Asymptotic Performance of M-ary Orthogonal Signals in Worst Case Partial-Band Interference and Rayleigh Fading

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Abstract—In this paper, we show that for large M the symbol error probability of an orthogonal signal set transmitted over a channel with partial-band Gaussian interference is

\[
P_e = \lim_{M \to \infty} P_e(M) = \begin{cases} 
1 & E_b/N_0 < \ln 2 \\
\frac{\ln 2}{E_b/N_0} & E_b/N_0 > \ln 2
\end{cases}
\]

where \(E_b\) is the transmitted bit energy and \(N_0\) is the average power spectral density of the interference. This is in contrast to the additive white Gaussian noise channel which has asymptotic probability of error going to zero for \(E_b/N_0 > \ln 2\). We also show that for a Rayleigh fading channel for large M the symbol error probability is \(P_e = 1 - e^{-2(E_b/N_0)}\). Finally, we provide numerical computations of the minimum \(E_b/N_0\) required to achieve a symbol error probability of \(10^{-5}\) to illustrate the asymptotic behavior described above.

I. INTRODUCTION

It is well known that when communicating over a white Gaussian noise channel an M-ary orthogonal signal set achieves capacity in that provided \(E_b/N_0 > \ln 2\) the error probability can be made arbitrarily small [1]. It is also known [2] that provided codes of small enough rates are used the signal-to-noise ratio needed for reliable communication in the presence of a partial-band interferer is the same as that of

Our simulation results have demonstrated the performance characteristics of an interconnection network employing gateway adapters, and have been especially useful in understanding the effects of congestion on throughput as determined by local network traffic levels, local network bus access mechanism, gateway bus access priority, gateway buffering capacity, and use of local network error correction.

Fig. 8. Effects of eliminating LAN error correction: hyperchannel bus access protocols: \(\lambda = 3.5, \rho = 6\).

Fig. 9. Effects of eliminating LAN error correction: LDDI bus access protocols: \(\lambda = 3.5, \rho = 6\).

improvement with \(\rho = 1\), but is greater for lower gateway priorities.

On the other hand, eliminating LAN error correction can have an adverse effect on throughput as lost or damaged gateway packets cannot be retransmitted by the gateway and, consequently, the entire internet message is lost. Our simulation experiments indicate that collisions rarely occur when prioritized HYPERchannel is used, and eliminating error correction does not adversely affect throughput when using the prioritized HYPERchannel protocol. The detrimental effect of collisions on throughput when fair HYPERchannel is used can be observed in Figs. 7 and 8. With the fair HYPERchannel protocol gateway packets are frequently lost to collisions and, consequently, LAN error correction results in greater throughput than results from its omission. The fair LDDI protocol which avoids collisions and their adverse effects achieves higher throughput levels without LAN error correction than with error correction, as Fig. 9 demonstrates.

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white Gaussian noise. Thus, one might conjecture that $M$ orthogonal signals would also provide reliable communication in the presence of a partial-band interferer. In this paper, we show that for worst case partial-band Gaussian jamming the error probability approaches $2(\frac{E_b}{N_0}/N_t)$ for $E_b/N_t > 1$ in 2 and thus $M$-ary orthogonal signals do not achieve capacity. For partial-band interference the error probability for $M = 2, 4, 8, 16$, and 32 was determined by Houston [3]. (The numerical results in Houston are slightly off for $M = 16$ and $M = 32$.) In [4], Crepeau and McGregor noted that while the exact bit error probability (from Houston's results) decreases with $M$ (for $M = 2, 4, 8, 16$, and 32), the union bound increased with $M$. We will show the exact bit error probability is a decreasing function of $M$ for small $M$ ($M = 2, 4, 8, 16$, and 32) but then increases for large $M$ ($M = 128, 256$, etc.) although very slowly. The signal-to-noise ratio required for symbol error probability of $10^{-5}$ decreases for $M$ less than 64 and increases thereafter.

For comparison purposes, we also derive the asymptotic form of the error probability for a Rayleigh faded channel. This was calculated previously by Lindsey [5] and approximated (for large signal-to-noise ratios) by Sussman [6]. For large signal-to-noise ratio ($> 10$ dB) this is well approximated by $\ln 2(\frac{E_b}{N_t}/N_t)$ which is the same asymptotic limit as for the partial-band Gaussian jamming case. However, numerical examination indicates that the error probability is higher for the fading cases than for the partial-band cases for finite $M$.

**II. PARTIAL-BAND GAUSSIAN INTERFERENCE**

The effect of a partial-band Gaussian interference is to add Gaussian noise in each of the $M$ dimensions with probability $\rho$ and no noise with probability $1 - \rho$. Let $E$ denote the energy of the transmitted signal and $E_b = E/\log_2 M$. Let $N_t$ denote the equivalent one-sided white noise power spectral density. When jamming noise is present the noise power spectral density is $N_t/\rho$. Let $P_e(E_b/N_t, M)$ be the symbol error probability of an $M$-ary orthogonal signal set on an additive white Gaussian noise channel. Since the results in this paper apply equally well in both the coherent and noncoherent detection cases, we will allow $P_e(E_b/N_t, M)$ to indicate either case. For coherent detection [1]

$$P_e \left( \frac{E_b}{N_t}, M \right) = 1 - \int_{-\infty}^{\infty} \Phi(u) \left( \frac{u^2 + \frac{2E_b}{N_t} \log_2 M}{2} \right)^{M-1} \sqrt{\frac{\pi}{2}} \exp \left( -\frac{u^2}{2} \right) du$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution and for noncoherent detection

$$P_e \left( \frac{E_b}{N_t}, M \right) = 1 - \int_{0}^{\infty} u \exp \left( -\frac{u^2 + \frac{2E_b}{N_t} \log_2 M}{2} \right) \cdot \mathcal{I}_0 \left( \frac{\sqrt{2E_b \log_2 M}}{N_t} u \right) \left( 1 - \exp \left( -\frac{u^2}{2} \right) \right)^{M-1} \cdot \mathcal{I}_0 \left( \frac{\sqrt{2E_b \log_2 M}}{N_t} u \right) du$$

where $\mathcal{I}_0(\cdot)$ is the modified Bessel function of the zeroth order.

It is well known that [1]

$$\lim_{M \to \infty} P_e \left( \frac{E_b}{N_t}, M \right) = \begin{cases} 1 & E_b/N_t < \ln 2 \\ 0 & E_b/N_t > \ln 2 \end{cases}$$

in either the coherent or the noncoherent case.

**III. RAYLEIGH FADING**

For comparison purposes, we also consider the asymptotic performance for large values of $M$ of a Rayleigh faded channel. These results have also been derived by Lindsey [5]. We are interested in nonselective fading. If $R$ is a Rayleigh distributed random variable which represents the fading then the error probability, $P_{e,R}(E_b/N_0, M)$, for Rayleigh fading can be expressed as

$$P_{e,R} \left( \frac{E_b}{N_0}, M \right) = E \left[ P_e \left( \frac{E_b}{N_0}, M \right) R^2 - \frac{N_0}{N_t} \right]$$

where $P_e(E_b/N_0, M)$ is the error probability of $M$ orthogonal signals in additive white Gaussian noise with power spectral density $N_0/2$ (again, we treat both the coherent and noncoherent cases together). To determine the limiting performance
for large $M$, we apply the dominated convergence theorem

$$
\lim_{M \to \infty} P_{EF} \left( \frac{E_b}{N_0}, M \right) = \lim_{M \to \infty} E \left( \frac{E_b R^2}{N_0}, M \right) = I(R < \sqrt{\ln 2/(E_b/N_0)})
$$

where $I(R < \alpha)$ is the indicator function. Thus,

$$
\lim_{M \to \infty} P_{EF} \left( \frac{E_b}{N_0}, M \right) = \int_0^{\sqrt{\ln 2/(E_b/N_0)}} 2re^{-r^2/2} dr
$$

For large signal-to-noise ratio an excellent approximation is obtained by using $e^{-r^2} = (1 - x)^4$ for small $x$. Thus, for large $E_b/N_0$

$$
\lim_{M \to \infty} P_{EF} \left( \frac{E_b}{N_0}, M \right) = \frac{\ln 2}{E_b/N_0} \quad (8)
$$

which is the same as that shown earlier for partial-band jamming. In Table I, we give numerical results for various values of $E_b/N_0$ and $M$.

IV. CONCLUSION

For worst case partial-band jamming the error probability performance (for fixed $E_b/N_0$) becomes worse with increasing $M$ (for $M > 16$). The asymptotic probability of error is not zero for any $E_b/N_0$ ($> \ln 2$) but decreases inverse linearly with $M$ in the fading case, the error probability performance (for fixed $E_b/N_0$) improves with $M$ for noncoherent detection but worsens with $M$ for coherent detection. For large $E_b/N_0$ the performance of the Rayleigh fading channel asymptotically approaches the same limit as the worst case partial-band jammed channel. However, for values of $M$ at least up to 4096, the partial-band jammed channel does better. While it is unlikely that a M-ary orthogonal signal set with $M > 1024$ will be used in a practical situation these results suggest an important theoretical problem, namely, what signal set achieves reliable communication.

APPENDIX A

Proposition:

$$
\lim_{M \to \infty} \max_{0 \leq \rho \leq 1} P_e \left( \frac{E_b}{N_0}, M \right) = \begin{cases} 1, & E_b/N_0 < \ln 2 \\
\ln 2, & E_b/N_0 > \ln 2. \end{cases}
$$

Proof: To prove the proposition, we obtain an upper bound on

$$
\limsup_{M \to \infty} \max_{0 \leq \rho \leq 1} P_e \left( \frac{E_b}{N_0}, M \right)
$$

and then an identical lower bound on

$$
\liminf_{M \to \infty} \max_{0 \leq \rho \leq 1} P_e \left( \frac{E_b}{N_0}, M \right).
$$

This will prove the limit exists and is equal to the bounds. Clearly, $\max_{0 \leq \rho \leq 1} P_e(E_b/N_0, M) \leq P_e(E_b/N_0, M)$ so that for $E_b/N_0 < \ln 2$,

$$
\liminf_{M \to \infty} \max_{0 \leq \rho \leq 1} P_e \left( \frac{E_b}{N_0}, M \right) \geq \liminf_{M \to \infty} P_e \left( \frac{E_b}{N_0}, M \right) = 1.
$$

For $E_b/N_0 > \ln 2$ and any $\epsilon > 0$

$$
\max_{0 \leq \rho \leq 1} P_e \left( \frac{E_b}{N_0}, M \right) \leq \left( \frac{\ln 2 - \epsilon}{E_b/N_0} \right) P_e(\ln 2 - \epsilon, M).
$$

Thus,

$$
\liminf_{M \to \infty} \max_{0 \leq \rho \leq 1} P_e \left( \frac{E_b}{N_0}, M \right) \leq \liminf_{M \to \infty} \frac{\ln 2 - \epsilon}{E_b/N_0} P_e(\ln 2 - \epsilon, M) = \frac{\ln 2 - \epsilon}{E_b/N_0}
$$

for $E_b/N_0 > \ln 2$. Since this is true for any $\epsilon > 0$ we have proved that

$$
\liminf_{M \to \infty} \max_{0 \leq \rho \leq 1} P_e \left( \frac{E_b}{N_0}, M \right) \leq \begin{cases} 1, & E_b/N_0 < \ln 2 \\
\ln 2, & E_b/N_0 > \ln 2. \end{cases}
$$

To obtain the upper bound, we first observe that $P_e(E_b/N_0, M) \leq 1$ for $0 \leq \rho \leq 1$. Now, let $\rho_M$ be the value of $\rho$ that achieves the maximum in

$$
\max_{0 \leq \rho \leq 1} P_e \left( \frac{E_b}{N_0}, M \right).
$$

Then for sufficiently large $M$, $\rho_M \leq \ln 2/E_b/N_0 \leq M$. If this were not the case then for every $M$ there would be an $M_1 > M$ and $\rho_{M_1} > 0$ for which

$$
\rho_M P_e \left( \frac{E_b}{N_0}, \rho_{M_1}, M_1 \right) \geq \rho P_e \left( \frac{E_b}{N_0}, \rho, M \right)
$$

for all $\rho$, $0 \leq \rho \leq 1$. (A.1)

For any $\epsilon > 0$ the right-hand side of (A.1) for sufficiently large $M (M > M_1)$ can be made greater than $-\epsilon$. Since the left-hand side of (A.1) is obviously approaching $0$ for $\rho_M > 0$, we have a contradiction. Thus, for sufficiently large $M$ and $E_b/N_0 > \ln 2 \max_{0 \leq \rho \leq 1} \rho P_e(E_b/N_0, \rho, M) = \max_{0 \leq \rho \leq 1} \rho P_e(E_b/N_0, M) \leq \gamma$. This then proves that

$$
\limsup_{M \to \infty} \max_{0 \leq \rho \leq 1} P_e \left( \frac{E_b}{N_0}, M \right) \leq \begin{cases} 1, & E_b/N_0 < \ln 2 \\
\ln 2, & E_b/N_0 > \ln 2. \end{cases}
$$

and the proposition follows.
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