, 2\pi]$. In general, the received signal is given by $r(t) = \Sigma_{k=0}^{K-1} \Sigma_{l=0}^{L-1} \alpha_{k,l} e^{j\theta_{k,l}} s_{k,l}(t - u_k T_s) + \Sigma_{l=0}^{L-1} \eta_l(t)$, where $\eta_l(t)$ is the lowpass equivalent of a white Gaussian noise process with two-sided power spectral density $N_0/2$, and $u_k T_s$ is the time between the start of a data bit of user 0 (the desired user) and that of user k, per Fig. 5. For the synchronous case, each of the $u_k, k = 1, \cdots, K-1$ will be assumed to be equal to zero. For the asynchronous case, $\mathcal{T} \triangleq \{u_k : k = 1, \cdots, K-1\}$ will be assumed to be an independently, identically distributed set of uniform random variables on [0, 1].

Without loss of generality, consider the decoding of bit 0 of user 0. Assuming both that the pulse shape $p(t)$ is chosen such that it has negligible energy outside $[0, T_s]$ and that perfect slot separation is achieved, the lowpass equivalent of the received signal from the lth subchannel for $t \in (0, T_s)$ is given by

$$r_{0,l}(t) = \sqrt{2P_c} \sum_{k=0}^{K-1} a_{k,l} \alpha_{k,l} e^{j(\omega_l (t - u_k T_s) + \theta_{k,l})} \cdot \left(b_k^0 p(t - u_k T_s) + b_k^{-1} p(t + T_s - u_k T_s)\right) + \eta_l(t).$$

For the conventional receiver, user 0 performs downconversion and matched filtering on the lth slot, as shown in Fig. 6, to obtain $\underline{r}_0^0 = [r_{0,0}^0 \, r_{0,1}^0 \cdots r_{0,L-1}^0]$, where

$$r_{0,l}^0 = \mathrm{Re}\left[\int_0^{T_s} a_{0,l} \sqrt{\frac{2}{T_s}} e^{j(\omega_l t + \theta_{0,l})} p(t) r_{0,l}(t)\, dt\right] \tag{1}$$

$$\begin{aligned} &= \sqrt{E_c} b_0^0 \alpha_{0,l} + \sqrt{E_c} \sum_{k=1}^{K-1} a_{0,l} a_{k,l} \alpha_{k,l} \\ &\quad \cdot \left(b_k^{-1} \int_0^{u_k} \hat{p}(1 - u_k + s) \hat{p}(s)\, ds \right. \\ &\quad \left. + b_k^0 \int_{u_k}^1 \hat{p}(s - u_k) \hat{p}(s) ds \right) \\ &\quad \cdot \cos(\omega_l u_k T_s + \theta_{0,l} - \theta_{k,l}) + \eta_l \end{aligned} \tag{2}$$

$\mathrm{Re}(\cdot)$ denotes the real part of a complex argument, $E_c \triangleq P_c T_s$ is the transmitted energy per user per slot per symbol period, η_l is a Gaussian random variable with mean 0 and variance $N_0/2$, and $\hat{p}(x) = p(x T_s)$ is a normalized version of the pulse shape.

B. Combining Factors

In the conventional receiver, user 0 seeks to make a decision on the value of b_0^0 based on the (perfect) fading estimates $\Upsilon_0 \triangleq \{\alpha_{0,l}, l = 0, 1, \cdots, L-1\}$ and the vector of observations

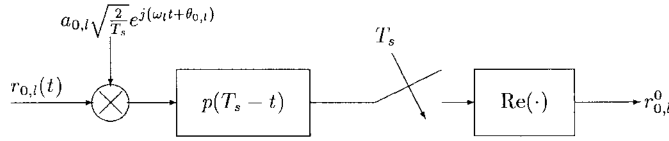

Fig. 6. Correlator demodulation on slot l.

$\underline{r}_0^0$ with knowledge only of the statistics of the fading of the other users and that the fading of the other users is independent of his/her fading. Because $p(\underline{r}_0^0 | b_0^0 = +1, \Upsilon_0)$ is not the density function of a vector of jointly Gaussian random variables in the general asynchronous case, the optimal combining factors are not obvious. Assume temporarily that user 0 also has knowledge of $\mathcal{T}$ and $\mathcal{B} \triangleq \{b_k^{-1}, b_k^0 : k = 1, \cdots, K-1\}$; the optimal decision is then based on the likelihood ratio [21]

$$\Lambda_0^0 = \frac{p\left(\underline{r}_0^0 | b_0^0 = +1, \Upsilon_0, \mathcal{T}, \mathcal{B}\right)}{p\left(\underline{r}_0^0 | b_0^0 = -1, \Upsilon_0, \mathcal{T}, \mathcal{B}\right)}.$$

Observe that $p\left(\underline{r}_0^0 | b_0^0 = +1, \Upsilon_0, \mathcal{T}, \mathcal{B}\right)$ and $p(\underline{r}_0^0 | b_0^0 = -1, \Upsilon_0, \mathcal{T}, \mathcal{B})$ are the probability density functions of Gaussian random vectors with independent components and parameters

$$E\left[r_{0,l}^0 | b_0^0 = +1, \Upsilon_0, \mathcal{T}, \mathcal{B}\right] = \sqrt{E_c} \alpha_{0,l}$$
$$E\left[r_{0,l}^0 | b_0^0 = -1, \Upsilon_0, \mathcal{T}, \mathcal{B}\right] = -\sqrt{E_c} \alpha_{0,l}$$

and

$$\begin{aligned} &\mathrm{var}\left[r_{0,l}^0 | b_0^0 = +1, \Upsilon_0, \mathcal{T}, \mathcal{B}\right] \\ &\quad = \mathrm{var}\left[r_{0,l}^0 | b_0^0 = -1, \Upsilon_0, \mathcal{T}, \mathcal{B}\right] \\ &\quad = E_c \frac{\mu_f^2}{2} \sum_{k=1}^{K-1} \left(b_k^{-1} \int_0^{u_k} \hat{p}(1 - u_k + s) \hat{p}(s)\, ds \right. \\ &\qquad \left. + b_k^0 \int_{u_k}^1 \hat{p}(s - u_k) \hat{p}(s)\, ds \right)^2 + \frac{N_0}{2} \end{aligned}$$

where $\mu_f^2 = E[\alpha_{k,l}^2]$. Let

$$\sigma^2(\mathcal{T}, \mathcal{B}) \triangleq E_c \frac{\mu_f^2}{2} \sum_{k=1}^{K-1} \left(b_k^{-1} \int_0^{u_k} \hat{p}(1 - u_k + s) \hat{p}(s)\, ds + b_k^0 \int_{u_k}^1 \hat{p}(s - u_k) \hat{p}(s)\, ds \right)^2 + \frac{N_0}{2}.$$

Note that $\sigma^2(\mathcal{T}, \mathcal{B})$ is independent of l. Thus,

$$\ln \Lambda_0^0 = \frac{4\sqrt{E_c} \sum_{l=0}^{L-1} r_{0,l}^0 \alpha_{0,l}}{2\sigma^2(\mathcal{T}, \mathcal{B})}.$$

A sufficient statistic for the decision on b_0^0 is $\Sigma_{l=0}^{L-1} r_{0,l}^0 \alpha_{0,l}$. Since this combining is optimal with the extra knowledge of $\mathcal{T}$ and $\mathcal{B}$ and does not depend on $\mathcal{T}$ and $\mathcal{B}$, it must also be optimal when the receiver only knows the fading variables of the desired user.

C. Receiver Performance

The derivation in this section is similar to the derivation found in [14, p. 723] for the single-user case. Using maximum-ratio combining, the decision variable for bit 0 of user 0 is given by

$$\begin{aligned} y_0^0 &= \sum_{l=0}^{L-1} r_{0,l}^0 \alpha_{0,l} \\ &= \sqrt{E_c} \sum_{l=0}^{L-1} \alpha_{0,l}^2 b_0^0 + \sum_{l=0}^{L-1} a_{0,l}\alpha_{0,l}\sqrt{E_c} \\ &\quad \cdot \sum_{k=1}^{K-1} \alpha_{k,l} a_{k,l} \cos(\omega_l u_k T_s + \theta_{0,l} - \theta_{k,l}) \\ &\quad \cdot \Bigg(b_k^{-1} \int_0^{u_k} \hat{p}(1-u_k+s)\hat{p}(s)\, ds \\ &\qquad + b_k^0 \int_{u_k}^1 \hat{p}(s-u_k)\hat{p}(s)\, ds \Bigg) + \sum_{l=0}^{L-1} \alpha_{0,l}\eta_l \end{aligned} \tag{3}$$

and the optimal receiver performs a zero-threshold decision on y_0^0.

Let the total fading of the first user be given by $F_L = \Sigma_{l=0}^{L-1} \alpha_{0,l}^2$, and the received signal-to-background noise ratio be denoted by $\Gamma \triangleq (E_s/N_0)\mu_f^2$ where $E_s = LE_c$. The probability of error will be found by approximating y_0^0 as Gaussian when conditioned on Υ_0 and b_0^0; the resulting expression will be exact in the synchronous case but only an approximation in the general asynchronous case. Realizing that F_L is chi-square with $2L$ degrees of freedom, the bit-error probability for the conventional receiver is approximated as

$$\begin{aligned} P_e^a(K,L,\Gamma) &\approx \tilde{P}_e^a(K,L,\Gamma) \\ &= \int_0^\infty Q(\sqrt{2\overline{\gamma}_a(K,L,\Gamma)f})\, \frac{f^{L-1}}{(L-1)!}\, e^{-f}\, df \end{aligned} \tag{4}$$

where $Q(x) \triangleq \int_x^\infty (1/\sqrt{2\pi})e^{-(y^2/2)}\, dy$, $\overline{\gamma}_a(K,L,\Gamma) = (\Gamma/(\psi\Gamma(K-1)+L))$, and ψ depends on the shape of the signaling waveform as

$$\psi = E_{u_k}\left[\left(\int_0^{u_k} \hat{p}(1-u_k+s)\hat{p}(s)\, ds\right)^2 + \left(\int_{u_k}^1 \hat{p}(s-u_k)\hat{p}(s)\, ds\right)^2\right]. \tag{5}$$

For the synchronous system, the exact error probability is given as

$$P_e^s(K,L,\Gamma) = \int_0^\infty Q(\sqrt{2\overline{\gamma}_s(K,L,\Gamma)f})\, \frac{f^{L-1}e^{-f}}{(L-1)!}\, df \tag{6}$$

where the average signal-to-interference (SIR) per subchannel $\overline{\gamma}_s(K,L,\Gamma)$ is defined as

$$\overline{\gamma}_s(K,L,\Gamma) = \frac{\frac{E_s}{L}\mu_f^2}{(K-1)\frac{E_s}{L}\mu_f^2 + N_0} = \frac{\Gamma}{\Gamma(K-1)+L}. \tag{7}$$

The integral of (4) can be evaluated similarly to the single-user case of [14, p. 723] to yield a finite series with positive terms, which is useful for numerical evaluation. However, the result is not amenable to analytic optimization in the general case, and thus much of the optimization work in [8] and [22] is based directly on (4). Finally, an explicit relation between the approximation to the bit-error probability of the conventional receiver for the asynchronous system and the bit-error probability of the conventional receiver for the synchronous system is given by

$$\tilde{P}_e^a(K,L,\Gamma) = P_e^s(\psi K + 1 - \psi, L, \Gamma).$$

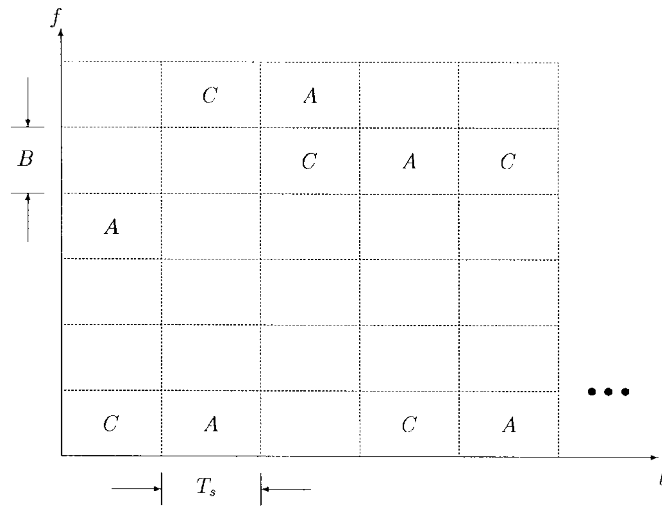

Fig. 7. Allocation of subchannels in an FH/CDMA system to users "A" and "C."

IV. Equivalences

A. Exclusive Allocation Schemes

A fast frequency-hopping system sends a data bit by dividing the energy across L different hops, as shown in Fig. 7. If the hopping patterns of all users are coordinated so that users do not hop onto the same subchannel at the same time, this is denoted "orthogonal hopping" and is a means of exclusive multiple access. If the amount of frequency diversity available in the system is high (i.e., the number of slots available for hopping is large and the coherence bandwidth is not large), it can be assumed that hops are faded independently, and each user achieves Lth-order diversity. This system is mathematically equivalent to a system in the class introduced here with $K = 1$ and the same diversity per user L.

B. Maximum Resource Sharing Schemes

In this section, the performance of schemes in the proposed class that employ maximum resource sharing is compared to DS/CDMA systems.

1) MC Systems: In this section, the performance of maximum resource sharing ($L = N$) systems in the proposed class is considered. Per Section III, the probability of error of such a system is given by $P_e^a(K,N,\Gamma) \approx E_{\hat{F}_N}[Q(\sqrt{2\overline{\gamma}_a(K,N,\Gamma)N\hat{F}_N})]$, where $\hat{F}_N \triangleq (F_N/N\mu_f^2)$. In general, a multiuser communication system has a large total bandwidth W available to it, and thus N is large. Thus, it is of interest to examine the performance of maximum resource sharing systems when N is large. To do this, the user density λ is fixed, and L is allowed to tend to infinity. The following proposition can be established.

Proposition 1: Consider a binary coherent MC system as described in Section II with asynchronous and independently faded users and parameter ψ derived from the pulse shape. Then, for any $\Gamma \in (0, \infty), \lambda > 0$, consider the following.

1)
$$\lim_{L\to\infty} P_e^a(\lambda L, L, \Gamma) = Q\left(\sqrt{\frac{2\Gamma}{\psi\Gamma\lambda + 1}}\right). \tag{8}$$

2) Let $z_0^0 \triangleq (y_0^0/\sqrt{L}\mu_f)$ be the normalized decision statistic, and assume $b_0^0 = +1$. Then, as $L \to \infty$,
$$z_0^0 \triangleq \frac{y_0^0}{\sqrt{L}\mu_f} \xrightarrow{\mathcal{D}} N\left(\sqrt{E_s\mu_f^2}, \frac{\lambda\mu_f^2 E_s\psi}{2} + \frac{N_0}{2}\right) \tag{9}$$
where the symbol $\mathcal{D}$ over a relation will imply the relation holds in distribution.

The proof of Proposition 1 is contained in Appendix A. Note that convergence for the general asynchronous system is shown without invoking the conditional Gaussian approximation. A special case of the proposition yields convergence in the synchronous case
$$\lim_{L\to\infty} P_e^s(\lambda L, L, \Gamma) = Q\left(\sqrt{\frac{2\Gamma}{\Gamma\lambda + 1}}\right). \tag{10}$$
Equations (8) and (10) show convergence of the MC system as the bandwidth becomes large to the Gaussian approximation to the bit-error probability of the conventional receiver in an unfaded DS/CDMA system with the same user density λ.

In the next section, maximum resource sharing systems in the proposed class will be compared to DS/CDMA. To perform this comparison for finite N requires the definition of a pulse shape so that the bandwidths of the two systems can be compared exactly. Thus, assume a finite bandwidth W and that the MC system employs the Nyquist pulse shape $\mathrm{sinc}(t/T_s)$. Although this is impractical, this gives an exact bandwidth definition that will be useful for the comparison below. The number of users per second per hertz is given by λ, and the bit-error probability of the conventional receiver in a synchronous system is given by
$$P_e^s(\lambda N, N, \Gamma) = E_{\hat{F}_N}\left[Q\left(\sqrt{\frac{2\Gamma\hat{F}_N}{\Gamma\left(\lambda - \frac{1}{N}\right) + 1}}\right)\right]. \tag{11}$$

2) DS/CDMA: The representative of the maximum resource sharing systems considered here will be DS/CDMA. In DS/CDMA, each of the users occupies the entire bandwidth all of the time by forming the wideband signal
$$x_k(t) = \sum_{i=-\infty}^{\infty} b_k^i \sum_{n=0}^{N_c-1} a_{k,n}^i c(t - nT_c - iT_s) \tag{12}$$
where $c(t)$ is the chip pulse shape, $a_{k,n}^i$ is the nth chip of the ith bit of the spreading sequence of user k, N_c is the processing gain, and $T_c = (T_s/N_c)$ is the chip period. Let $c(t)$ be the Nyquist pulse shape $\mathrm{sinc}(t/T_c)$ so that the DS/CDMA system can be compared to the MC system described above, under the same pulse-shape assumption.

Next, consider the performance of the DS/CDMA system over the frequency-selective fading channel when each user is employing random spreading sequences and the optimal single-user RAKE receiver [14]. Since this performance as derived in [22] was derived independently for a similar system and appeared in [7], the derivation is omitted. If the multipath delay spread is $\tau_{\max}$, which will be assumed to be less than T_s, the bit-error probability of the RAKE receiver under the conditional Gaussian assumption is given by
$$\tilde{P}_e^{ds}(K, N_c, \phi_c(\cdot)) = E_{\tilde{F}_L}\left[Q\left(\sqrt{\frac{2\tilde{\Gamma}_L\tilde{F}_L}{\frac{\tilde{\Gamma}_L}{N_c}(K-1) + 1}}\right)\right] \tag{13}$$
where $L = \lfloor \tau_{\max}W + 1\rfloor$
$$\tilde{F}_L \triangleq \frac{1}{L\tilde{\mu}_{f,L}^2} \sum_{l=0}^{L-1} \left|h_0\left(\frac{l}{N_c}T_s\right)\right|^2 \tag{14}$$
$$\tilde{\mu}_{f,L}^2 \triangleq \frac{1}{L} \sum_{l=0}^{L-1} E\left[\left|h_0\left(\frac{l}{N_c}\,T_s\right)\right|^2\right] = \frac{1}{L}\sum_{l=0}^{L-1} \phi_c\left(\frac{l}{N_c}\,T_s\right) \tag{15}$$
and the received SNR is given by $\tilde{\Gamma}_L \triangleq (E_s\tilde{\mu}_{f,L}^2/N_0)$.

It is instructive to compare the above result to the bit-error probability of the maximum resource sharing $(L = N)$ system from the MC framework with the same bandwidth $W = (N_c/T_s)$ and same user density λ on the fading channel. For the MC system, $L = (W/(\Delta f)_c) = W\tau_{\max}$, and is approximately the same as for the DS/CDMA system. Thus, the comparison can be done by comparing (11) with the substitution $\lambda = (K/N_c)$ to (13).

The performance of the two systems (for large K) is identical except that the MC system bit-error probability depends on the normalized fading variable $\hat{F}_L$, while the DS/CDMA system bit-error probability depends on the normalized fading variable $\tilde{F}_L$.[2] Thus, the desired equivalence of a system in the proposed class to a DS/CDMA system has not been established because identical fading variables comprise $\hat{F}_L$, while $\tilde{F}_L$ is composed of fading variables with unequal second moments unless the multipath intensity profile is rectangular. This appears to give the MC system slightly better performance, but this conclusion must be taken with caution. For the case that the multipath intensity profile is not rectangular, an MC system cannot achieve L perfectly independent and identically faded channels if it is operating on the same *contiguous* bandwidth as the DS/CDMA system; it can only do this by employing noncontiguous bandwidth. Thus, the goal of this section is not to establish the exact relation between a standard MC system and a DS/CDMA system, but instead to demonstrate that a

[2] This observation is noted independently in [7] for similar systems when comparing MC/DS/CDMA and DS/CDMA systems.

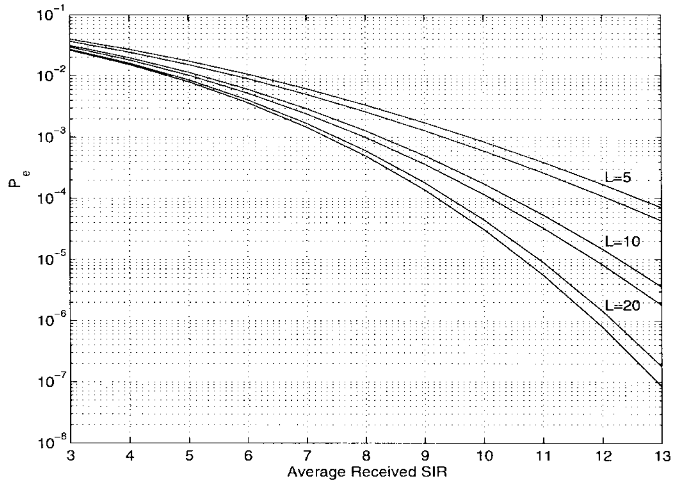

Fig. 8. Comparison of average probability of error (P_e) expressions for systems operating with instantaneous SIR's $\overline{S}\hat{F}_L$ and $\overline{S}\tilde{F}_L$, where $\overline{S}$ is the average received SIR. In each of the three groups, the top curve corresponds to the average DS/CDMA system error probability, which operates with instantaneous SIR $\overline{S}\tilde{F}_L$, and the lower curve corresponds to the average error probability of the proposed MC system, which operates with instantaneous SIR $\overline{S}\hat{F}_L$. For the curves corresponding to instantaneous SIR $\overline{S}\tilde{F}_L$, an exponentially decaying multipath intensity profile is considered. It is evident that for large L, the curves are nearly identical, particularly for large error probabilities.

DS/CDMA system fits into the *MC framework* presented in this paper. Toward this end, Fig. 8 shows the convergence of a DS/CDMA system to the MC framework presented in this paper. Note that as the bandwidth of the DS/CDMA system increases, the bit-error probability of the DS/CDMA system converges to that of the MC system; this observation is made rigorous by the following proposition.

Proposition 2: Consider a synchronous DS/CDMA system as described above with spreading factor N_c and independently faded users operating over a frequency-selective fading channel with additive white Gaussian background noise and bounded continuous multipath delay profile $\phi_c(\cdot)$, such that $\int_0^{\tau_{\max}} \phi_c(\tau)\, d\tau > 0$ and $\phi_c(\tau) = 0$, for $\tau > \tau_{\max}$. Then,

$$\lim_{N\to\infty} \tilde{P}_e^{ds}(\lambda N_c, N_c, \phi_c(\cdot)) = Q\left(\sqrt{\frac{2\Gamma}{\Gamma\lambda + 1}}\right) \tag{16}$$

where $\lambda = (K/N_c)$ and

$$\Gamma \triangleq \frac{E_s}{N_0} \int_0^{\tau_{\max}} \phi_c(\tau)\, d\tau. \tag{17}$$

The proof of Proposition 2 is contained in Appendix B. Note that the convergence in the above proposition can be shown without the Gaussian assumption on the interference. This can be accomplished with a technique similar to that used in the proof of Proposition 1. The proposition establishes the performance equality of systems in the proposed class to DS/CDMA systems as the bandwidth becomes large.

C. Hybrid Schemes

Although the framework presented here does not explicitly consider DS spreading within a slot, spreading can easily be incorporated as in the MC/DS/CDMA of [6] and [23]. This spreading would expand the bandwidth of each slot if the data rate were kept constant, but since it is assumed that the bandwidth B over which the channel can be assumed to be nonselectively faded is fixed, the spreading must necessarily mean that each user reduces his/her transmission rate by a factor equal to the time-domain spreading within a slot. Thus, assuming a spreading factor of N_c, each user must send N_c data streams (and thus appear as N_c users) to keep his/her data rate the same. If it is assumed that a given user employs orthogonal signals for these N_c streams on a given slot, the SIR on each slot for a synchronous system when there are K users sharing L slots is

$$\frac{\Gamma}{\Gamma\dfrac{(KN_c - N_c)}{N_c} + L} = \frac{\Gamma}{\Gamma(K-1) + L} \tag{18}$$

and thus it is equivalent (in an SIR and diversity sense) to a system in the proposed class. Furthermore, the spreading within a slot of the MC system of [6] with independent fading between slots is mathematically identical to a hybrid DS/FH system with independent hops.

V. Conclusions

In this paper, a class of systems has been introduced that can be used to compare the performance of different types of bandwidth allocation in coherent multiuser systems operating over fading channels. The class was motivated by the MC tenet of avoiding complex equalization by transmitting on narrowband subchannels. The desire for simple user separation at the receiver motivates the use of soft-decision repetition coding of independent subchannels. These two assumptions define the class of systems. It is clear from the introduction of the class of systems that it contains systems which employ both exclusive time-bandwidth allocation and completely shared time-bandwidth allocation. However, the main result of this paper is that a number of practically considered methods of multiple access fall under this framework, thus supporting the assumptions used in its derivation. This framework can be used to optimize between exclusive allocation systems (FH/CDMA with orthogonal hopping patterns), hybrid allocation systems (hybrid FH/DS/CDMA or MC/DS/CDMA), and shared allocation schemes (DS/CDMA), as shown in the sequel [8].

Appendix A
Proof of Proposition 1

It is sufficient to prove the second item. Recall

$$\begin{aligned} z_0^0 = {} & \sqrt{E_s\mu_f^2}\, \frac{1}{\mu_f^2 L} \sum_{l=0}^{L-1} \alpha_{0,l}^2 b_0^0 + \sqrt{E_s\mu_f^2}\, \frac{1}{\mu_f^2 L} \\ & \cdot \sum_{l=0}^{L-1} a_{0,l}\alpha_{0,l} \sum_{k=1}^{\lambda L-1} a_{k,l}\alpha_{k,l} \\ & \cdot \cos(\theta_{0,l} - \theta_{k,l} + \omega_l u_k T_s) \\ & \cdot \left(b_k^{-1} \int_0^{u_k} \hat{p}(1 - u_k + t)\hat{p}(t)\, dt \right. \\ & \left. + b_k^0 \int_{u_k}^1 \hat{p}(t - u_k)\hat{p}(t)\, dt \right) + \sum_{l=0}^{L-1} \alpha_{0,l}\eta_l. \end{aligned}$$

z_0^0 is Gaussian when conditioned on Υ_0, $\mathcal{B}$, and $\mathcal{T}$. Thus, see the defined equation, shown at the bottom of the page. By the strong law of large numbers [24, p. 260], $\hat{F}_L \to 1$ with probability 1

$$\frac{1}{\lambda L-1}\sum_{k=1}^{\lambda L-1}\left(b_k^{-1}\int_0^{u_k}\hat{p}(1-u_k+s)\hat{p}(s)\,ds + b_k^0\int_{u_k}^1\hat{p}(s-u_k)\hat{p}(s)\,ds\right)^2 \to \psi$$

with probability 1, and

$$G_L(x|\hat{F}_L,\mathcal{B},\mathcal{T}) \to Q\left(\frac{x-\sqrt{E_s\mu_f^2}}{\sqrt{\frac{\lambda E_s\mu_f^2\psi}{2}+\frac{N_0}{2}}}\right)$$

with probability 1. Thus, by the bounded convergence theorem [25, p. 214]

$$E_{\hat{F}_L,\mathcal{B},\mathcal{T}}[G_L(x|\hat{F}_L,\mathcal{B},\mathcal{T})] \to Q\left(\frac{x-\sqrt{E_s\mu_f^2}}{\sqrt{\frac{\lambda E_s\mu_f^2\psi}{2}+\frac{N_0}{2}}}\right).$$

Appendix B
Proof of Proposition 2

First, consider the convergence of $\tilde{F}_L$ as defined in (14). Unlike the fading variables for the MC system, the distributions of the fading variables change as N_c is incremented because the sampling times change. A limiting result for triangular arrays will be required. Let $X_{N_c,l} \triangleq |h_0((l/N_c)T_s)|^2$. Then, $\tilde{F}_L$ can be written as

$$\tilde{F}_L = 1 + \frac{\sqrt{\sum_{l=0}^{L-1}\mathrm{var}[X_{N_c,l}]}}{\sum_{l=0}^{L-1}E[X_{N_c,l}]}\frac{1}{\sqrt{\sum_{l=0}^{L-1}\mathrm{var}[X_{N_c,l}]}} \cdot \sum_{l=0}^{L-1}(X_{N_c,l}-E[X_{N_c,l}]). \quad (19)$$

The central limit theorem can be applied to the triangular array $\{X_{N_c,l}\}$ if it satisfies the Lindeberg condition [26, p. 326]; that is, $\forall\epsilon>0$

$$\lim_{N_c\to\infty}\frac{1}{\sum_{m=0}^{L-1}\mathrm{var}[X_{N_c,m}]}\sum_{l=0}^{L-1}\int_{|y-E[X_{N_c,l}]|\geq\epsilon\sqrt{\Sigma_{m=0}^{L-1}\mathrm{var}[X_{N_c,m}]}} \cdot (y-E[X_{N_c,l}])^2 p_{X_{N_c,l}}(y)\,dy = 0.$$

Note that $\mathrm{var}[X_{N_c,l}] = (\phi_c((l/N_c)T_s))^2$. Now consider any $\epsilon>0$. Since $\sqrt{\Sigma_{m=0}^{L-1}\mathrm{var}[X_{N_c,m}]} \to \infty$ as $N_c \to \infty$, and $\exists C$ such that $\forall N_c, \forall l$, $E[X_{N_c,l}] \leq C$ (since $\phi_c(\cdot)$ is bounded), it must be true that $\exists N_0$ such that $\forall N_c \geq N_0$, $\forall l$, $|y-E[X_{N_c,l}]| \geq \epsilon\sqrt{\Sigma_{l=0}^{L-1}\mathrm{var}[X_{N_c,l}]}$ only if $y-E[X_{N_c,l}] \geq \epsilon\sqrt{\Sigma_{l=0}^{L-1}\mathrm{var}[X_{N_c,l}]}$. Thus, convergence to zero must be shown for

$$\lim_{N_c\to\infty}\frac{e^{-1}}{\sum_{m=0}^{L-1}\left(\phi_c\left(\frac{m}{N_c}T_s\right)\right)^2}\sum_{l=0}^{L-1}\int_{u\geq\epsilon\sqrt{\Sigma_{m=0}^{L-1}\mathrm{var}[X_{N_c,m}]}} \frac{u^2}{E[X_{N_c,l}]}e^{-(u/E[X_{N_c,l}])}\,du. \quad (20)$$

Furthermore, since $E[X_{N_c,l}]<C, \forall N_c, \forall l$, $\exists u_0$ such that $\forall N_c, \forall l$, $\forall u\geq u_0$, $(1/E[X_{N_c,l}])e^{-(u/E[X_{N_c,l}])} \leq (1/2C)e^{(-u/2C)}$. Thus, the right-hand side of (20) can be upper bounded by (21), shown at the top of the next page. Recognizing $\lim_{L\to\infty}(1/L)\,\Sigma_{l=0}^{L-1}\,(\phi_c((l/N_c)T_s))^2$ as the Riemann integral of a bounded continuous function on a closed interval, the integral exists [27, p. 193], is finite and is nonzero by the assumption $\int_0^{\tau_{\max}}\phi_c(\tau)\,d\tau>0$, and thus (21) goes to zero as required. Thus, the central limit theorem holds for the triangular array $\{X_{N_c,l}\}$. The term in front of the triangular array in (19) can be written as

$$\frac{\frac{1}{\sqrt{L}}\sqrt{\frac{1}{L}\sum_{l=0}^{L-1}\left(\phi_c\left(\frac{l}{N_c}T_s\right)\right)^2}}{\frac{1}{L}\sum_{l=0}^{L-1}\phi_c\left(\frac{l}{N_c}T_s\right)}.$$

If the $1/\sqrt{L}$ in the numerator is ignored, both the numerator and denominator converge to finite constants as the Riemann integrals of bounded continuous functions, and the denomina-

$$G_L(x|\hat{F}_L,\mathcal{B},\mathcal{T}) \triangleq P\left\{z_0^0 \geq x|\hat{F}_L,\mathcal{B},\mathcal{T}\right\} = Q\left(\frac{x-\sqrt{E_s\mu_f^2}\hat{F}_L}{\sqrt{\frac{E_s\mu_f^2\hat{F}_L(\lambda L-1)}{2L}\frac{1}{\lambda L-1}\sum_{k=1}^{\lambda L-1}\left(b_k^{-1}\int_0^{u_k}\hat{p}(1-u_k+t)\hat{p}(t)\,dt+b_k^0\int_{u_k}^1\hat{p}(t-u_k)\hat{p}(t)\,dt\right)^2+\hat{F}_L\frac{N_0}{2}}}\right)$$

$$\lim_{N_c\to\infty} \frac{e^{-1}}{\sum_{m=0}^{L-1}\left(\phi_c\left(\frac{m}{N_c}\,T_s\right)\right)^2} \sum_{l=0}^{L-1} \int_{u\geq\epsilon\sqrt{\Sigma_{m=0}^{L-1}\ \mathrm{var}[X_{N_c,m}]}} \frac{u^2}{2C}\, e^{-(u/2C)}\, du$$
$$= \lim_{N_c\to\infty} \frac{e^{-1}}{\frac{1}{L}\sum_{m=0}^{L-1}\left(\phi_c\left(\frac{m}{N_c}\,T_s\right)\right)^2} e^{-(\epsilon/2C)\sqrt{\Sigma_{m=0}^{L-1}\ \mathrm{var}[X_{N_c,m}]}} \left(\epsilon^2 \sum_{m=0}^{L-1}\mathrm{var}[X_{N_c,m}] + 4C\epsilon\sqrt{\sum_{m=0}^{L-1}\mathrm{var}[X_{N_c,m}]} + 8C^2\right) \tag{21}$$

tor limit is nonzero. Thus, the factor $1/\sqrt{L}$ drives the entire second term of (19) to converge in distribution to zero, and $\tilde{F}_L \xrightarrow{\mathcal{D}} 1$. Also, $\tilde{\Gamma}_L \to \Gamma$. Thus,

$$\frac{2\tilde{\Gamma}_L\tilde{F}_L}{\frac{\tilde{\Gamma}_L}{N_c}(K-1)+1} \xrightarrow{\mathcal{D}} \frac{2\Gamma}{\Gamma\lambda+1}$$

and by the special mapping theorem [28, p. 31], $P_e^{ds}(\lambda N_c, N_c, \phi_c(\cdot)) \to Q(\sqrt{(2\Gamma/(\Gamma\lambda+1)})$.

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Dennis L. Goeckel (S’89–M’92), for a photograph and biography, see p. 855 of the June 1999 issue of this TRANSACTIONS.

Wayne E. Stark (S’77–M’82–SM’94–F’98), for a photograph and biography, see p. 1746 of the November 1999 issue of this TRANSACTIONS.