Error Probability for Direct-Sequence Spread-Spectrum Multiple-Access Communications—Part I: Upper and Lower Bounds

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Abstract—**Upper and lower bounds on the average probability of error are obtained for direct-sequence spread-spectrum multiple-access communications systems with additive white Gaussian noise channels. The bounds, which are developed from convexity properties of the error probability function, are valid for systems in which the maximum multiple-access interference does not exceed the desired signal and the signature sequence period is equal to the duration of the data pulse. The tightness of the bounds is examined for systems with a small number of simultaneously active transmitters. This is accomplished by comparisons of the upper and lower bounds for several values of the system parameters. The bounds are also compared with an approximation based on the signal-to-noise ratio and with the Chernoff upper bound.**

INTRODUCTION

DURING the past few years there has been considerable interest in efficient methods for obtaining approximations and bounds for the average probability of error in asynchronous direct-sequence spread-spectrum multiple-access (SSMA) communications systems. Among the published contributions to this problem are the approximation based on the signal-to-noise ratio (SNR) [6, p. 798], the bounds based on moment-space techniques [14], approximations based on series-expansion methods and Gauss quadrature rules ([5] and [13]), approximations based on the integration of the characteristic function [3] (see also [4]), and our preliminary versions of the bounds obtained in the present paper ([1] and [11]). Each of the proposed methods has its advantages and disadvantages, and the choice of method for a given application ultimately depends on the system parameters, the required accuracy, and the available computing equipment. Some of the methods require fairly sophisticated computer software (e.g., [14]), while others are very easy to apply. In particular, the approximation based on the SNR can be evaluated from the tabulated correlation parameters (e.g., [2], [10], and [12]) without the use of a computer.

One of the key issues is whether a *bound* on the error probability is required or an *approximation* will suffice. Generally speaking, it is much easier to obtain an approximation than a bound, but upper and lower bounds together supply more information. For instance, in order to guarantee that a particular error rate specification is or is not attainable for a given set of system parameters, bounds on the probability of error are required. Moreover, any bound is also an approximation, and an upper and a lower bound together furnish not only an approximation but also a bound on the resulting error in the approximation.

The present paper is devoted to *bounds* on the average probability of error. These bounds are conceptually simpler than the moment-space bounds given in [14] and we have found that they are easier to evaluate for small to moderate values of K, the number of simultaneously active transmitters. The relative simplicity of the bounds we present is evident from the fact that the paper gives all of the necessary details to enable the reader to compute these bounds. The evaluation of our bounds does not require the determination of convex hulls, the solution of sets of nonlinear equations, or the computation of high-order moments of the multiple-access interference. Moreover, the numerical results presented in [1] show that our bounds are much tighter than the second-moment bound of [14], and we have found that the improved bounds given in the present paper are also tighter than the single-exponential bounds of [14].

The main disadvantage with the bounds presented in this paper is that the computational requirements increase exponentially in K. Thus, these bounds are not suitable for systems with a large number of simultaneously active transmitters. However, they are suitable for packet radio systems and other applications involving bursty data transmission, and they are also suitable for hybrid frequency-hopped direct-sequence SSMA systems. It should be noted that for SSMA systems with relatively few chips per bit (e.g., 31), the number of simultaneously active transmitters must necessarily be small in order to achieve satisfactory performance.

As shown in [8], the bounds developed in this paper can also be modified to give bounds on the probability of error for spread-spectrum communications over certain specular multipath channels. The multipath interference is handled in the same way as the multiple-access interference is handled in the present paper. The main change that must be made is that crosscorrelations are replaced by autocorrelations.

SYSTEM MODEL

This paper is concerned with bounds on the average probability of error for an asynchronous binary direct-sequence

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spread-spectrum multiple-access (SSMA) communications system with an additive white Gaussian noise channel. The model that is employed in the present paper is taken from [6]. This model has been used in most of the recent performance analyses for asynchronous binary direct-sequence SSMA communications (e.g., [1], [3], [11], [13], and [14]), and it has been generalized [7] to provide a model for asynchronous quaternary direct-sequence SSMA (e.g., [4], [5], and [9]). Although some of the results that we obtain here can be extended in a straightforward manner to quaternary systems, we restrict attention to binary systems throughout the paper.

The received signal in the asynchronous binary direct-sequence SSMA system is the sum of K spread-spectrum signals $s_k(t-\tau_k)$, $1 \leqslant k \leqslant K$, plus an additive white Gaussian noise process $n(t)$ which has (two-sided) spectral density $\frac{1}{2}N_0$. For the model of [6], the spread-spectrum signal $s_k(t-\tau_k)$ is given by

$$s_k(t-\tau_k) = \sqrt{2P}a_k(t-\tau_k)b_k(t-\tau_k)\cos(\omega_c t + \varphi_k) \quad (1)$$

where $a_k(\cdot)$ is the code waveform, $b_k(\cdot)$ is the data signal, τ_k is a time-delay parameter which accounts for propagation delay and the lack of synchronism between the signals, and φ_k is the phase angle for the kth carrier (the time delay for the carrier has been absorbed in φ_k). The reader is referred to [6] for a detailed description of the code and data waveform. Basically, $a_k(t)$ is a periodic infinite sequence of nonoverlapping rectangular pulses which are called code pulses or chips. Each code pulse has duration T_c. The amplitude of the nth pulse is $a_n^{(k)}$, where $a_n^{(k)}$ is $+1$ or -1 for each n and where $(a_n^{(k)}) = \cdots, a_{-1}^{(k)}, a_0^{(k)}, a_1^{(k)}, \cdots$ is a periodic sequence with period p. The data signal $b_k(t)$ is a sequence of nonoverlapping rectangular pulses, each of which has duration T. The amplitude of the lth pulse is denoted by $b_l^{(k)}$. We assume that there are exactly N full code pulses in each data pulse, and therefore it must be that $T = NT_c$. We also assume that N is a integer multiple of p. Several of the properties of the code and data waveforms are summarized in a compact form by $a_k(t) = a_j^{(k)}$ for $jT_c \leqslant t < (j+1)T_c$ and $b_k(t) = b_l^{(k)}$ for $lT \leqslant t < (l+1)T$.

In this paper we are concerned with the *average* probability of error. Consequently, the parameters $b_l^{(k)}$, τ_k, and φ_k are treated as random variables. We assume that the collection of all of these parameters (i.e., for $-\infty < l < \infty$ and $1 \leqslant k \leqslant K$) is a set of mutually independent random variables and that $P(b_l^{(k)} = +1) = P(b_l^{(k)} = -1) = \frac{1}{2}$ for each l and k. From these basic assumptions and certain properties of the SSMA system we can draw the following conclusions. First, because of the symmetry of the problem we may restrict attention to the output of the receiver for signal $s_1(t-\tau_1)$. Second, since only relative time delays and phase angles are important, we may set $\tau_1 = \varphi_1 = 0$. The parameters τ_k and φ_k are then the time delay and phase angle for the kth signal relative to the first. Third, the properties of an SSMA system and the stationarity of the noise $n(t)$ permit us to consider only time delays modulo T and phase angles modulo 2π, rather than the absolute values of these parameters.

If $b_0^{(1)}$ is the data bit for the first signal during the interval $[0, T]$, then the output of a correlation receiver matched to the first signal is the random variable

$$Z = \eta + (\tfrac{1}{2}P)^{1/2}T\left\{b_0^{(1)} + \sum_{k=2}^{K} I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k)\right\} \quad (2)$$

where $\boldsymbol{b}_k = (b_{-1}^{(k)}, b_0^{(k)})$ and the channel noise component η is given by

$$\eta = \int_0^T n(t)a_1(t)\cos\omega_c t\,dt. \quad (3)$$

The multiple-access interference $I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k)$ which appears in (2) is defined in terms of the continuous-time partial crosscorrelation functions

$$R_{k,i}(\tau) = \int_0^\tau a_k(t-\tau)a_i(t)\,dt, \qquad 0 \leqslant \tau \leqslant T \quad (4a)$$

and

$$\hat{R}_{k,i}(\tau) = \int_\tau^T a_k(t-\tau)a_i(t)\,dt, \qquad 0 \leqslant \tau \leqslant T \quad (4b)$$

which are defined in [6]. For the present analysis we need consider only $i = 1$ and $k > 1$. It is shown in [6] and [7] that the multiple-access interference component is given by

$$I_{k,1}(\boldsymbol{b}_k, \tau, \varphi) = T^{-1}[b_{-1}^{(k)}R_{k,1}(\tau) + b_0^{(k)}\hat{R}_{k,1}(\tau)]\cos\varphi \quad (5)$$

for $0 \leqslant \tau < T$ and $0 \leqslant \varphi < 2\pi$, and that for *rectangular* chip waveforms the continuous-time partial crosscorrelation functions are given by

$$R_{k,i}(\tau) = C_{k,i}(l-N)T_c + [C_{k,i}(l+1-N) - C_{k,i}(l-N)](\tau - lT_c) \quad (6a)$$

and

$$\hat{R}_{k,i}(\tau) = C_{k,i}(l)T_c + [C_{k,i}(l+1) - C_{k,i}(l)](\tau - lT_c) \quad (6b)$$

where $l = \lfloor \tau/T_c \rfloor$ and where $C_{k,i}$ is the aperiodic crosscorrelation function which is defined by

$$C_{k,i}(l) = \begin{cases} \sum_{j=0}^{N-1-l} a_j^{(k)}a_{j+l}^{(i)}, & 0 \leqslant l \leqslant N-1, \quad (7a) \\ \sum_{j=0}^{N-1+l} a_{j-l}^{(k)}a_j^{(i)}, & 1-N \leqslant l < 0 \quad (7b) \end{cases}$$

and $C_{k,i}(l) = 0$ for $|l| \geqslant N$. Notice that from (5)-(7) the multiple-access interference is a linear function of τ on the interval $[lT_c, (l+1)T_c]$ provided the chip waveform is a rectangular pulse. Furthermore, it depends upon φ only through the term $\cos \varphi$, a property which holds for other types of chip waveforms as well [7], [9].

AVERAGE PROBABILITY OF ERROR

The receiver which is matched to the first signal produces the output Z at time T. This receiver is not optimum for making a decision on the data symbol $b_0^{(1)}$ (i.e., Z is not a sufficient statistic), since the total interference is not a white Gaussian noise process. However, this type of receiver is employed in nearly all direct-sequence spread-spectrum systems. It is relatively easy to implement and we believe that its performance is very close to that of the optimal receiver (at least for large N).

The actual bit decision is made as follows. If $Z \geqslant 0$ the decision is that $b_0^{(1)} = +1$; otherwise, the decision is that $b_0^{(1)} = -1$. Thus, an error occurs if $Z \geqslant 0$ when in fact $b_0^{(1)} = -1$ or if $Z < 0$ when $b_0^{(1)} = +1$. Because of the symmetry of the problem, these two types of errors occur with equal probability. Thus, we may assume in all that follows that $b_0^{(1)} = +1$; that is, a positive data pulse is sent by the first transmitter during the time interval $[0, T]$.

The conditional probability of error given that $b_0^{(1)} = +1$ is a function of the data symbols $\boldsymbol{b} = (\boldsymbol{b}_2, \boldsymbol{b}_3, \cdots, \boldsymbol{b}_K)$, the delays $\boldsymbol{\tau} = (\tau_2, \tau_3, \cdots, \tau_K)$, and the phase angles $\boldsymbol{\varphi} = (\varphi_2, \varphi_3, \cdots, \varphi_K)$. Since η is a zero-mean Gaussian random variable with variance $\frac{1}{4}N_0T$, the conditional probability of error for a given $\boldsymbol{b}, \boldsymbol{\tau}$, and $\boldsymbol{\varphi}$ is

$$P_{e,1}(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}) = Q\left(\alpha\left[1 + \sum_{k=2}^{K} I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k)\right]\right) \tag{8}$$

where the function Q is defined by

$$Q(y) = (2\pi)^{-1/2} \int_y^\infty e^{-(1/2)x^2}\, dx$$

and the parameter α is given by $\alpha = (2PT/N_0)^{1/2}$. Notice that if there is no multiple-access interference (e.g., if $K = 1$), then the error probability is just

$$Q(\alpha) = Q([2PT/N_0]^{1/2}) = Q([2E_b/N_0]^{1/2})$$

where E_b is the energy per data bit.

The *average probability of error* $\bar{P}_{e,1}$ is the expected value of $P_{e,1}(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi})$. Assuming that the time delays (modulo T) are uniformly distributed on $[0, T]$ and the phase angles (modulo 2π) are uniformly distributed on $[0, 2\pi]$, then

$$\bar{P}_{e,1} = (8\pi T)^{1-K} \Sigma_{\boldsymbol{b}} \iint P_{e,1}(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi})\, d\boldsymbol{\tau}\, d\boldsymbol{\varphi} \tag{9}$$

where $\Sigma_{\boldsymbol{b}}$ denotes the sum over all $\boldsymbol{b} = (\boldsymbol{b}_2, \boldsymbol{b}_3, \cdots, \boldsymbol{b}_K)$ such that $\boldsymbol{b}_k = (b_{-1}^{(k)}, b_0^{(k)})$ with $b_l^{(k)} \in \{-1, +1\}$.

Although $\int d\boldsymbol{\tau}$ is a multidimensional integral over $[0, T]^{K-1}$ and $\int d\boldsymbol{\varphi}$ is a multidimensional integral over $[0, 2\pi]^{K-1}$, each of these integrals can be replaced by a sequence of one-dimensional integrals. This sequence can be bounded recursively to provide bounds on $\bar{P}_{e,1}$.

First, we define

$$G_2(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}) = Q\left(\alpha\left[1 + \sum_{k=2}^{K} I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k)\right]\right). \tag{10}$$

Next, for $2 \leqslant n \leqslant K$ let

$$G_{n+1}(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}) = (8\pi T)^{-1} \Sigma_{\boldsymbol{b}_n} \int_0^{2\pi} \int_0^T G_n(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi})\, d\tau_n\, d\varphi_n \tag{11}$$

where $\Sigma_{\boldsymbol{b}_n}$ denotes the sum over all $\boldsymbol{b}_n \in \{-1, +1\}^2$. Notice that $G_n(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ depends on $\boldsymbol{b}_k$, τ_k, and φ_k only if $k \geqslant n$. In particular, $G_{K+1}(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ does not depend on $\boldsymbol{b}$, $\boldsymbol{\tau}$, and $\boldsymbol{\varphi}$ at all. Indeed, we see from (9) that

$$G_{K+1}(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}) = \bar{P}_{e,1}. \tag{12}$$

The bounds presented in this paper depend primarily on the fact that $Q(x)$ is convex for $x \geqslant 0$. As a result, the bounds are valid for spread-spectrum multiple-access systems for which

$$\sum_{k=2}^{K} I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k) \geqslant -1 \tag{13}$$

for all $\boldsymbol{b}$, $\boldsymbol{\tau}$, and $\boldsymbol{\varphi}$. Because of symmetry properties of the multiple-access interference this condition is equivalent to

$$\left|\sum_{k=2}^{K} I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k)\right| \leqslant 1 \tag{14}$$

for all $\boldsymbol{b}, \boldsymbol{\tau}$, and $\boldsymbol{\varphi}$, which is the requirement that the maximum multiple-access interference must be less than the desired signal component at the output of the correlation receiver. This restriction is imposed in all that follows.

The symmetry property mentioned above is due to the following relationships:

$$I_{k,1}(-\boldsymbol{b}_k, \tau_k, \varphi_k) = -I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k), \tag{15a}$$

$$I_{k,1}(\boldsymbol{b}_k, \tau_k, 2\pi - \varphi_k) = I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k), \tag{15b}$$

and

$$I_{k,1}(\boldsymbol{b}_k, \tau_k, \pi - \varphi_k) = -I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k). \tag{15c}$$

Because of these properties we can replace (11) by

$$G_{n+1}(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}) = (2\pi T)^{-1} \Sigma_{\boldsymbol{b}_n} \int_0^{(1/2)\pi} \int_0^T G_n(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi})\, d\tau_n\, d\varphi_n. \tag{16}$$

LOWER BOUNDS FOR THE AVERAGE PROBABILITY OF ERROR

The first step in obtaining a lower bound on $\bar{P}_{e,1}$ is to consider (16) with $n = 2$ and develop a lower bound $G_3{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ for $G_3(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$. The bound $G_3{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ must have the property that it is suitable for use in (16). This limits the types of lower bounds that can be considered. The next step is to obtain a suitable lower bound $G_4{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ for $G_4(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$, and so on. The approach that we develop gives a bound $G_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ on $G_n(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ with the property that the dependence of $G_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ on b_n, τ_n, and φ_n is of the same form for each n. Hence, the bound on $\bar{P}_{e,1}$ is derived from a sequence of essentially identical bounds on $G_n(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ for $n = 3, 4, \cdots, K + 1$.

The sequence is set up as follows. Let $G_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ be given and define

$$\hat{G}_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi}) = (4T)^{-1} \Sigma_{b_n} \int_0^T G_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})\, d\tau_n. \tag{17}$$

Since $G_n(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi}) \geq G_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ for each $\boldsymbol{b}$, $\boldsymbol{\tau}$, and $\boldsymbol{\varphi}$, then

$$G_{n+1}(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi}) \geq (2/\pi) \int_0^{(1/2)\pi} \hat{G}_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})\, d\varphi_n. \tag{18}$$

In order to obtain $G_{n+1}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ we make use of the fact that for any positive integer J

$$\hat{G}_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi}) = (4T)^{-1} \Sigma_{b_n} \sum_{l=0}^{N-1} \sum_{j=0}^{J-1} \int_{\Delta(l,j)}^{\Delta(l,j+1)} G_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})\, d\tau_n \tag{19}$$

where $\Delta(l, i) = (l + J^{-1} i) T_c$ for $0 \leq l < N$ and $0 \leq i \leq J$, and for any positive integer M

$$G_{n+1}(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi}) \geq (2/\pi) \sum_{m=0}^{M-1} \int_{\psi(m)}^{\psi(m+1)} \hat{G}_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})\, d\varphi_n \tag{20}$$

where $\psi(m) = \frac{1}{2} m\pi/M$ for $0 \leq m \leq M$. We then derive $G_{n+1}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ by obtaining lower bounds on the integrals of (19) and (20). These lower bounds are presented in Appendix A.

The lower bound on $\bar{P}_{e,1}$ is in terms of the interference spectrum $S_k{}^L(i)$ defined as follows. Let $\mathcal{S}_k{}^L(i)$ be the set

$$\mathcal{S}_k{}^L(i) = \{(b_k, l, j): 2JNI_{k,1}(b_k, \bar{\Delta}(l, j), 0) = i\} \tag{21}$$

where

$$\bar{\Delta}(l, j) = [l + J^{-1}(j + \tfrac{1}{2})]\, T_c \tag{22}$$

for $0 \leq l < N$ and $0 \leq j < J$. Let $S_k{}^L(i) = |\mathcal{S}_k{}^L(i)|$, the cardinality of the set $\mathcal{S}_k{}^L(i)$. Notice that $S_k{}^L(i) = 0$ for $|i| > 2JN$, and that the symmetry properties (i.e., (15)) of the interference function $I_{k,1}$ imply that $S_k{}^L(i) = S_k{}^L(-i)$. Next, for $0 \leq m < M$ define

$$\gamma(m) = M(\pi JN)^{-1} \{\sin[(m+1)\pi/2M] - \sin[m\pi/2M]\}. \tag{23}$$

Finally, let $\Sigma_{\boldsymbol{i}}$ denote the sum over all $\boldsymbol{i} = (i_2, i_3, \cdots, i_K)$ such that $|i_k| \leq 2JN$ for $1 < k \leq K$, and let $\Sigma_{\boldsymbol{m}}$ denote the sum over all $\boldsymbol{m} = (m_2, m_3, \cdots, m_K)$ such that $0 \leq m_k < M$ for $1 < k \leq K$.

The lower bound is then

$$\bar{P}_{e,1} \geq (4JNM)^{1-K} \Sigma_{\boldsymbol{i}} \left\{ \prod_{n=2}^{K} S_n{}^L(i_n) \right\} \cdot \Sigma_{\boldsymbol{m}} Q\left(\alpha \left[1 + \sum_{k=2}^{K} i_k \gamma(m_k) \right] \right). \tag{24}$$

This bound is valid for each choice of the positive integers J and M. Larger values of J and M give tighter bounds at the expense of increased computation. As pointed out in Appendix A, the right-hand side of (24) can also be viewed as an approximation to $\bar{P}_{e,1}$, which is obtained from an application of a rectangular integration rule to the integrals of (19) and (20). Hence, the difference between $\bar{P}_{e,1}$ and the lower bound of (24) converges to zero as $J \to \infty$ and $M \to \infty$. Moreover, the approach that we have taken guarantees the convergence is monotonic.

UPPER BOUNDS FOR THE AVERAGE PROBABILITY OF ERROR

In order to obtain an upper bound on $\bar{P}_{e,1}$ we use a procedure which is analogous to that developed in the previous section. This amounts to finding a sequence of upper bounds $G_n{}^U(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ on the quantities $G_n(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ for $n = 3, 4, \cdots, K + 1$. Thus, (17) and (18) are replaced by

$$\hat{G}_n{}^U(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi}) = (4T)^{-1} \Sigma_{b_n} \int_0^T G_n{}^U(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})\, d\tau_n \tag{25}$$

and

$$G_{n+1}(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi}) \leq (2/\pi) \int_0^{(1/2)\pi} \hat{G}_n{}^U(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})\, d\varphi_n, \tag{26}$$

respectively.

Expressions (19) and (20) apply if we simply replace the lower bounds $\hat{G}_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ and $G_n{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ by the corresponding upper bounds $\hat{G}_n{}^U(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ and $G_n{}^U(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$, and reverse the inequality in (20). In Appendix B we obtain upper bounds on the integrals

$$\int_{\Delta'(l,j)}^{\Delta'(l,j+1)} G_n{}^U(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})\, d\tau_n \tag{27}$$

where $\Delta'(l,j) = [l + (j/J')]T_c$, and

$$\int_{\psi'(m)}^{\psi'(m+1)} \hat{G}_n{}^U(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi})\,d\varphi_n \tag{28}$$

where $\psi'(m) = \frac{1}{2}m\pi/M'$.

The upper bound on the average probability of error is given in terms of an interference spectrum $S_k{}^U(i)$. This spectrum is defined as $S_k{}^U(i) = |\mathcal{S}_k{}^U(i)|$, where $\mathcal{S}_k{}^U(i)$ is the set

$$\mathcal{S}_k{}^U(i) = \{(\boldsymbol{b}_k, l, j): 2J'NI_{k,1}(\boldsymbol{b}_k, \Delta'(l,j), 0) = i\}. \tag{29}$$

In this case J' can be of the form $\frac{1}{2}L$ for some integer L, but it is sufficient for our purposes to restrict J' to be an integer. Let $\Sigma_i{}'$ and $\Sigma_m{}'$ be defined in the same way as Σ_i and Σ_m with J and M replaced by J' and M'. The upper bound is

$$\bar{P}_{e,1} \leqslant (8J'NM')^{1-K}\Sigma_i{}'\left\{\prod_{n=2}^{K} S_n{}^U(i_n)\right\} \cdot \Sigma_m{}' Q\left(\alpha\left[1 + \sum_{k=2}^{K}(i_k/2J'N)\cos\left(\tfrac{1}{2}i_k\pi/M'\right)\right]\right) \tag{30}$$

which is valid for all positive integers J' and M'. As with the lower bound, the upper bound given by (30) becomes tighter as J' and M' increase. Indeed, if $\bar{P}_{e,1}^U$ denotes the right-hand side of (30), then the error $\bar{P}_{e,1}^U - \bar{P}_{e,1}$ is nonnegative and it decreases monotonically to zero as $J' \to \infty$ and $M' \to \infty$.

GENERALIZED CHEBYSHEV BOUNDS

We examine a class of upper bounds which includes the Chebyshev and Chernoff bounds. This class is derived from the generalized Chebyshev inequality

$$P\{X \geqslant \beta\} \leqslant \frac{E\{h(X)\}}{h(\beta)} \tag{31}$$

which holds for any nonnegative, nondecreasing function h, any real number β such that $h(\beta) > 0$, and any random variable X. We consider two such functions: 1) the function h_1 defined by $h_1(x) = x^2$ for $x \geqslant 0$ and $h_1(x) = 0$ for $x < 0$, and 2) the function h_2 defined by $h_2(x) = \exp(sx)$, where s is an arbitrary nonnegative real number. The latter case is the well-known Chernoff bound, which can be optimized by minimizing the upper bound with respect to s.

From (2) and (15) it follows that $\bar{P}_{e,1}$ can be written as

$$\bar{P}_{e,1} = P\left\{\eta' + \sum_{k=2}^{K} I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k) \geqslant 1\right\} \tag{32}$$

where η' is a zero-mean Gaussian random variable with variance α^{-1}. In (32) η', $\boldsymbol{b}$, τ, φ are all random variables and $\bar{P}_{e,1}$ is the average probability of error as defined in (9).

Let $\beta = +1$ and define

$$X = \eta' + \sum_{k=2}^{K} I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k). \tag{33}$$

Application of (31) with $h(x) = h_1(x)$ requires the computation of $E\{h_1(X)\}$. From [6] and the fact that the distribution of X is symmetric about its mean we see that

$$E\{h_1(X)\} = \tfrac{1}{2}E(X^2) = \tfrac{1}{2}\,(\mathrm{SNR}_1)^{-2} \tag{34}$$

where SNR_1 is given in [6] for rectangular pulses and in [7] and [9] for general pulse shapes.

The computation of $E\{h_2(X)\}$ is more complicated so the details are omitted. The result is

$$E\{h_2(X)\} = E\{e^{sX}\} = e^{(\frac{1}{2}s^2/\alpha)}\prod_{k=2}^{K}\left[\frac{1}{2T}\int_0^T \{I_0(s[R_{k,1}(\tau) + \hat{R}_{k,1}(\tau)]/T) + I_0(s[R_{k,1}(\tau) - \hat{R}_{k,1}(\tau)]/T)\}\,d\tau\right] \tag{35}$$

where I_0 is the modified Bessel function of order zero.

The general Chebyshev bounds can also be evaluated for *random* signature sequences which are defined as follows. For each k, the sequence $(a_n{}^{(k)})$ is a sequence of independent identically distributed random variables with $P\{a_j{}^{(k)} = +1\} = P\{a_j{}^{(k)} = -1\} = \frac{1}{2}$ for each j. Also, for each $i \neq k$, $a_l{}^{(i)}$ and $a_j{}^{(k)}$ are independent. The resulting bound is $\bar{P}_e \leqslant \frac{1}{2}(\overline{\mathrm{SNR}})^{-2}$, where

$$\overline{\mathrm{SNR}} = \left\{\frac{N_0}{2E_b} + \frac{K-1}{NT_c{}^3}\int_0^{T_c} \mathcal{R}^2(\tau)\,d\tau\right\}^{-1/2}. \tag{36}$$

The function $\mathcal{R}$ is the aperiodic autocorrelation function for the chip waveform [9]. By evaluating $E\{h_2(X)\}$ and applying the result to (31) gives

$$\bar{P}_e \leqslant \min_{s \geqslant 0} \exp\{-s + \tfrac{1}{2}s^2/\alpha\} \cdot \left[\frac{1}{T_c}\int_0^{T_c}\frac{2}{\pi}\int_0^{\pi/2}[f(s,\tau,\varphi)]^N\,d\varphi\,d\tau\right]^{K-1} \tag{37a}$$

where

$$f(s,\tau,\varphi) = \cosh(sT^{-1}\mathcal{R}(\tau)\cos\varphi) \cdot \cosh(sT^{-1}\mathcal{R}(T_c - \tau)\cos\varphi). \tag{37b}$$

These bounds are specialized to binary PSK direct-sequence systems by letting $\mathcal{R}(\tau) = \tau$ for $0 \leqslant \tau \leqslant T_c$, which is the aperiodic autocorrelation function for the rectangular chip waveform.

NUMERICAL RESULTS

In this section we present some representative numerical results which will give an indication of the tightness of the bounds. Numerical values are given for the bounds for various choices of signature sequences. For $K = 2$ and $K = 3$ the signature sequences are m-sequences, and for $K = 4$ the Gold sequences of period 31 are employed since there are only three nonreciprocal m-sequences of period 31. The phases of the m-sequences employed for our numerical results are the auto-optimal least-sidelobe-energy (AO/LSE) phases given in [10] or the phases which maximize the SNR [2]. The shift-register tap connections and initial loadings are given in [2, Table 5] for the phases which maximize the SNR and in [10, Fig. A.1] for the AO/LSE phases.

Tables of numerical values for the bounds are here for various values of the parameters K, N, and E_b/N_0. In order to give an indication of the amount of computation required in each case, we have specified the values of J and M used in (24) to compute the lower bound and the values of J' and M' used in (30) to compute the upper bound.

In Table I we give values for the lower and upper bound [(24) and (30)] denoted by $\bar{P}_{e,1}^L$ and $\bar{P}_{e,1}^U$, respectively, as a function of J and M for $K = 2$ and $N = 31$. For the data in this table, $J' = J$ and $M' = M$. The sequences are the first two AO/LSE sequences given in [10, Fig. A.1(a)]. Since K is only 2, we can approximate the integrals using an algorithm based on Simpson's rule. This calculation gives 2.3975×10^{-5} as the approximate value of $\bar{P}_{e,1}$. Notice that the lower bound converges faster than the upper bound. This is due to the type of bounds being used for the integrals in (19), (20), (27), and (28). For example, the integral in (28) is bounded using the rectangular rule which is the weakest bound employed. This accounts for the slower convergence in the upper bound. For $K = 3$ and $N = 31$ the bounds are given in Table II where again $J' = J$ and $M' = M$.

In Table III the upper and lower bounds are given for several values of K and N. In Tables IV, V, and VI the bounds are given for $K = 2$, 3, and 4, respectively, and $N = 31$. Also tabulated is the approximation $\tilde{P}_{e,1} = Q(\mathrm{SNR}_1)$ suggested in [6]. The values of J, M, J', and M' were chosen to give moderate computation for each case. The sequences for Tables IV and V are the first two or three AO/LSE m-sequences of [10, Fig. A.1(a)]. For Table VI the sequences are four Gold sequences that have the property that the maximum interference is less than the desired signal. The shift-register tap connection for these sequences is 3551 (in the notation of [10]), and the initial loadings (in octal notation) for sequences 1 through 4 are 1756, 0355, 0432, and 1306, respectively.

The importance of the selection of the phases of signature sequences for direct-sequence SSMA systems is illustrated in Fig. 1. In this figure the upper and lower bounds are shown for two different sets of phases of the *same set* of three m-sequences of period 31. These phases, which are given in [2], give the minimum and maximum possible SNR. Notice that at a bit error rate of 10^{-5}, the difference in performance is about 2 dB. Thus, proper choice of the phases of the signature sequences can significantly improve the efficiency of a direct-sequence SSMA system.

TABLE I
BOUNDS FOR $K = 2, N = 31$, AND $E_b/N_0 = 10$ dB

J	M	$\bar{P}_{e,1}^L$	$\bar{P}_{e,1}^U$	
20	10	2.322	2.689	$(\times 10^{-5})$
40	20	2.378	2.534	$(\times 10^{-5})$
60	30	2.389	2.485	$(\times 10^{-5})$
80	40	2.393	2.463	$(\times 10^{-5})$
160	80	2.396	2.429	$(\times 10^{-5})$
300	150	2.397	2.414	$(\times 10^{-5})$

TABLE II
BOUNDS FOR $K = 3, N = 31$, AND $E_b/N_0 = 12$ dB

J	M	$\bar{P}_{e,1}^L$	$\bar{P}_{e,1}^U$	
8	2	5.31	36.20	$(\times 10^{-6})$
8	4	6.24	23.70	$(\times 10^{-6})$
16	4	8.95	19.87	$(\times 10^{-6})$
16	6	9.22	18.04	$(\times 10^{-6})$
32	6	10.20	16.05	$(\times 10^{-6})$
32	12	10.38	13.41	$(\times 10^{-6})$
64	12	10.65	13.24	$(\times 10^{-6})$
64	16	10.68	12.63	$(\times 10^{-6})$

TABLE III
BOUNDS FOR $E_b/N_0 = 12$ dB

K	N	J, M	J', M'	$\bar{P}_{e,1}^L$	$\bar{P}_{e,1}^U$	
2	31	300, 150	300, 150	4.81	4.86	$(\times 10^{-7})$
3	31	64, 16	64, 16	1.07	1.26	$(\times 10^{-7})$
4	31	32, 4	8, 16	4.01	6.16	$(\times 10^{-5})$
2	127	160, 40	80, 80	3.94	3.98	$(\times 10^{-8})$
3	127	12, 16	16, 12	1.17	1.42	$(\times 10^{-7})$
2	255	120, 40	40, 80	1.98	2.00	$(\times 10^{-8})$

TABLE IV
BOUNDS AND APPROXIMATION FOR $K = 2, N = 31$ ($J = 160$, $M = 100; J' = 80, M' = 120$)

E_b/N_0 (dB)	$\bar{P}_{e,1}^L$	$\bar{P}_{e,1}^U$	$\tilde{P}_{e,1}$	
4	1.44	1.45	1.44	$(\times 10^{-2})$
6	3.35	3.36	3.34	$(\times 10^{-3})$
8	4.22	4.24	4.17	$(\times 10^{-4})$
10	2.40	2.42	2.34	$(\times 10^{-5})$
12	4.81	4.89	5.13	$(\times 10^{-7})$
14	2.28	2.34	4.43	$(\times 10^{-9})$

TABLE V
BOUNDS AND APPROXIMATION FOR $K = 3$ AND $N = 31$ ($J = J' = 32, M = M' = 15$)

E_b/N_0 (dB)	$\bar{P}_{e,1}^L$	$\bar{P}_{e,1}^U$	$\tilde{P}_{e,1}$	
4	1.66	1.69	1.66	$(\times 10^{-2})$
6	4.58	4.77	4.56	$(\times 10^{-3})$
8	8.43	9.11	8.13	$(\times 10^{-4})$
10	1.06	1.21	0.92	$(\times 10^{-4})$
12	1.04	1.29	0.70	$(\times 10^{-5})$
14	9.68	13.32	4.40	$(\times 10^{-7})$

TABLE VI
BOUNDS AND APPROXIMATION FOR $K = 4$ AND $N = 31$ ($J = 16$, $M = 4$; $J' = 8, M' = 8$)

$\mathcal{E}_b/N_0$ (dB)	$\bar{P}^L_{e,1}$	$\bar{P}^U_{e,1}$	$\tilde{P}_{e,1}$	
4	1.87	1.99	1.88	$(\times 10^{-2})$
6	5.85	6.67	5.94	$(\times 10^{-3})$
8	1.36	1.74	1.38	$(\times 10^{-3})$
10	2.44	3.74	2.48	$(\times 10^{-4})$
12	3.74	7.29	3.84	$(\times 10^{-5})$
14	5.37	14.19	6.16	$(\times 10^{-6})$

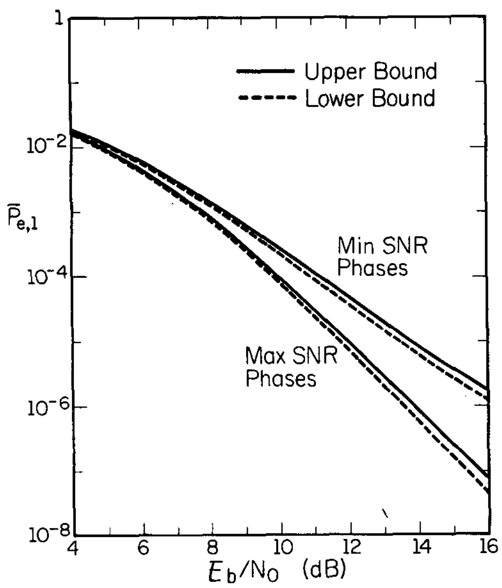

Fig. 1. Upper and lower bounds for two sets of sequences with different phases ($K = 3, N = 31$).

Numerical results for $K = 3$ and $N = 127$ are shown in Table VII for the AO/LSE sequences given in [10]. In addition to the bounds of (24) and (30), the approximation $\tilde{P}_{e,1}$ and the Chernoff bound $\bar{P}^C_{e,1}$ are shown. Results on the Chernoff bound $\bar{P}_e{}^C$ for random sequences of length 127 are presented in Table VIII for $K = 3$. As expected, the Chernoff bound is not very tight. Numerical values for the approximation $\tilde{P}_e = Q(\overline{\text{SNR}})$ for random sequences are also given. We found the Chebyshev bound was far too loose to be of any interest, so detailed numerical results are not presented. As an illustration, the Chebyshev bound is 1.26×10^{-2} for random sequences of length 127 with $K = 3$ and $\mathcal{E}_b/N_0 = 14$ dB.

We close by giving some comparisons between the bounds presented in this paper and the monent-space bounds of [14]. The numerical results presented in [1] for the second-moment bounds indicate that the bounds presented in this paper are tighter even for small values of J. In addition, we have found that our bounds are tighter than the single-exponential moment-space bounds (even for relatively small values of J and M). For $K = 2$ and $N = 31$ we computed the single-exponential bounds and the bounds presented in this paper. The two signature sequences are specified by the feedback connections 45 and 67 with loadings 06 and 36, respectively. These are two sequences from the set obtained in [2]. For $\mathcal{E}_b/N_0 = 12$ dB the upper and lower single-exponential bounds are 7.830×10^{-7} and 5.401×10^{-7}, respectively. The difference between the upper and lower bounds is 2.429×10^{-7}. Our bounds for $J' = J = 20$ and $M' = M = 10$ are 9.020×10^{-7} and 6.871×10^{-7}, so the difference is 2.149×10^{-7}. For $J' = J = 40$ and $M' = M = 80$ we find that our bounds are 7.680×10^{-7} and 7.309×10^{-7}, which is a difference of only 0.371×10^{-7}.

TABLE VII
BOUNDS, APPROXIMATION, AND CHERNOFF BOUND FOR $K = 3$ AND $N = 127$ ($J = 12, M = 16$; $J' = 16, M' = 12$)

$\mathcal{E}_b/N_0$ (dB)	$\bar{P}^L_{e,1}$	$\bar{P}^U_{e,1}$	$\tilde{P}_{e,1}$	$\bar{P}^C_{e,1}$	
4	1.35	1.36	1.35	8.67	$(\times 10^{-2})$
6	2.86	2.91	2.86	22.0	$(\times 10^{-3})$
8	2.95	3.07	2.94	27.5	$(\times 10^{-4})$
10	1.11	1.22	1.08	12.6	$(\times 10^{-5})$
12	1.17	1.41	0.98	16.3	$(\times 10^{-7})$
14	3.03	4.35	1.62	51.8	$(\times 10^{-10})$

TABLE VIII
APPROXIMATION AND CHERNOFF BOUND FOR RANDOM SEQUENCES OF LENGTH 127 WITH $K = 3$

$\mathcal{E}_b/N_0$ (dB)	$\tilde{P}_e$	$\bar{P}_e{}^C$
4	1.35×10^{-2}	4.33×10^{-2}
6	2.85×10^{-3}	1.10×10^{-2}
8	2.41×10^{-4}	1.37×10^{-3}
10	1.05×10^{-5}	6.51×10^{-5}
12	9.29×10^{-8}	1.02×10^{-6}
14	1.44×10^{-10}	6.69×10^{-9}

APPENDIX A

In this Appendix we present the lower bounds for the integrals that appear in (19) and (20). These are obtained as key steps in the derivation of (24).

The first step is to consider (17)-(20) for $n = 2$. We let $G_2{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi}) = G_2(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$ and observe from (10) and (13a) that

$$\hat{G}_2{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi}) = (4T)^{-1} \Sigma_{b_2} \cdot \int_0^T Q(\alpha[1 + \beta_3 + I_{2,1}(b_2, \tau_2, \varphi_2)])\, d\tau_2 \tag{A.1}$$

where β_3 does not depend on b_2, τ_2, or φ_2. The convexity of Q on $[0, \infty)$ implies that

$$T_c{}^{-1} J \int_{\Delta(l,j)}^{\Delta(l,j+1)} G_2{}^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})\, d\tau_2 \geqslant Q(\alpha[1 + \beta_3 + \hat{I}_{2,1}(b_2, \varphi_2; l, j)]) \tag{A.2}$$

where

$$\hat{I}_{2,1}(b_2, \varphi_2; l, j) = T_c{}^{-1} J \int_{\Delta(l,j)}^{\Delta(l,j+1)} I_{2,1}(b_2, \tau_2, \varphi_2)\, d\tau_2. \tag{A.3}$$

From a comparison of (10) and (A.1) it is easy to see that the quantity β_3 is given by

$$\beta_3 = \sum_{k=3}^{K} I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k). \tag{A.4}$$

Furthermore, if we define

$$\Gamma_k(l,j) = \int_{\Delta(l,j)}^{\Delta(l,j+1)} R_{k,1}(\tau)\, d\tau \tag{A.5}$$

and

$$\hat{\Gamma}_k(l,j) = \int_{\Delta(l,j)}^{\Delta(l,j+1)} \hat{R}_{k,1}(\tau)\, d\tau \tag{A.6}$$

for $2 \leqslant k \leqslant K$, then we see from (5) that

$$\hat{I}_{2,1}(\boldsymbol{b}_2, \varphi_2; l, j) = J(T_cT)^{-1}\{b_{-1}^{(2)}\Gamma_2(l,j) + b_0^{(2)}\hat{\Gamma}_2(l,j)\}\cos\varphi_2. \tag{A.7}$$

Notice that we have not made use of (6) in arriving at (A.7). Instead, (13)-(20) and (A.1)-(A.7) are valid for arbitrary time-limited pulses such as the sine pulse considered in [7] and [9]. For the rectangular pulse it follows that

$$\Gamma_k(l,j) = J^{-1}T_cR_{k,1}(\bar{\Delta}(l,j)) \tag{A.8}$$

and

$$\hat{\Gamma}_k(l,j) = J^{-1}T_c\hat{R}_{k,1}(\bar{\Delta}(l,j)) \tag{A.9}$$

because of the linear form of $R_{k,i}(\tau)$ and $\hat{R}_{k,i}(\tau)$ as demonstrated in (6). Alternatively, we can work directly with (A.3) and deduce from (5) and (6) that for the rectangular pulse waveform

$$\hat{I}_{2,1}(\boldsymbol{b}_2, \varphi_2; l, j) = I_{2,1}(\boldsymbol{b}_2, \bar{\Delta}(l,j), \varphi_2) = T^{-1}[b_{-1}^{(2)}R_{2,1}(\bar{\Delta}(l,j)) + b_0^{(2)}\hat{R}_{2,1}(\bar{\Delta}(l,j))]\cos\varphi_2. \tag{A.10}$$

If we now combine the above results for the rectangular pulse with (A.2) we have the lower bound

$$T_c^{-1}J\int_{\Delta(l,j)}^{\Delta(l,j+1)} G_2(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi})\, d\tau_2 \geqslant Q(\alpha[1+\beta_3+I_{2,1}(\boldsymbol{b}_2, \bar{\Delta}(l,j), \varphi_2)]), \tag{A.11}$$

which can then be employed in (19) to obtain a lower bound for $\hat{G}_2^L(\boldsymbol{b}, \boldsymbol{\tau}, \boldsymbol{\varphi})$. We can simplify the computation of the right-hand side of (19) by replacing the triple sum (over $\boldsymbol{b}_2$, l, and j) by a single sum as follows. First, observe that the quantity $T_c^{-1}\{b_{-1}^{(2)}R_{2,1}(lT_c) + b_0^{(2)}\hat{R}_{2,1}(lT_c)\}$ takes on integer values between $-N$ and N (this follows from (6) and properties of the aperiodic correlation function). Consequently, (6) and (A.10) imply that $(2JT/T_c)I_{2,1}(\boldsymbol{b}_2, \bar{\Delta}(l,j), 0)$ takes on integer values between $-2JN$ and $2JN$. Since $T = NT_c$ we define $S_2^L(i)$ as in (21). Let $S_2^L(i) = |S_2^L(i)|$, the number of elements in $S_2^L(i)$. From (19) and (A.11) we have

$$\hat{G}_2^L(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}) \geqslant (2N')^{-1}\sum_{i=-N'}^{N'} S_2^L(i) \cdot Q(\alpha[1+\beta_3+(i/N')\cos\varphi_2]) \tag{A.12}$$

where $N' = 2JN$. It follows from (20), (23), the convexity of Q on $[0, \infty)$, and the symmetry of the multiple-access interference that if

$$G_3^L(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}) = (2N'M)^{-1}\sum_{i=-N'}^{N'} S_2^L(i)\sum_{m=0}^{M-1} Q(\alpha[1+\beta_3+i\gamma(m)]) \tag{A.13}$$

then $G_3(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}) \geqslant G_3^L(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi})$, as required.

This completes the derivation of the lower bound for $n = 2$. The next step in obtaining a lower bound for $\bar{P}_{e,1}$ is to employ (A.13) with β_3 as defined in (A.4) to (17)-(20) with $n = 3$. For example, (17) yields

$$\hat{G}_3^L(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}) = (2N'M)^{-1}\Sigma_{i_2}\Sigma_{m_2}S_2(i_2) \cdot H_3(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}; i_2, m_2) \tag{A.14}$$

where

$$H_3(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}; i, m) = (4T)^{-1}\Sigma_{\boldsymbol{b}_3}\int_0^T Q(\alpha[1+\beta_4(i,m) + I_{3,1}(\boldsymbol{b}_3, \tau_3, \varphi_3)]\ d\tau_3 \tag{A.15}$$

for $|i| \leqslant N'$ and $0 \leqslant m \leqslant M-1$. We use Σ_{i_n} and Σ_{m_n} to denote the sums from $i_n = -N'$ to N' and $m_n = 0$ to $M-1$, respectively. The quantity $\beta_4(i,m)$ is defined by

$$\beta_4(i,m) = i\gamma(m) + \sum_{k=4}^{K} I_{k,1}(\boldsymbol{b}_k, \tau_k, \varphi_k). \tag{A.16}$$

Therefore, it does not depend on $\boldsymbol{b}_3$, τ_3, or φ_3.

A comparison of (A.15) with (A.1) shows that we can obtain a lower bound on $H_3(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi}; i, m)$ in exactly the same way as we obtained the lower bound on $\hat{G}_2^L(\boldsymbol{b},\boldsymbol{\tau},\boldsymbol{\varphi})$; namely, we apply the procedure outlined in (A.2)-(A.12). This gives a lower bound of the form

$$\hat{G}_3{}^L(b,\tau,\varphi) \geq M^{-1}(2N')^{-2}\Sigma_{i_2}\Sigma_{i_3}S_2(i_2)S_3(i_3)\Sigma_{m_2} \cdot Q(\alpha[1+\beta_4(i_2,m_2)+(i_3/N')\cos\varphi_3]). \quad (A.17)$$

We then proceed as in (A.13) to obtain

$$G_4{}^L(b,\tau,\varphi) = (2N'M)^{-2}\Sigma_{i_2}\Sigma_{i_3}S_2(i_2)S_3(i_3)\Sigma_{m_2}\Sigma_{m_3} \cdot Q(\alpha[1+\beta_4(i_2,m_2)+i_3\gamma(m_3)]). \quad (A.18)$$

The pattern should be clear at this point, so the reader should be able to verify the final expression (24).

Notice that the lower bound of (A.11) is also the approximation to the integral on the left-hand side of (A.11) that is obtained by applying a rectangular integration rule using the value of the function at the midpoint of the interval. Similarly, the lower bound of (A.13) can be viewed as an application of a rectangular integration rule to the integrals of the right-hand side of (20) using the value of the function at some point (not necessarily the midpoint) of the interval $[\psi(m), \psi(m+1)]$. Thus, the error $G_3(b,\tau,\varphi) - G_3{}^L(b,\tau,\varphi)$ in the bound of (A.13) is nonnegative. Moreover, it converges monotonically to zero as $J \to \infty$ and $M \to \infty$. The same conclusion holds for the error $\bar{P}_{e,1} - \bar{P}_{e,1}^L$ in the lower bound, where $\bar{P}_{e,1}^L = G_{K+1}^L(b,\tau,\varphi)$ is just the right-hand side of (24).

APPENDIX B

In this Appendix we present upper bounds for the integrals given in (27) and (28). First, the special case $n = 2$ is handled by letting $\hat{G}_2{}^U(b,\tau,\varphi) = \hat{G}_2{}^L(b,\tau,\varphi)$, which is given by (A.1). If the chip waveform is the rectangular pulse, then $\hat{G}_2{}^U(b,\tau,\varphi)$ is a convex function of τ_2 on the interval $[lT_c, (l+1)T_c]$ and so

$$T_c{}^{-1}J\int_{\Delta'(l,j)}^{\Delta'(l,j+1)} G_2{}^U(b,\tau,\varphi)\,d\tau_2 \leq \tfrac{1}{2}Q(\alpha[1+\beta_3+I_{2,1}(b_2,\Delta'(l,j),\varphi_2)]) + \tfrac{1}{2}Q(\alpha[1+\beta_3+I_{2,1}(b_2,\Delta'(l,j+1),\varphi_2)]). \quad (B.1)$$

The inequality in (B.1) follows from the fact that the (normalized) integral of a convex function on any subinterval is not greater than the average of the values of the function at the two endpoints of the subinterval. Notice also that the right-hand side of (B.1) is just the trapezoidal approximation to the integral; therefore the difference between the right-hand side and the left-hand side of (B.1) converges monotonically to zero as $J' \to \infty$.

We then proceed as in (A.10)–(A.12) to obtain

$$\hat{G}_2{}^U(b,\tau,\varphi) \leq (2N'')^{-1}\sum_{i=-N''}^{N''} S_2{}^U(i) \cdot Q(\alpha[1+\beta_3+(i/2J'N)\cos\varphi_2]) \quad (B.2)$$

where $N'' = 2J'N$. It is easy to show that the quantity

$$Q(\alpha[1+\beta_3-(i/2J'N)\cos\varphi]) + Q(\alpha[1+\beta_3+(i/2J'N)\cos\varphi])$$

is a decreasing function of φ on $[0, \frac{1}{2}\pi]$. Thus

$$(2M/\pi)\int_{\psi'(m)}^{\psi'(m+1)} \hat{G}_2{}^U(b,\tau,\varphi)\,d\varphi_2 \leq (2N'')^{-1}\sum_{i=-N''}^{N''} S_2{}^U(i)Q(\alpha[1+\beta_3+(i/2J'N) \cdot \cos\psi'(m)]). \quad (B.3)$$

Thus we let

$$G_3{}^U(b,\tau,\varphi) = (2N''M')^{-1}\sum_{i=-N''}^{N''} S_2{}^U(i) \cdot \sum_{m=0}^{M'-1} Q(\alpha[1+\beta_3+(i/2J'N)\cos\psi'(m)]) \quad (B.4)$$

which is analogous to (A.13).

Notice that the right-hand side of (B.3) can be obtained from an application of the rectangular integration rule to the left-hand side. For this approximation we use the value of the function at the left endpoint [i.e., at $\psi'(m)$] of the interval $[\psi'(m), \psi'(m+1)]$. Thus, the error in the resulting approximation converges monotonically to zero as $M' \to \infty$.

The procedure for continuing on to $n = 3$ is the same as outlined in Appendix A, especially (A.14)–(A.18). By following the same pattern for $n = 4, 5, \cdots, K+1$, the reader can verify the bound of (30).

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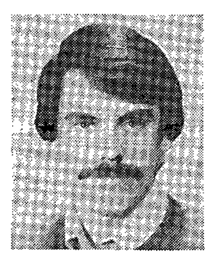

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