Coding for Frequency-Hopped Spread-Spectrum Communication with Partial-Band Interference—Part II: Coded Performance

WAYNE E. STARK, MEMBER, IEEE

Abstract—**The performance of codes in a frequency-hopped spread-spectrum communication system with partial-band interference is investigated. The performance measure considered is the decoded bit error probability. A simplified interference model and worst-case partial-band Gaussian noise interference model is considered with the interference noise statistically independent of the transmitted signal. We consider soft decision receivers with side information and hard decision receivers with and without side information.**

I. INTRODUCTION

THE design of error-correcting codes for the additive white Gaussian noise channel is well understood. Coding for channels for which the noise is not Gaussian and not white is much less well understood. One such channel that arises in practice is that of a partial-band interference or pulsed interference channel. For example, a frequency-hopped spread-spectrum communication system that is operating in the presence of a partial-band jammer has interference that is on part of the time (when the signal is transmitted in a jammed band) and off part of the time (when the signal is transmitted in an unjammed band). Another such channel is a frequency-hopped multiple-access channel. In addition to the interference, there may be some white Gaussian noise that causes additional errors. One of the issues involved in the design of a communication system operating in partial-band interference, and in particular the error-correction coding to be used, is that of side information availability. If the decoder has knowledge of which symbols were received when interference is present, then these symbols carry less weight in deciding which codeword is transmitted than those received when there is no interference present. If the decoder does not have this side information, then no such weighting can be made. In this paper we present analytical methods for determining the error probability of codes on such channels both when side information is present and when side information is unavailable.

Several other researchers have analyzed the error probability of some codes considered here (e.g., [1]-[4]). The analysis in these papers has been largely that of determining the bit error probability of convolutional codes, possibly in conjunction with repetition codes and with soft decisions, for which case a union-Chernoff bound has been employed. Also, the effect of background noise has largely been ignored. In these papers there is lacking a comparison between the performance of several different receivers. In this paper we determine what gains can be achieved using side information with soft decisions over hard decisions with side information and over hard decisions without side information. We also consider the effect of background noise in addition to the interference and determine under what conditions can the background noise be safely neglected. We consider three classes of codes: repetition codes, convolutional codes, and Reed-Solomon codes (RS codes). New analytical results are also presented for repetition codes on i) two state channels for which background noise is present and ii) M-ary symmetric channels. For repetition codes with a soft decision receiver, we use an exact value for the error probability. Trumpis [5], in an unpublished manuscript, has derived an exact expression for the error probability of repetition codes with soft decision (square-law combining) decoding. Here we write the error probability as an integral which may be easily evaluated by standard numerical integration techniques. When evaluating the error probability of convolutional codes, we use exact values for the error probability between two codewords instead of using the Chernoff bound. For the dual-k convolutional codes we present a new class of upper bounds that approaches the union bound without the Chernoff bound. These new bounds are computationally tractable and offer considerable improvement over the existing bounds. For Reed-Solomon codes we consider two cases: the channel that cannot declare erasures and the channel that can declare erasures (but does not make errors). The later channel corresponds to the case of side information available.

From the analysis we make the following general conclusions. First, background noise can (conservatively) be neglected if the signal-to-background noise ratio is larger than the *average* signal-to-jamming noise ratio by about 3 dB and, in many cases, if the average signal-to-jamming noise ratio is the same or even less than the signal-to-background noise ratio, the background noise can be neglected. For large or small signal-to-jamming noise ratios, the performance of a hard decision decoder is the same with and without side information. The difference between hard decisions with side information and soft decisions with side information is on the order of 1-3 dB, whereas the difference between hard decisions with side information and without side information varies considerably (between 0 and 8 dB). One final conclusion we draw is that a partial-band jammer can be neutralized, provided codes of small enough rates are used.

In Section II we introduce the basic channel model we will use to model interference in a frequency-hopped spread-spectrum communication system. The error probability of the above three classes of codes is determined both with and without side information. The techniques developed in Section II are applied in Section III to determine the error probability of codes on partial-band interference and hard decisions. We consider the case of side information available and no side information available. The effect of background noise is also considered. In Section IV the error probability of a soft decision decoder with side information available is examined for repetition codes and for convolutional codes. Finally, in Section V we draw conclusions and compare the

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The author is with the Department of Electrical and Computer Engineering, University of Michigan, Ann Arbor, MI 41809.

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error probability of actual codes to the channel capacity and cutoff rate which were determined in Part I of this paper. This is done for the three cases mentioned above: soft decisions with side information, hard decisions with side information, and hard decisions without side information.

II. Interference Model

In this section we describe a channel model for interference in a slow frequency-hopped spread-spectrum communication system and evaluate the error probability of some codes when used on this channel. We will consider a system which transmits m symbols during each hop. The model is general enough to include background noise and channel memory in a frequency-hopped spread-spectrum communication system with partial-band interference. The channel model consists of two states 0 and 1. When the state of the channel is 0, each input symbol is transmitted over an M-ary symmetric channel with crossover probability $p_0/(M-1)$ (i.e., symbol error probability p_0). When the state of the channel is 1, each symbol is transmitted over an M-ary symmetric channel with crossover probability $p_1/(M-1)$. The state of the channel can change only on every mth symbol. The sequence of channel states is a sequence of independent, identically distributed random variables. The probability of the channel being in state 1 is ρ and the probability of being in state 0 is $1-\rho$. The case of a strong interferer and no background noise is $p_0 = 0$ and $p_1 = (M-1)/M$ with ρ being the probability that a symbol is subject to interference. The model with $p_0 = 0$ and $p_1 = (M-1)/M$ also gives an upper bound to the error probability of any coded system using hard decisions (when there is no background noise). The models with $p_1 = (M-1)/M$ can be used to evaluate the minimum fraction of the band that a jammer must occupy to be effective (cause the bit error probability to be larger than the desired error probability), irrespective of jamming power or noise distribution [6]. In the next three subsections we will evaluate the error probabilities of three classes of codes on this channel.

A. Repetition Codes

First we determine the error probability of repetition codes on this channel with maximum likelihood decoding. For simplicity we will only consider the case of $m = 1$. The results, though, will be applicable for the case $m > 1$ if interleaving of codewords is done. Calculations of error probability for $m > 1$ without interleaving are quite tedious and will not be examined here. These calculations have been performed in [7] for the special case of $p_0 = 0$ and $p_1 = 1/2$. If the channel state is not available to the decoder, then the overall channel is an M-ary symmetric channel with error probability $\rho p_1 + (1-\rho)p_0$. A maximum likelihood decoder will decide that 0 was transmitted if there are more zeros than any other symbol in the received word. If there is a tie between two or more symbols, the optimal decoder can choose any of those symbols. The error probability of a *binary* repetition code of length n on a binary symmetric channel ($M = 2$) with maximum likelihood decoding without side information is

$$P_{e,b} = \sum_{l=1}^{n} w_l \bar{p}^l (1-\bar{p})^{n-l} \tag{1}$$

where w_l is the number of error patterns of weight l that will cause a symbol error and

$$\bar{p} = \rho p_1 + (1-\rho) p_0. \tag{2}$$

For $M = 2$, w_l is given by

$$w_l = \begin{cases} \binom{n}{l} & l \geq e \\ \frac{1}{2}\binom{n}{n/2} & l = n/2,\ n \text{ even} \\ 0 & l < e,\ n \text{ odd};\ l < e-1,\ n \text{ even} \end{cases}$$

where e is the least integer greater than or equal to $(n+1)/2$. For $M > 2$, w_l can be determined from the results in Appendix A.

When the state of the channel is known to the decoder, then the maximum likelihood decoder is more complicated than just comparing the number of zeros and ones. The maximum likelihood decoder for the two-state channel, with each state being a binary symmetric channel, is derived in Appendix B. The bit error probability is also derived in Appendix B. Let d_0 be the number of 0's received when the channel is in state 0, d_1 be the number of 0's received when the channel is in state 1, l be the number of times out of n that channel is in state 1,

$$\delta = d_0 \ln \frac{p_0}{1-p_0} + d_1 \ln \frac{p_1}{1-p_1} \tag{3}$$

and

$$\alpha_l = \frac{l}{2} \ln \frac{p_1}{1-p_1} + \frac{n-l}{2} \ln \frac{p_0}{1-p_0}. \tag{4}$$

The maximum likelihood decoding rule is to decide 1 if $\delta > \alpha_l$ and to decide 0 otherwise. The bit error probability for maximum likelihood decoding is then given by

$$\begin{aligned} P_{e,b} = \sum_{l=0}^{n} \binom{n}{l} \rho^l (1-\rho)^{n-l} & \cdot \Bigg[\sum_{\delta > \alpha_l} \binom{l}{d_1}\binom{n-l}{d_0} \\ & \cdot (1-p_1)^{d_1} (p_1)^{l-d_1} (1-p_0)^{d_0} (p_0)^{n-l-d_0} \\ & + \frac{1}{2} \sum_{\delta = \alpha_l} \binom{l}{d_1}\binom{n-l}{d_0} \\ & \cdot (1-p_1)^{d_1} (p_1)^{l-d_1} (1-p_0)^{d_0} (p_0)^{n-l-d_0} \Bigg]. \end{aligned} \tag{5}$$

When $p_0 = 0$, the error probability for maximum likelihood decoding simplifies to

$$P_{e,b} = \rho^n \pi_n \tag{6}$$

where π_n is the error probability of a repetition code of length n on a binary symmetric channel with crossover probability p_1. This is given by (1) with $\bar{p}$ replaced by p_1. The error probability of an M-ary repetition code on the two-state channel with side information available is, in general, much more complicated to derive, even for small n. When there is no background noise ($p_0 = 0$) and side information is available, the analysis is simplified considerably. In this case the maximum likelihood decoder only looks at the symbols that are not jammed (if any) and chooses the output to be that symbol (they all must be the same since there is no background noise). If all the symbols are jammed, the decoder chooses the symbol which was received the most (ties are resolved by randomly selecting one of the contending symbols). Thus, there is an error only if all symbols are transmitted over the jammed channel. This happens with probability ρ^n. The probability of a symbol error, given that all symbols are jammed, is the same as the symbol error probability of an M-ary repetition code on an M-ary symmetric channel with symbol error probability p_1.

B. Convolutional Codes

Next we consider the error probability of convolutional codes with maximum likelihood decoding. When side information is unavailable, the calculation of error probability of binary convolutional codes is straightforward, since the channel is just a binary symmetric channel with error probability $\bar{p}$ given in (2). Bounds on the bit error probability for a rate $1/n$ code are given in [8] as

$$P_{e,b} \leq \sum_{j=d_f}^{\infty} w_j P_j \tag{7}$$

where w_j is the number of paths with information weight j, and P_j is the error probability between two codewords which differ in j symbols and d_f is the free distance of the code. Notice that P_j is just the error probability of a repetition code of length j which is given in (1) with $j = n$. For most codes w_j is known for only the first few values of j. In [9] (see also [8]) these are tabulated for some good codes. For these codes we must truncate the series in (7) and get an approximation to the error probability. One class of codes for which w_j is known for all j are the *dual-k* rate $1/v$ convolutional codes [9]. These codes have symbol alphabets of size 2^k and are thus well suited for M-ary modulation. For these codes the bit error probability on an M-ary symmetric channel can be bounded by [10]

$$P_{e,b} \leq (2^{k-1}) \sum_{j=0}^{\infty} (j+1) \sum_{l=0}^{j} \binom{j}{l} a^l b^{j-l} q_{2v+vj-l} \tag{8}$$

where $a = v$, $b = 2^k - 1 - v$, $M = 2^k$, and q_j is the error probability between two codewords of a repetition code of length j on an M-ary symmetric channel. This can be shown to be given by (see Appendix A)

$$\begin{aligned} q_n = &\sum_{\substack{j,k \\ j<k \\ j+k \leq n}} \binom{n}{j} (1-p)^j \\ &\cdot \left(\frac{p}{M-1}\right)^k \left(\frac{M-2}{M-1} p\right)^{n-j-k} \\ &+ \frac{1}{2} \sum_{j=0}^{[n/2]} \binom{n}{j,\, j} (1-p)^j \\ &\cdot \left(\frac{p}{M-1}\right)^j \left(\frac{M-2}{M-1} p\right)^{n-2j} \end{aligned} \tag{9}$$

where p is the symbol error probability of the channel. When the Bhattacharyya bound [11] is used ($q_j \leq D^j$), (8) reduces to the familiar form [10]

$$P_{e,b} \leq \frac{(2^{k-1}) D^{2v}}{1 - vD^{v-1} - (2^k - 1 - v) D^v)^2} \tag{10}$$

where

$$D = \left(\frac{M-2}{M-1}\right) p + 2\sqrt{p(1-p)/(M-1)}\,. \tag{11}$$

In [10] Odenwalder obtained an improved upper bound for the dual-3 rate 1/2 code. Here we obtain a class of upper bounds for any dual-k rate $1/v$ convolutional code. We first break the summation over j in (8) into two sums:

$$\begin{aligned} P_{e,b} \leq (2^{k-1}) \Bigg\{ &\sum_{j=0}^{J} (j+1) \sum_{l=0}^{j} \binom{j}{l} a^l b^{j-l} P_{2v+vj-l} \\ &+ \sum_{j=J+1}^{\infty} (j+1) \sum_{l=0}^{j} \binom{j}{l} a^l b^{j-l} P_{2v+vj-l} \Bigg\}. \end{aligned} \tag{12}$$

Now notice that in the second term of (12), the coefficient of $P_{2v+vj-l}$ is positive, so that we can further upper bound P_e by bounding $P_{2v+vj-l}$ by $D^{2v+vj-l}$. Thus,

$$\begin{aligned} P_{e,b} \leq (2^{k-1}) \Bigg\{ &\sum_{j=0}^{J} (j+1) \sum_{l=0}^{j} \binom{j}{l} a^l b^{j-l} P_{2v+vj-l} \\ &+ \sum_{j=J+1}^{\infty} (j+1) \sum_{l=0}^{j} \binom{j}{l} a^l b^{j-l} D^{2v+vj-l} \Bigg\}. \end{aligned} \tag{13}$$

Now the second term in (13) can be summed, provided $c = aD^{v-1} + bD^v < 1$ to yield

$$\begin{aligned} P_{e,b} \leq (2^{k-1}) \Bigg\{ &\sum_{j=0}^{J} (j+1) \sum_{l=0}^{j} \binom{j}{l} a^l b^{j-l} P_{2v+vj-l} \\ &+ D^{2v} c^{J+1} [1 + (J+1)(1-c)]/(1-c)^2 \Bigg\}. \end{aligned} \tag{14}$$

The condition that $c < 1$ is necessary for the convergence of the series that upper bounds the error probability in both (10) and (14). This condition is met for error probabilities of practical importance.

When side information is available, the error probability of binary convolutional codes is still given by (7); however, P_j is the error probability between two codewords of a repetition code with side information available, given by (5) with $n = j$. For dual-k codes, (14) is still valid for the channel with side information when P_j is the error probability between two codewords of a length j repetition code for that channel and D is the Bhattacharyya parameter for the channel with side information, which is given by

$$\begin{aligned} D = \rho &\left[\left(\frac{M-2}{M-1}\right) p_1 + 2\sqrt{p_1(1-p_1)/(M-1)} \right] \\ &+ (1-\rho) \left[\left(\frac{M-2}{M-1}\right) p_0 + 2\sqrt{p_0(1-p_0)/(M-1)} \right]. \end{aligned}$$

C. Reed-Solomon Codes

The last code we consider for this interference model is the Reed-Solomon code. Since Reed-Solomon codes are nonbinary, the code symbols must have alphabet size greater than 2. We will consider two cases. The two cases correspond to $m > 1$, $M = 2$ and $m = 1$, $M > 2$. We note that it is also possible to have $m > 1$ and $M > 2$. The analytical results presented below are general enough to include the case $m > 1$ and $M > 2$, but no numerical results will be given for that case. See [12], [6] for numerical results for that case. The Reed–Solomon codes we consider will have alphabet size M^m so that one code symbol (m M-ary symbols) is transmitted during each frequency-hop. We will assume that the decoder can detect with probability 1 when the number of errors and erasures is greater than the capability of the code. When the decoder detects that too many errors have occurred, the decoder outputs the received information symbols. The undetected error probability can be calculated using techniques in [13]. Let N denote the length of the RS code and K the number of information symbols per codeword. When side information is unavailable, the channel is an M^m-ary channel with code symbol error probability given in (15).

$$p_s = \rho[1-(1-p_1)^m] + (1-\rho)[1-(1-p_0)^m]. \tag{15}$$

The probability of a code symbol (m M-ary symbols) being in error at the output of the decoder with a bounded distance decoder [8, p. 19] may be calculated as

$$P_{e,s} = \sum_{j=e+1}^{N} \frac{j}{N} \binom{N}{j} p_s^j (1-p_s)^{N-j} \tag{16}$$

where $e = \lfloor (N-K)/2 \rfloor$ and $\lfloor x \rfloor$ is the largest integer less than or equal to x. The bit error probability can be determined from (16) using the techniques in [6] as

$$P_{e,b} = \frac{M}{2(M-1)} \frac{\rho p_1 + (1-\rho)p_0}{\rho(1-(1-p_1)^m) + (1-\rho)(1-(1-p_0)^m)} \cdot P_{e,s}. \tag{17}$$

With side information available, a possible decoding strategy (although not maximum likelihood decoding) is to erase those symbols which are received over the bad channel. With this strategy an erasure will occur with probability ρ. An (N, K) RS code is capable of correcting t errors and s erasures provided $s + 2t \leq N - K$. The decoded bit error probability can also be derived using techniques similar to those in [6] as

$$P_{e,b} = \frac{M}{2(M-1)} [p_0(1-\rho)P_0 + \rho p_1 P_1] \tag{18}$$

where

$$P_0 = \sum_{\substack{s+2(t+1)>N-K \\ s+t<N}} \binom{N-1}{s,\ t} \rho^s((1-\rho)(1-(1-p_0)^m))^t \cdot (1-\rho-(1-\rho)(1-(1-p_0)^m))^{N-1-s-t} \tag{19}$$

and

$$P_1 = \sum_{\substack{s+1+2t>N-K \\ s+t<N}} \binom{N-1}{s,\ t} \rho^s((1-\rho)(1-(1-p_0)^m))^t \cdot (1-\rho-(1-\rho)(1-(1-p_0)^m))^{N-1-s-t}. \tag{20}$$

When $p_0 = 0$, $p_1 = 1/2$, and $M = 2$, (18) simplifies to

$$P_{e,b} = \frac{1}{2} \sum_{s=N-K+1}^{N} \frac{s}{N} \binom{N}{s} \rho^s (1-\rho)^{N-s}. \tag{21}$$

This decoding technique will not be effective when the jammer has the freedom to choose ρ. This is because $\rho = 1$ will cause every symbol to be erased. For the channel for which ρ is fixed, such as multiple access channels, this decoding technique will be superior to decoding without side information. A decoding algorithm that avoids this problem for jamming is given in [6].

D. Numerical Results and Discussion

The error probability of the various codes mentioned was evaluated for the simplified jamming model in which the unjammed error probability p_0 was zero and the jammed error probability p_1 was 1/2. In Fig. 1 the error probability of repetition codes of length 1, 3, 5, and 7 is shown with and without side information. For $n = 1$ (no coding) the error probability with and without side information is the same. In Fig. 2 the error probability of Reed–Solomon codes with $m = 5$ and 8 ($N = 2^m - 1$) and binary convolutional codes with constraint length 7 and 9 are shown for the case of side information available, while in Fig. 3 the corresponding results are shown when no side information is available. It is clear from the figures that for a specified bit error probability, the largest value of ρ such that the error probability is less than the specified error probability is larger with side information than without. It is also clear that with side information, the minimum fraction of band that must be jammed in order for the error probability to be greater than the specified error probability is nearly the same for convolutional codes and Reed–Solomon codes, while without side information (interleaving the convolutional codes) the Reed–Solomon codes can withstand larger values of ρ. This suggests that without side information, using binary codes with interleaving (of depth $\log_2 M$) on an M-ary channel is not a good idea. From these figures one can determine, for any coded system (with or without side information) and a specified error probability, the minimum fraction of band that must be jammed for the resulting error probability to be greater than the specified error probability. (See [6] for further discussion of this.) Results for other values of p_0 and p_1 will be given in the next section, where these error probabilities are related to signal-to-noise ratios.

III. Hard Decision Receiver

As in Part I, we consider an M-ary communication system in which there are two possible states 0 and 1 of the channel. The channel is in state 1 when the jammer is present during a particular frequency hop. The probability of symbol error, p_1, in this case is given by [14]

$$p_1 = \frac{1}{M} \sum_{j=2}^{M} (-1)^j \binom{M}{j} \exp\left\{-\frac{E}{N_0 + N_J/\rho}(1-1/j)\right\}. \tag{22}$$

where E is the energy per transmitted symbol and $N_J/2$ is the average noise spectral density. The channel is in state 0 when the jammer is absent during a particular frequency hop. The probability of symbol error, p_0, in this case is given by [14]

$$p_0 = \frac{1}{M} \sum_{j=2}^{M} (-1)^j \binom{M}{j} \exp\left\{-\frac{E}{N_0}(1-1/j)\right\}. \tag{23}$$

The probability of the channel being in state 1 is ρ. The probability of the channel being in state 0 is $1 - \rho$. When the receiver has side information, the decoder knows the state of the channel for each symbol sent.

A. No Side Information

We now determine the bit error probability of the codes considered in Section II when no side information is

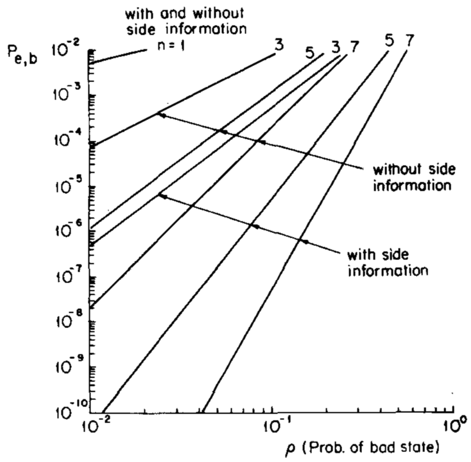

Fig. 1. Bit error probability for repetition codes of length 1, 3, 5, and 7 on two-state channels with and without side information.

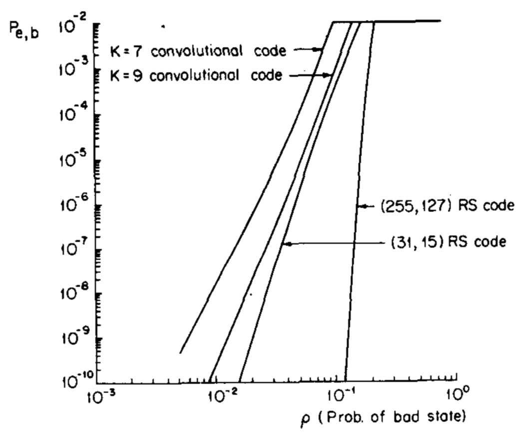

Fig. 2. Bit error probability for Reed–Solomon codes and upper bound on bit error probability for convolutional codes on two-state channel without side information.

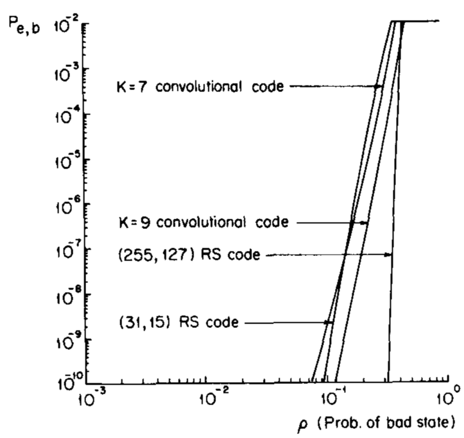

Fig. 3. Bit error probability for Reed–Solomon codes and upper bound on bit error probability for convolutional codes on two-state channel with side information.

available. First consider the repetition codes. We assume, temporarily, that there is one symbol transmitted during each frequency hop. If m symbols are sent, interleaving m codewords will give the same error probability, although this is not necessarily a good strategy. Without side information the coding channel is just an M-ary symmetric channel with symbol error probability $p_1\rho + p_0(1-\rho)$. The decoded bit error probability is then given by (1) for repetition codes. If we are interested in the worst-case error probability, then (1) is maximized over ρ: $0 \le \rho \le 1$. However, it is easily seen that maximizing $P_{e,b}$ is equivalent to maximizing $\bar{p}_s$. The ρ that maximizes (1) is inversely proportional to E/N_J with a constant term that depends on E/N_0:

$$\rho^* = \begin{cases} 1, & E/N_J < A(E/N_0) \\ \dfrac{A(E/N_0)}{E/N_J}, & E/N_J \ge A(E/N_0) \end{cases} \tag{24}$$

$$\bar{p}^* = \begin{cases} f(E/(N_0+N_J)), & E/N_J < A(E/N_0) \\ f(E/N_0) + \dfrac{B(E/N_0)}{E/N_J}, & E/N_J \ge A(E/N_0) \end{cases} \tag{25}$$

where

$$f(\alpha) = \frac{1}{M}\sum_{j=2}^{M}(-1)^j \binom{M}{j} e^{-\alpha(1-1/j)}.$$

For very large E/N_0 compared to E/N_J, (18) and (19) become

$$\rho^* = \begin{cases} 1, & E/N_J < A_M \\ \dfrac{A_M}{E/N_J}, & E/N_J \ge A_M \end{cases} \tag{26}$$

and

$$\bar{p}^* = \begin{cases} f(E/N_J), & E/N_J < A_M \\ \dfrac{B_M}{E/N_J}, & E/N_J \ge A_M. \end{cases} \tag{27}$$

In (24) and (25), $A(E/N_0)$ and $B(E/N_0)$ depend on E/N_0 and M, while in (28) and (29), A_M and B_M depend only on M. Numerical values for $A(E/N_0)$ and $B(E/N_0)$ are given in Table I for $M = 2$. This is a generalization of the work of Houston [15], who considered only the case of no background noise. The constants A_M and B_M may be determined from [15]. The error probability of convolutional codes can be easily evaluated by using (25) in (9) and then (9) in (7) or a truncated version of (7). The error probability of Reed–Solomon codes can be calculated by using $\bar{p}^*$ given by (25) in (16) for P_s.

B. Side Information Available

When side information is available, the error probabilities given in (22) and (23) can be used in (5) to determine the error probability of a repetition code and also of a convolutional code by then using (5) in (7). The error probability of Reed–Solomon codes that correct both errors and erasures can be determined using (19)–(21). However, since the Reed–Solomon decoding algorithm erases those symbols that are jammed and is not a maximum likelihood decoding algorithm, a strategy for the jammer is to jam the entire band, thereby causing every symbol to be erased, which causes the decoder to make an error with probability close to 1. Another

TABLE I
CONSTANTS FOR DETERMINING ERROR PROBABILITY FOR BINARY FSK WITH WORST-CASE PARTIAL BAND JAMMING AND BACKGROUND NOISE

E/N_0 (dB)	$A(E/N_0)$	$B(E/N_0)$
10.0	3.6112	0.46702
15.0	2.3017	0.39366
20.0	2.0842	0.37547
∞	2.0000	0.36788

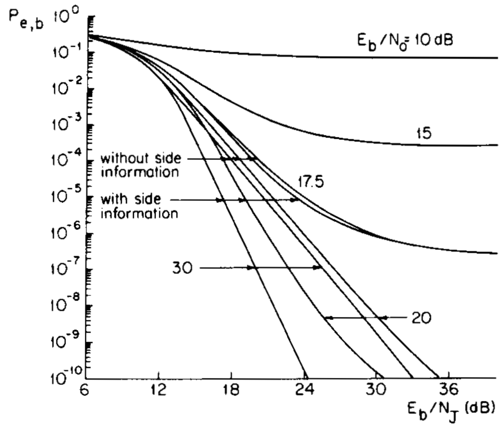

Fig. 4. Bit error probability for length 7 repetition codes on worst-case partial-band jamming channel with binary FSK.

decoding algorithm which uses side information that eliminates this problem is given in [16]. Another method of using the Reed–Solomon codes on channels with side information is to concatenate a repetition code and a Reed–Solomon code. The error probability of such a code can be easily calculated from the error probability formulas of each code separately.

C. Numerical Results and Discussion

The error probability of various codes was evaluated for the case of a hard decision receiver both with and without side information. In Fig. 4 the error probability of a binary repetition code of length 7 is shown for E_b/N_0 taking the values 10, 15, 20, and 30 dB. Notice that for very high signal-to-jamming noise ratios the background noise is dominant (even compared to a pulsed jammer), so the use of side information becomes superfluous. When the signal-to-jamming ratio is very small, then the jammer need not pulse to cause larger error probability, so again side information is not necessary. For the repetition code of length 7, the background noise has an effect only when the background noise power is smaller than the *average* jamming noise power. In Fig. 5 the approximate upper bound to the bit error probability of the constraint length 7 binary convolutional code from [10] is shown with and without side information. Notice that the error probability when E_b/N_0 is 15 dB differs from the error probability for no background noise ($E_b/N_0 = \infty$) by less than 1 dB for error probabilities between 10^{-3} and 10^{-5}. This is true both with and without side information. Also notice that for error probabilities in this range, the availability of side information improves the performance by about 4 dB when E_b/N_0 is greater than 15 dB. When E_b/N_0 is less than 10 dB and the error probability is greater than 10^{-3}, there is essentially no difference between the performance of convolutional codes with and without side information available.

The error probability of an M-ary orthogonal code, or

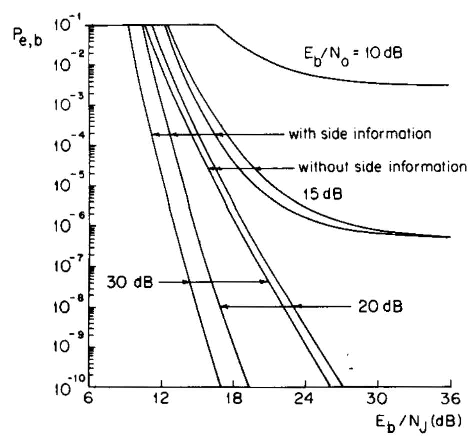

Fig. 5. Upper bound on bit error probability for constraint length 7 convolutional code with and without side information on worst-case partial-band jamming channel with binary FSK.

equivalently M-ary orthogonal signaling when side information is unavailable, is shown in Table II for E_b/N_0 of 5, 7.5, 10 and ∞ dB. Notice that for this code (or modulation) the effect of background noise is negligible for E_b/N_0 greater than around 10 dB. If we used another code such as a Reed–Solomon code or even a repetition code in conjunction with the M-ary signaling, then this would even further suppress the effect of the background noise. The error probability of M-ary repetition codes of length 1, 3, 5 and 7 are shown in Fig. 6 when no side information is available and in Fig. 7 when side information is available for the case of no background noise. In Fig. 8 the bit error probability of three different Reed–Solomon codes is shown when no side information is available. In Fig. 9 the error probability of dual-5 convolutional codes is shown both with and without side information. In calculating the error probability for dual-k codes we used (14) with $J = 10$. In Table III the bounds of (14) are shown for $J = 1$, 5, and 10 [denoted by $P_e(J)$] along with the upper bound of (10) [denoted by P_e(UB)]. Notice that as E_b/N_J becomes larger, the bounds for various values of J become nearly the same. When this happens, the upper bounds then are the union bound on the error probability. By comparing this to the union-Bhattacharyya (UB) bounds, we can determine the additional difference between the two. For the case of no background noise and worst-case jamming, the UB bound is a factor of about 9 worse than the union bound at E_b/N_J of 20 dB or about 4 dB worse at error probability of 10^{-5}.

IV. SOFT DECISIONS

We now consider the error probability of codes that employ a soft decision receiver. We do not consider maximum likelihood decoding any more, since in an actual implementation of a decoder this would involve highly nonlinear processing (e.g., computing Bessel functions, etc.). Instead we consider the more practical case of square-law combining. With soft decisions the jammer has as a possible strategy transmitting very narrow high-amplitude pulses. Just one pulse with large enough amplitude can cause the error probability between two codewords to be nearly 1/2. Thus, with soft decisions it seems likely that a very low duty cycle jammer would be optimum [2]. On the other hand, such a jammer would be quite easy to detect and those symbols that are jammed could be erased. This would force the jammer to have a higher duty cycle and, thus, use smaller amplitude pulses. In this part we will make the following simplifying assumptions. First, the decoder has perfect side information about the presence of a jammer. Second, there is

TABLE II
BIT ERROR PROBABILITY FOR 32-ARY FSK WITH PARTIAL-BAND INTERFERENCE

E_b/N_J (dB)	E_b/N_0 5.0 dB	7.5 dB	10.0 dB	∞ dB
0.0	2.3105×10^{-1}	2.0345×10^{-1}	1.8998×10^{-1}	1.7586×10^{-1}
5.0	7.6083×10^{-2}	6.4342×10^{-2}	6.0077×10^{-2}	5.4347×10^{-2}
10.0	2.5483×10^{-2}	2.0351×10^{-2}	1.8998×10^{-2}	1.7586×10^{-2}
15.0	9.4818×10^{-3}	6.4395×10^{-3}	6.0077×10^{-3}	5.5613×10^{-3}
20.0	4.4218×10^{-3}	2.0404×10^{-3}	1.8998×10^{-3}	1.7586×10^{-3}
25.0	2.8217×10^{-3}	6.4925×10^{-4}	6.0077×10^{-4}	5.5613×10^{-4}
30.0	2.3157×10^{-3}	2.0933×10^{-4}	1.8998×10^{-4}	1.7586×10^{-4}
35.0	2.1557×10^{-3}	7.0222×10^{-5}	6.0077×10^{-5}	5.5613×10^{-5}
40.0	2.1051×10^{-3}	2.6231×10^{-5}	1.8998×10^{-5}	1.7586×10^{-5}
45.0	2.0891×10^{-3}	1.2329×10^{-5}	6.0078×10^{-6}	5.5613×10^{-6}

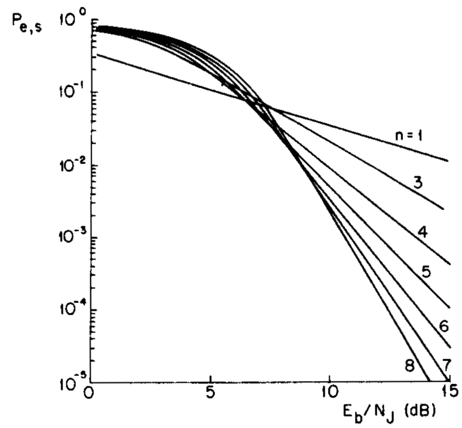

Fig. 6. Symbol error probability for repetition codes on channel with hard decisions and no side information on worst-case partial-band jamming channel and 32-ary FSK ($E_b/N_0 = \infty$).

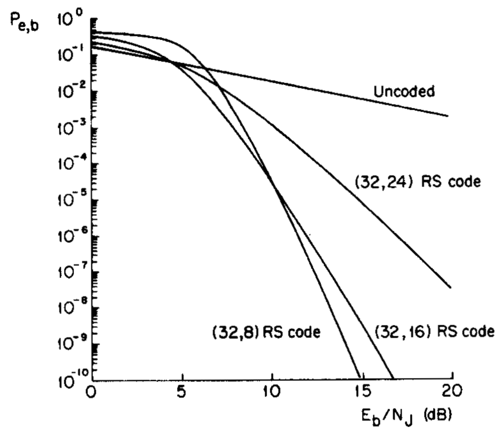

Fig. 8. Error probability for Reed–Solomon codes on channel without side information on worst-case partial-band jamming channel and 32-ary FSK ($E_b/N_0 = \infty$).

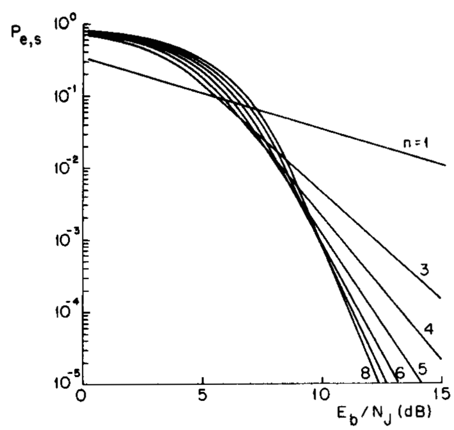

Fig. 7. Symbol error probability for repetition codes on channel with hard decisions and side information on worst-case partial-band jamming channel and 32-ary FSK ($E_b/N_0 = \infty$).

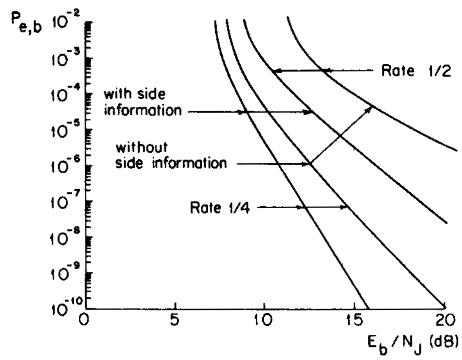

Fig. 9. Upper bound on bit error probability for dual-k convolutional codes with and without side information on the worst-case partial-band jamming channel with 32-ary FSK ($E_b/N_0 = \infty$).

no background noise present. Together these two assumptions imply that the decoder will make an error in comparing two codewords only if all the symbols where the two codewords differ are jammed. Since just one symbol is not jammed, it is received over a noise-free channel and, thus, will not be in error. If all such symbols are jammed, the decoder uses square-law combining of the code symbols to make a decision on which codeword was transmitted. We note here that the analysis can be done for other cases (no side information, background noise present) but the results are quite complicated. We do not consider the error probability of block codes (Reed–Solomon, etc.) besides the repetition code, since in practice soft decision decoding is very difficult to implement. See [17] for some numerical results concerning soft decision decoding of R–S codes.

A. Repetition Codes

The error probability of repetition codes of length L can be calculated for the channel with soft decisions and side information as follows. Since we are assuming square-law combining, the receiver adds the outputs of the square-law detectors only if all symbols have been jammed. Let Y_{ij} be

TABLE III
UPPER BOUNDS ON BIT ERROR PROBABILITY FOR 32-ARY FSK WITH PARTIAL-BAND INTERFERENCE (HARD DECISIONS, NO SIDE INFORMATION) USING DUAL-5 RATE 1/2 CONVOLUTIONAL CODES

E_b/N_J (dB)	$P_e(J)$			
	$P_e(1)$	$P_e(5)$	$P_e(10)$	$P_e(\mathrm{UB})$
11.0	0.9942×10^{-1}	0.6456×10^{-1}	0.3339×10^{-1}	0.1110×10^{-0}
12.0	0.1260×10^{-1}	0.4562×10^{-1}	0.2080×10^{-2}	0.1798×10^{-1}
14.0	0.8378×10^{-3}	0.2492×10^{-3}	0.2228×10^{-3}	0.2111×10^{-2}
16.0	0.1077×10^{-3}	0.4853×10^{-4}	0.4807×10^{-4}	0.4431×10^{-3}
18.0	0.2026×10^{-4}	0.1290×10^{-4}	0.1288×10^{-4}	0.1165×10^{-3}
20.0	0.4993×10^{-5}	0.3926×10^{-5}	0.3926×10^{-5}	0.3450×10^{-4}
22.0	0.1476×10^{-5}	0.1304×10^{-5}	0.1304×10^{-5}	0.1101×10^{-4}
24.0	0.4903×10^{-6}	0.4599×10^{-6}	0.4599×10^{-6}	0.3701×10^{-5}

the output of the square-law detector for symbol i on the jth symbol of the repetition code, $1 \leq j \leq L, 0 \leq i \leq M - 1$. When the input code symbol X_j takes the value l and the jammer is present, the density function of Y_{ij} is given by [18]

$$p(y_{i,j}|X_j=l)=\begin{cases} f_1(y_{i,j}, \Lambda), & y_{i,j}\geq 0,\ i=l \\ g_1(y_{i,j}), & y_{i,j}\geq 0,\ i\neq l \\ 0, & y_{i,j}<0 \end{cases} \tag{28}$$

where

$$f_L(x, y)=\left(\frac{x}{yL}\right)^{(L-1)/2} e^{-(x+yL)} I_{L-1}(\sqrt{4xyL}),$$

$$g_L(x)=x^{L-1}e^{-x}/(L-1)!,$$

$\Lambda = E_\rho/N_J$, and I_k is the modified Bessel function of order k. In obtaining (28) we have normalized Y_{ij} in [18] by dividing by $\sqrt{N_J T/2\rho}$. The decoder then computes Z_i, $0 \leq i \leq M - 1$ where

$$Z_i=\sum_{j=1}^{L} Y_{ij} \tag{29}$$

and L is the length of the repetition code or the level of diversity used. The density of Z_i is well known [19] to be given by

$$p(z_i|X_j=l,\ 1\leq j\leq L)=\begin{cases} f_L(y_{i,j}, \Lambda), & z_i\geq 0,\ i=l \\ g_L(y_{i,j}), & z_i\geq 0,\ i\neq l \\ 0, & z_i<0. \end{cases} \tag{30}$$

The decoder then makes a decision that symbol l was transmitted if $Z_l = \max \{Z_i\colon 0\leq i\leq M - 1\}$. The probability of error is then the probability that all L symbols are jammed times the probability of error given that all L symbols are jammed:

$$\begin{aligned} P_{e,M}(L, \rho) &= \rho^L P\{Z_k > Z_l \text{ for some } k\neq l|X_j=l,\ 1\leq j\leq n\} \\ &= \rho^L[1-P\{Z_l> Z_k \text{ for all } k\neq l|X_j=l\}] \\ &= \rho^L\left[1-\int_0^\infty p(z_l|X_j=l)\left(1-\sum_{m=0}^{L-1}\frac{z_l^m}{m!}e^{-z_l}\right)^{M-1} dz_l\right]. \end{aligned} \tag{31}$$

This expression can be evaluated using standard numerical integration techniques to determine $P_{e,M}(L, \rho)$. When $M = 2$, the integral in (31) can be evaluated to yield

$$P_{e,2}(L, \rho)=\frac{1}{2}\rho^L \exp\{-EL\rho/2N_J\}\sum_{i=0}^{L-1}\frac{(EL\rho/2N_J)^i}{i!(L+i-1)!} \cdot \sum_{k=i}^{L-1}\frac{(k+L-1)!}{(k-i)!2^{k+L-1}}. \tag{32}$$

This reduces to the standard result [20] of Marcum when $\rho = 1$ (i.e., an additive white Gaussian noise channel). The error probability against the worst-case jamming strategy can be found by maximizing (31) over ρ. The form of the optimal jamming strategy is for the jammer to have a duty factor ρ that is inversely proportional to the signal-to-noise ratio:

$$\rho^*=\begin{cases} 1 & E/N_J<A_{L,M} \\ \dfrac{A_{L,M}}{E/N_J} & E/N_J\geq A_{L,M}. \end{cases} \tag{33}$$

When $\rho = \rho^*$, the error probability has the form

$$P_{e,M}(L, \rho^*)=\begin{cases} P_{e,M}(L, 1) & E/N_J<A_{L,M} \\ \dfrac{B_{L,M}}{(E/N_J)^L} & E/N_J\geq A_{L,M} \end{cases} \tag{34}$$

where $A_{L,M}$ and $B_{L,M}$ are constants and are given in Table IV for $M = 2$ and $M = 32$ and various values of L.

B. Convolutional Codes

The error probability of convolutional codes can be determined using the union bound and the formulas for the error probability of repetition codes. The error probability of binary convolutional codes is calculated using (7) with P_j replaced by $P_{e,2}(j, \rho)$ given in (32). For dual-k codes we use (14) and (32) with D given by [21]

$$D=\rho\frac{\exp[-w(E\rho/N_J)/(1+w)]}{1-w^2}$$

where

$$w=\frac{\sqrt{(E\rho/N_J)^2+12E\rho/N_J+4}-E\rho/N_J-2}{4}$$

to obtain an upper bound on the error probability. The worst-

TABLE IV
CONSTANTS FOR DETERMINING WORST-CASE ERROR PROBABILITY FOR M-ARY SIGNALING AND PARTIAL-BAND JAMMING

L	M	$A_{L,M}$	$B_{L,M}$
1	2	2.00	0.368
3	2	2.54	0.523
4	2	2.63	0.682
5	2	2.70	0.912
6	2	2.74	1.239
7	2	2.78	1.702
8	2	2.80	2.359
9	2	2.82	3.289
10	2	2.84	4.611
11	2	2.85	6.493
12	2	2.86	9.175
13	2	2.87	13.005
1	32	3.61	1.704
2	32	3.44	3.014
3	32	3.32	5.052
4	32	3.23	8.180
5	32	3.17	1.293×10^1
6	32	3.12	2.007×10^1
7	32	3.08	3.072×10^1
8	32	3.06	4.653×10^1
9	32	3.03	6.985×10^1
10	32	3.01	1.041×10^2

case error probability is calculated by maximizing these expressions over ρ.

C. Numerical Results

We present numerical results for several different codes when binary FSK and 32-ary FSK are used. First we show in Fig. 10 the bit error probability for repetition codes of length 1, 2, 3, 5, 7, and 11 for binary FSK modulation. In Fig. 11 we show the bit error probability of the constraint length 7 binary convolutional codes of rate 1/2 for the cases of hard decisions, no side information; hard decisions, side information available; and soft decisions, side information available. From this we see that for this code, soft decisions are better than hard decisions by about 3 dB when side information is available, and that side information is better than no side information by 4.5 dB when hard decisions are made and the desired error probability is 10^{-5}. In Fig. 12 the symbol error probabilities of repetition codes of length 1, 3, 5 and 7 are shown for 32-ary FSK. In Fig. 13 the bit error probabilities for dual-5 convolutional codes of rate 1/2, 1/4, 1/6, and 1/8 are shown.

V. CONCLUSIONS

The error probability of various coding schemes on channels with partial-band interference has been investigated. Analytical techniques have been used to evaluate the error probability of the codes. New bounds have been found for dual-k convolutional codes, which are significantly better than the previously used bounds. The use of side information in decoding has been investigated to determine the gain achievable when side information is present. In Table V we list the values of E_b/N_J necessary for bit error probability of 10^{-5} for soft decisions with side information and for hard decisions both with and without side information. For Reed–Solomon codes with soft decision, we mean soft decisions when combining the outputs for each diversity transmission and not of the code itself. Thus, for diversity 1, soft decisions and hard decision decoding have the same performance when

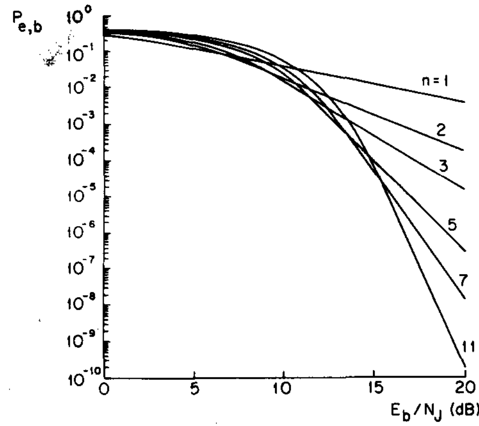

Fig. 10. Bit error probability for binary repetition codes with soft decision decoding with side information and worst case partial-band jamming using binary FSK ($E_b/N_0 = \infty$).

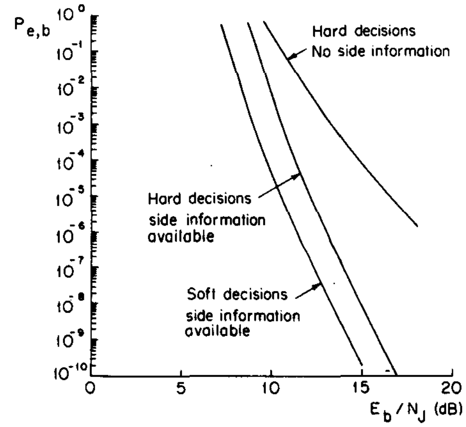

Fig. 11. Bit error probabilities for rate 1/2, constraint length 7 convolutional codes with binary FSK on worst-case partial-band jamming channel.

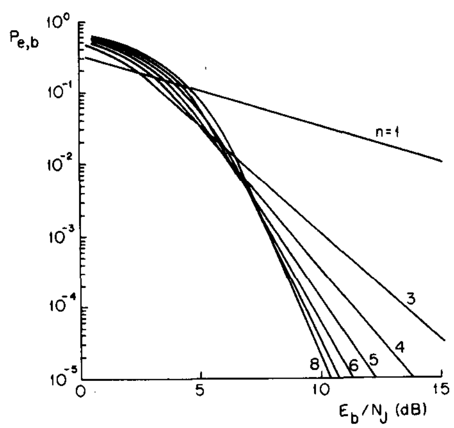

Fig. 12. Symbol error probability for repetition codes on channel with soft decisions and side information on worst-case partial-band jamming channel and 32-ary FSK ($E_b/N_0 = \infty$).

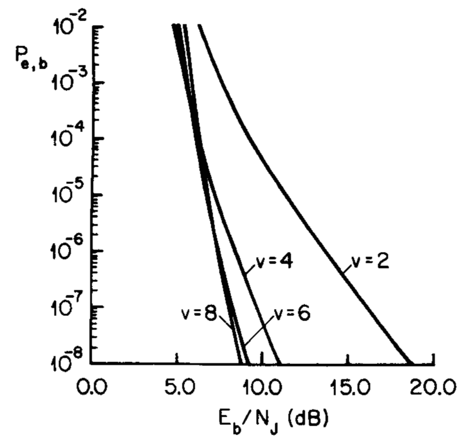

Fig. 13.

TABLE V
SIGNAL-TO-NOISE RATIO REQUIRED FOR BIT ERROR PROBABILITY OF 10^{-5} FOR VARIOUS CODES AND CHANNELS

Code	**Rate (bits/dimension)**	**$(E_b/N_0)_{10^{-6}}$ dB (ρ^*)**		
		Soft Decisions Side Information	**Hard Decisions Side Information**	**Hard Decisions No Side Information**
Uncoded (Binary FSK)	1/2=0.50	45.66 (5.4×10^{-6})	45.66 (5.4×10^{-6})	45.66 (5.4×10^{-6})
K=7, Binary Convolutional	1/4=0.250	10.46 (0.51)	11.96 (0.53)	16.73 (0.09)
K=9, Binary Convolutional	1/4=0.250	9.57 (0.51)	11.04(0.66)	14.52 (0.14)
Binary repetition code, n=3	1/6=0.167	20.50 (.068)	21.34 (.069)	27.81 (9.93×10^{-8})
Binary repetition code, n=5	1/10=0.100	16.91 (.275)	17.95 (.280)	22.63 (.055)
Binary repetition code, n=7	1/14=0.071	15.92 (.496)	17.05 (.509)	20.91 (.127)
Binary repetition code, n=9	1/18=.055	15.67 (.687)	16.85 (.707)	19.28 (.212)
Binary repetition code, n=11	1/22=0.045	15.70 (.844)	16.91 (.869)	18.67 (.299)
Binary repetition code, n=13	1/26=0.038	15.84 (.972)	17.07 (1.00)	18.32 (.382)
M-ary repetition code, n=1	5/32=0.156	42.45 (4.11×10^{-6})	42.45(4.11×10^{-6})	42.45 (4.11×10^{-6})
M-ary repetition code, n=3	5/96=0.0521	15.84 (5.20×10^{-8})	17.89 (8.36×10^{-8})	25.20 (6.55×10^{-6})
M-ary repetition code, n=5	5/160=0.0312	11.65 (0.22)	13.56 (0.21)	17.23 (.068)
M-ary repetition code, n=7	5/224=0.022	10.32 (0.40)	12.27 (0.28)	14.36 (.185)
Dual-5 Rate 1/2	5/64=0.0781	11.40(0.077)	13.86(0.067)	18.41(0.021)
Dual-5 Rate 1/3	5/96=0.0521	8.41(0.240)	10.73(0.212)	12.97(0.109)
Dual-5 Rate 1/4	5/128=0.0391	7.35(0.415)	9.60(0.373)	10.99(0.230)
Dual-5 Rate 1/5	5/160=0.0312	6.94(0.576)	9.16(0.521)	10.11(0.352)
Dual-5 Rate 1/6	5/192=0.0260	6.83(0.716)	9.01(0.650)	9.68(0.466)
Dual-5 Rate 1/7	5/224=0.0223	6.85(0.835)	9.02(0.760)	9.49(0.568)
Dual-5 Rate 1/8	5/356=0.0195	6.95(0.93)	9.11(0.855)	9.43(0.659)
Dual-5 Rate 1/9	5/288=0.0174	7.09(1.000)	9.23(0.938)	9.44(0.739)
Dual-5 Rate 1/10	5/320=0.0156	7.23(1.000)	9.39(1.000)	9.50(0.809)
(32,8) RS code diversity 1	40/32/32=0.039062	9.02(0.568)	9.02(0.568)	10.49(0.258)
(32,16) RS code diversity 2	40/32/32=0.039062	7.30(0.627)	9.50(0.640)	13.65(0.125)
(32,24) RS code diversity 2	40/32/32=0.039062	7.26(0.567)	8.24(0.650)	11.85(0.189)

side information is present. From this table and other numerical results presented here, we have the following general conclusions about the worst-case partial-band jamming channel. When side information is available, the difference between hard and soft decisions is between 1 and 3 dB. The difference between hard decisions with and without side information varies quite a bit. For low rate codes the difference is very low, while for high rate codes the difference between decoders with and those without side information can be considerable (on the order of 6 dB or more). For fixed rate codes, the difference between side information and no side information decreases as the length of the code increases. These conclusions agree with those in [18], based on capacity and cutoff rate as the performance measure. In Table V we also list the minimum required signal-to-noise ratio necessary for reliable communications for various code rates considered previously. We also list the minimum required signal-to-noise ratio for a Reed–Solomon coded system which also employs diversity (see [6]). As a comparison, the minimum signal-to-noise ratio to achieve capacity for soft decisions with side information is 6.72 dB at rate 0.24 bits per dimension. Binary convolutional codes of rate 1/2 achieve error probability 10^{-5} with signal-to-noise ratio of 9.57 dB and rate of 0.25 bits per dimension. For binary FSK with hard decisions and side information available, the capacity is achieved with signal-to-noise ratio of 7.82 dB at rate 0.250 bits per dimension, while binary convolutional codes require 11.96 dB at the same rate. For hard decisions without side information, the signal-to-noise ratio for capacity is 7.98 dB, while convolutional codes require 16.73 dB. As can be seen, binary convolutional codes are about 2 dB away from capacity with soft decisions and side information, and more than 8 dB away from capacity with hard decisions and no side information.

The models used in this paper are quite general and can be used to evaluate the error probability of codes in many other systems, such as a frequency-hopped multiple-access system where the interference results from signals of two or more users occupying the same frequency at the same time. In that case the two-state model of Section II would be appropriate, with the bad state being a hit from one or more other users. The analysis can also be easily extended to include fading channels.

Appendix A

Error Probability for Repetition Codes on M-ary Symmetric Channels

In this Appendix we derive the error probability for repetition codes on an M-ary symmetric channel (MSC). Let p be the probability of a symbol error on an MSC and let $q = 1 - p$. Also let X represent the input to the channel ($X \in \{0, 1, \cdots, M-1\}$) and Y the output. Then the probability that $Y = y$ given that $X = x$ is given by

$$P\{Y=y|X=x\} = \begin{cases} q, & x=y \\ p/(M-1), & x\neq y. \end{cases}$$

Assume the information symbol to be transmitted is $X = 0$. The repetition code sends this symbol n times. The decoder counts the number of times each symbol was received and chooses the one that had the largest count, as the transmitted symbol. Let Y_i, $0 \le i \le M-1$, be the number of times that symbol i was received. For $n = 1$ the symbol error probability $P_{e,s}$ (1) is just p; for $n = 2$ the error probability $P_{e,s}$ (2) can be computed by considering the probability of correct decision $P_{c,s}$:

$$P_{c,s}(n) = 1 - P_{e,s}(n).$$

This can be computed as

$$P_{c,s}(n) = P\{Y_0 = 2\} + P\{Y_0 = 1,\ Y_j = 1,\ \text{some } j \neq 0\}/2.$$

The first term is the probability that both symbols transmitted were received correctly and is equal to $(1-p)^2$. The second term is the probability that a tie occurred, which is decided randomly between $X = 0$ and $X = j$. This is given by $(1-p)p$ so that

$$P_{c,s}(2) = (1-p)^2 + p(1-p) = 1-p.$$

For $n = 3$ we have the probability of correctly decoding $P_{c,s}$ (3) given by

$$\begin{aligned} P_{c,s}(3) &= P\{Y_0 \geq 2\} \\ &\quad + 1/3P\{Y_0=1,\ Y_i=1,\ Y_j=1,\ i\neq 0,\ j\neq 0,\ i\neq j\} \\ &= q^3 + 3pq^2 + qp\left(\frac{M-2}{M-1}p\right) \\ &= q^3 + 3pq^2 + \left(\frac{M-2}{M-1}\right)p^2q. \end{aligned}$$

For $n = 4$ we have

$$\begin{aligned} P_{c,s}(4) &= q^4 + 4q^3p + 6q^2p\left(\frac{M-2}{M-1}p\right) \\ &\quad + 3q^2p\frac{p}{M-1} + qp\left(\frac{M-2}{M-1}p\right)\left(\frac{M-3}{M-1}p\right) \\ &= q^4 + 4q^3p + \left[6\left(\frac{M-2}{M-1}\right) + 3\left(\frac{1}{M-1}\right)\right]q^2p^2 \\ &\quad + \frac{(M-2)(M-3)}{(M-1)^2}qp^3. \end{aligned}$$

For $n = 5$

$$\begin{aligned} P_{c,s}(5) &= q^5 + 5q^4p + 10q^3p^2 + \left[10\frac{(M-2)(M-3)}{(M-1)^2}\right. \\ &\quad \left. + 15\frac{M-2}{(M-1)^2}\right]q^2p^3 + \frac{(M-2)(M-3)(M-4)}{(M-1)^3}qp^4. \end{aligned}$$

For $n = 6$

$$\begin{aligned} P_{c,s}(6) &= q^6 + 6q^5p + 15q^4p^2 + \left[60\frac{(M-2)}{(M-1)^2}\right. \\ &\quad \left. + 20\frac{(M-2)(M-3)}{(M-1)^2} + \frac{10}{(M-1)^2}\right]q^3p^3 \\ &\quad + \left[15\frac{(M-2)(M-3)(M-4)}{(M-1)^3} + 45\frac{(M-2)(M-3)}{(M-1)^3}\right. \\ &\quad \left. + 15\frac{M-2}{(M-1)^3}\right]q^2p^4 \\ &\quad + \frac{(M-2)(M-3)(M-4)(M-5)}{(M-1)^4}qp^5. \end{aligned}$$

Notice in each case that $P_{c,s}(n)$ is expressed as

$$P_{c,s}(n)=\sum_{i=0}^{n-1} a_i q^{n-i} p^i.$$

For $n = 7$ the coefficients are

i	a_i
0	1
1	7
2	21
3	35
4	$35\left[6\frac{(M-2)(M-3)}{(M-1)^3}+\frac{(M-2)(M-3)(M-4)}{(M-1)^3}+5\frac{(M-2)}{(M-1)^3}\right]$
5	$21\left[\frac{(M-2)(M-3)(M-4)(M-5)}{(M-1)^4}+5\frac{(M-2)(M-3)(M-4)}{(M-1)^4}+5\frac{(M-2)(M-3)}{(M-1)^4}\right]$
6	$\frac{(M-2)(M-3)(M-4)(M-5)(M-6)}{(M-1)^5}$.

For any M and n with $p = (M-1)/M$, $P_{e,s}(n) = (M-1)/M$. We can calculate an upper bound on $P_{e,s}(n)$ for any n by applying the union bound technique. This is calculated as follows:

$$P_{e,s}(n)\le(M-1)P_{e,s}^{(2)}(n)$$

where $P_{e,s}^{(2)}(n)$ is the error probability between two codewords of a repetition code of length n. Assume without loss of generality that the two codewords are the all zero's codeword and the all one's codeword. Then the probability of error for this code with two codewords is

$$P_{e,s}^{(2)}(n)=P\{Y_0>Y_1\}+\frac{1}{2}P\{Y_0=Y_1\}$$

$$P_{e,s}^{(2)}(n)=\sum_{\substack{j,k\\ j<k\\ j+k\le n}}\binom{n}{j,\,k}(1-p)^j\cdot\left(\frac{p}{M-1}\right)^k\left(\frac{M-2}{M-1}p\right)^{n-j-k}+1/2\sum_{j=0}^{[n/2]}\binom{n}{j,\,j}(1-p)^j\cdot\left(\frac{p}{M-1}\right)^j\left(\frac{M-2}{M-1}p\right)^{n-2j}.$$

A simpler bound can be obtained by using the Bhattacharyya bound on $P_{e,s}^{(2)}(n)$. This bound is

$$P_{e,s}^{(2)}(n)\le D^n$$

where D is given as

$$D=\left(\frac{M-2}{M-1}\right)p+2\sqrt{p(1-p)/M-1}\ .$$

Appendix B

Repetition Codes on Two-State Channels

Consider a two-state channel described below. The channel when the state is 0 is a binary symmetric channel with bit error probability p_0. The channel when the state is 1 is a binary symmetric channel with bit error probability p_1. A sequence of channel symbols $x_1, x_2, \cdots, x_n$ is sent over a sequence of channels. The state of the channel at any time j, S_j, is a random variable. The sequence $\{S_j\}_{j=-\infty}^{\infty}$ is a sequence of i.i.d. random variables with $P\{S_n = 1\} = \rho$ and $P\{S_n = 0\} = 1-\rho$. A repetition code is used on this channel. When side information is not available, the overall channel is a BSC with error probability $\rho p_1 + (1-\rho)p_0$, and the error probability of a repetition code is a straightforward calculation. With side information the maximum likelihood decoding rule is calculated as follows. Let d_0 = number of 0's received on channel 0, d_1 = number of 0's received on channel 1, l = number of times out of n that channel 1 was used,

$$p(y,\ s|x=(111\cdots 11))=\rho^l(1-\rho)^{n-l}p_1^{d_1}(1-p_1)^{l-d_1}p_0^{d_0}(1-p_0)^{n-l-d_0}$$

$$p(y,\ s|x=(000\cdots 00))=\rho^l(1-\rho)^{n-l}(1-p_1)^{d_1}p_1^{l-d_1}(1-p_0)^{d_0}p_0^{n-l-d_0}.$$

Assume first that $p_0 \neq 1/2$ and $p_1 > p_0$. The maximum likelihood decision rule is then

$$\Lambda \triangleq \frac{p(y,\ s|x=(111\cdots 11))}{p(y,\ s|x=(000\cdots 00))}=\left(\frac{p_1}{1-p_1}\right)^{2d_1-l}\left(\frac{p_0}{1-p_0}\right)^{2d_0-(n-l)}\overset{1}{\underset{0}{\gtrless}}1$$

or

$$\delta\triangleq d_1\ \ln\frac{p_1}{1-p_1}+d_0\ \ln\frac{p_0}{1-p_0}\overset{1}{\underset{0}{\gtrless}}\frac{l}{2}\ \ln\frac{p_1}{1-p_1}+\frac{n-l}{2}\ \ln\frac{p_0}{1-p_0}.$$

An error occurs if 0 is transmitted and

$$\ln\Lambda>0$$

or

$$\delta>\alpha_l\triangleq\frac{l}{2}\ \ln\frac{p_1}{1-p_1}+\frac{n-l}{2}\ \ln\frac{p_0}{1-p_0}.$$

The probability of error P_e is then given by

$$P_e = \sum_{l=0}^{n} \binom{n}{l} \rho^l (1-\rho)^{n-l} \cdot \left[\sum_{\delta > \alpha_l} \binom{l}{d_1} \binom{n-l}{d_0} \cdot (1-p_1)^{d_1} p_1^{\,l-d_1} (1-p_0)^{d_0} p_0^{\,n-l-d_0} + \frac{1}{2} \sum_{\delta = \alpha_l} \binom{l}{d_1} \binom{n-l}{d_0} (1-p_1)^{d_1} p_1^{\,l-d_1} \cdot (1-p_0)^{d_0} p_0^{\,n-l-d_0} \right].$$

If $p_0 = 0$ or $p_1 = 1/2$ then the decision rule becomes

$$2d_0 - (n-l) \underset{1}{\overset{0}{\gtrless}} 0 \qquad l \neq n$$

$$2d_1 - l \underset{1}{\overset{0}{\gtrless}} 0 \qquad l = n.$$

That is, if $l < n$, examine only the $n - l$ bits that are received over channel 0. If there are more than half 0's then decide 0, otherwise decide 1. If $l = n$, then decide 1 if there are more 1's received than 0's.

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Wayne E. Stark (S'77–M'78), for a photograph and biography, see p. 774 of the August 1985 issue of this TRANSACTIONS.