Proof of the Theorem in NSF Proposal "Multi-job Production Systems Engineering"

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Theorem: Consider a two-machine serial line defined by work-based model (i)-(v) and operating in the synchronous SJP regime. Then, the expressions for the performance metrics of this system are:

$$\begin{aligned} PR\left(\frac{w_i}{W_i}\right) &= e_2\left(1 - Q\left(\lambda_1, \mu_1, \lambda_2, \mu_2, N\frac{w_i}{W_i}\right)\right) \\ &= e_1\left(1 - Q\left(\lambda_2, \mu_2, \lambda_1, \mu_1, N\frac{w_i}{W_i}\right)\right), \end{aligned} \tag{1}$$

$$TP\left(\frac{w_i}{W_i}\right) = \frac{60W_i}{w_i} PR\left(\frac{w_i}{W_i}\right), \tag{2}$$

$$WIP\left(\frac{w_i}{W_i}\right) = \begin{cases} \dfrac{D_5\left(\frac{w_i}{W_i}\right)}{D_2\left(\frac{w_i}{W_i}\right)+D_3\left(\frac{w_i}{W_i}\right)+D_4\left(\frac{w_i}{W_i}\right)}, & \text{if } e_1 \neq e_2, \\ \dfrac{\left(\frac{D_2\left(\frac{w_i}{W_i}\right)}{2}+D_4\left(\frac{w_i}{W_i}\right)\right)N\frac{w_i}{W_i}}{D_2\left(\frac{w_i}{W_i}\right)+D_3\left(\frac{w_i}{W_i}\right)+D_4\left(\frac{w_i}{W_i}\right)}, & \text{if } e_1 = e_2, \end{cases} \tag{3}$$

$$ST_2\left(\frac{w_i}{W_i}\right) = e_2 Q\left(\lambda_1, \mu_1, \lambda_2, \mu_2, N\frac{w_i}{W_i}\right), \tag{4}$$

$$BL_1\left(\frac{w_i}{W_i}\right) = e_1 Q\left(\lambda_2, \mu_2, \lambda_1, \mu_1, N\frac{w_i}{W_i}\right), \tag{5}$$

where (λ_i, μ_i), $i = 1, 2$ are the breakdown and repair rates of the machines, respectively, N is the buffer capacity, and

$$Q\left(\lambda_1, \mu_1, \lambda_2, \mu_2, N\frac{w_i}{W_i}\right) = \begin{cases} \dfrac{(1-e_1)(1-\phi)}{1-\phi e^{-\beta N \frac{w_i}{W_i}}}, & \text{if } \frac{\lambda_1}{\mu_1} \neq \frac{\lambda_2}{\mu_2}, \\ \dfrac{\lambda_1(\lambda_1+\lambda_2)(\mu_1+\mu_2)}{(\lambda_1+\mu_1)\left[(\lambda_1+\lambda_2)(\mu_1+\mu_2)+\lambda_2\mu_1(\lambda_1+\lambda_2+\mu_1+\mu_2)N\frac{w_i}{W_i}\right]}, & \text{if } \frac{\lambda_1}{\mu_1} = \frac{\lambda_2}{\mu_2}, \end{cases} \tag{6}$$

$$e_i = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad \phi = \frac{e_1(1-e_2)}{e_2(1-e_1)}, \quad \beta = \frac{(\lambda_1+\lambda_2+\mu_1+\mu_2)(\lambda_1\mu_2 - \lambda_2\mu_1)}{(\lambda_1+\lambda_2)(\mu_1+\mu_2)},$$

$$D_1 = \frac{\mu_1 + \mu_2}{\lambda_1 + \lambda_2},$$

$$D_2\left(\frac{w_i}{W_i}\right) = \begin{cases} \left(2 + D_1 + \frac{1}{D_1}\right) \frac{e^{KN\frac{w_i}{W_i}}-1}{K\frac{w_i}{W_i}}, & \text{if } e_1 \neq e_2, \\ \\ \left(2 + D_1 + \frac{1}{D_1}\right) N\frac{w_i}{W_i}, & \text{if } e_1 = e_2, \end{cases}$$

$$D_3\left(\frac{w_i}{W_i}\right) = \frac{(\lambda_1 + \mu_1 + \lambda_2 + \mu_2)(\lambda_2 + \mu_1) + \lambda_1\mu_2 - \lambda_2\mu_1}{\lambda_2\mu_1(\lambda_1 + \mu_1 + \lambda_2 + \mu_2)\frac{w_i}{W_i}},$$

$$D_4\left(\frac{w_i}{W_i}\right) = \frac{(\lambda_1 + \mu_1 + \lambda_2 + \mu_2)(\lambda_2 + \mu_1) + \lambda_2\mu_1 - \lambda_1\mu_2}{\lambda_1\mu_2(\lambda_1 + \mu_1 + \lambda_2 + \mu_2)\frac{w_i}{W_i}} e^{KN\frac{w_i}{W_i}},$$

$$D_5\left(\frac{w_i}{W_i}\right) = \frac{D_2\left[1 + \left(KN\frac{w_i}{W_i} - 1\right)e^{KN\frac{w_i}{W_i}}\right]}{K\frac{w_i}{W_i}\left(e^{KN\frac{w_i}{W_i}} - 1\right)} + D_4 N\frac{w_i}{W_i},$$

$$K = \begin{cases} \frac{(\lambda_1+\lambda_2+\mu_1+\mu_2)(\lambda_2\mu_1-\lambda_1\mu_2)}{(\lambda_1+\lambda_2)(\mu_1+\mu_2)}, & \text{if } e_1 \neq e_2, \\ \\ 0, & \text{if } e_1 = e_2. \end{cases}$$

Proof: System under consideration can be described by a continuous time, mixed space Markov process with the states defined as a triple (h, s_1, s_2), where h is the state of the buffer and $s_i, i = 1, 2$ are the states of the first and the second machine, respectively. States in this problem are categorized into two groups: (1) boundary states $(0, s_1, s_2)$, (N, s_1, s_2), and (2) internal states $(h, s_1, s_2), 0 < h < N$. Boundary states are described by their probability mass functions whereas internal states are described by their density functions.

Transition rates of the process from state $(s_1 = i, s_2 = j)$ to state $(s_1 = k, s_2 = l)$, i.e., $\nu_{kl,ij}$, are as follows. Note that below, we use the notation τ as a shorthand to refer to $\tau := \frac{w_1}{W_1}\left(= \frac{w_2}{W_2}\right)$.

$$\begin{aligned} &\nu_{11,00} = 0, && \nu_{10,00} = \mu_1, && \nu_{01,00} = \mu_2, \\ &\nu_{11,01} = \mu_1, && \nu_{10,01} = 0, && \nu_{00,01} = \lambda_2, \\ &\nu_{11,10} = \mu_2, && \nu_{01,10} = 0, && \nu_{00,10} = \lambda_1, \\ &\nu_{10,11} = \lambda_2, && \nu_{01,11} = \lambda_1, && \nu_{00,11} = 0. \end{aligned}$$

While the discrete part of the Markov process at hand is described by the above transition rates, the continuous part, i.e., buffer occupancy, is characterized by

$$\left.\frac{dh(t)}{dt}\right|_{0<h(t)<N} = \begin{cases} \frac{1}{\tau}, & \text{if } s_1(t) = 1, s_2(t) = 0, \\ 0, & \text{if } s_1(t) = s_2(t), \\ -\frac{1}{\tau}, & \text{if } s_1(t) = 0, s_2(t) = 1, \end{cases} \tag{7}$$

$$\left.\frac{dh(t)}{dt}\right|_{h(t)=0} = \begin{cases} \frac{1}{\tau}, & \text{if } s_1(t) = 1, s_2(t) = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

$$\left.\frac{dh(t)}{dt}\right|_{h(t)=N} = \begin{cases} -\frac{1}{\tau}, & \text{if } s_1(t) = 0, s_2(t) = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

The following can be obtained for the steady state pdf for the internal states $h \in (0, N)$,

$$\begin{aligned}
f_{H,I,J}(h,1,1) &= f_{H,I,J}(h,1,0)\tfrac{\mu_2}{\lambda_1+\lambda_2} + f_{H,I,J}(h,0,1)\tfrac{\mu_1}{\lambda_1+\lambda_2}, \\
\tfrac{\partial f_{H,I,J}(h,1,0)}{\partial h} &= f_{H,I,J}(h,1,1)\lambda_2\tau - f_{H,I,J}(h,1,0)(\lambda_1+\mu_2)\tau + f_{H,I,J}(h,0,0)\mu_1\tau, \\
\tfrac{\partial f_{H,I,J}(h,0,1)}{\partial h} &= f_{H,I,J}(h,0,1)(\lambda_2+\mu_1)\tau - f_{H,I,J}(h,1,1)\lambda_1\tau - f_{H,I,J}(h,0,0)\mu_2\tau, \\
f_{H,I,J}(h,0,0) &= f_{H,I,J}(h,1,0)\tfrac{\lambda_1}{\mu_1+\mu_2} + f_{H,I,J}(h,0,1)\tfrac{\lambda_2}{\mu_1+\mu_2}.
\end{aligned} \tag{10}$$

Since $h(t)$ is continuous in t, events

$$A_1 = \{h(t) = h, h(t) \text{ is increasing in } t\}$$

and

$$A_2 = \{h(t) = h, h(t) \text{ is decreasing in } t\}$$

occur alternately with the equal frequencies. Therefore,

$$f_{H,I,J}(h,1,0) = f_{H,I,J}(h,0,1) \tag{11}$$

The stationary probabilities of the boundary states, $h = 0$ and $h = N$, are derived as follows. The evolution equations for these probabilities are:

$$\begin{aligned}
P_{0,11}(t+\delta t) &= P_{0,11}(t)[1-(\lambda_1+\lambda_2)\tau\delta t] + P_{0,01}(t)\mu_1\tau\delta t + o(\delta t), \\
P_{0,10}(t+\delta t) &= 0, \\
P_{0,01}(t+\delta t) &= P_{0,01}(t)[1-(\mu_1+\lambda_2)\tau\delta t] + P_{0,11}(t)\lambda_1\tau\delta t + P_{0,00}(t)\mu_2\tau\delta t + f_{H,I,J}(0,0,1,t)\delta t + o(\delta t), \\
P_{0,00}(t+\delta t) &= P_{0,11}(t)[1-(\mu_1+\mu_2)\tau\delta t] + P_{0,01}(t)\lambda_2\tau\delta t + o(\delta t), \\
P_{N,11}(t+\delta t) &= P_{N,11}(t)[1-(\lambda_1+\lambda_2)\tau\delta t] + P_{N,10}(t)\mu_2\tau\delta t + o(\delta t), \\
P_{N,10}(t+\delta t) &= P_{N,10}(t)[1-(\lambda_1+\mu_2)\tau\delta t] + P_{N,11}(t)\lambda_2\tau\delta t + P_{N,00}(t)\mu_1\tau\delta t + f_{H,I,J}(N,1,0,t)\delta t + o(\delta t), \\
P_{N,01}(t+\delta t) &= 0, \\
P_{N,00}(t+\delta t) &= P_{N,00}(t)[1-(\mu_1+\mu_2)\tau\delta t] + P_{N,10}(t)\lambda_1\tau\delta t + o(\delta t).
\end{aligned}$$

In the limit $\delta t \to 0$, these equations lead to the following steady state relationships:

$$\begin{aligned}
P_{0,11} &= \tfrac{\mu_1}{\lambda_1+\lambda_2}P_{0,01}, \\
P_{0,10} &= 0, \\
P_{0,01} &= \tfrac{1}{\mu_1+\lambda_2}\left[\tfrac{1}{\tau}f_{H,I,J}(0,0,1) + \lambda_1 P_{0,11} + \mu_2 P_{0,00}\right], \\
P_{0,00} &= \tfrac{\lambda_2}{\mu_1+\mu_2}P_{0,01}, \\
P_{N,11} &= \tfrac{\mu_2}{\lambda_1+\lambda_2}P_{N,10}, \\
P_{N,10} &= \tfrac{1}{\lambda_1+\mu_2}\left[\tfrac{1}{\tau}f_{H,I,J}(N,1,0) + \lambda_2 P_{N,11} + \mu_1 P_{N,00}\right], \\
P_{N,01} &= 0, \\
P_{N,00} &= \tfrac{\lambda_1}{\mu_1+\mu_2}P_{N,10}.
\end{aligned} \tag{12}$$

Solving (10) and (11), we obtain

$$\begin{aligned}
f_{H,I,J}(h,1,0) &= f_{H,I,J}(h,0,1) = C_0 e^{Kh\tau}, \\
f_{H,I,J}(h,1,1) &= C_0 D_1 e^{Kh\tau}, \\
f_{H,I,J}(h,1,1) &= \tfrac{C_0}{D_1} e^{Kh\tau},
\end{aligned} \tag{13}$$

where D_1 is defined in the theorem and C_0 is a free constant. Substituting (13) into (12) results in

$$\begin{aligned}
P_{0,11} &= \tfrac{(\mu_1+\mu_2)C_0}{\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)\tau}, \\
P_{0,10} &= 0, \\
P_{0,01} &= \tfrac{(\lambda_1+\lambda_2)(\mu_1+\mu_2)C_0}{\lambda_2\mu_1(\lambda_1+\lambda_2+\mu_1+\mu_2)\tau}, \\
P_{0,00} &= \tfrac{(\lambda_1+\lambda_2)C_0}{\mu_1(\lambda_1+\lambda_2+\mu_1+\mu_2)\tau}, \\
P_{N,11} &= \tfrac{(\mu_1+\mu_2)C_0}{\lambda_2\lambda_1(\lambda_1+\lambda_2+\mu_1+\mu_2)\tau}e^{KN\tau}, \\
P_{N,10} &= \tfrac{(\lambda_1+\lambda_2)(\mu_1+\mu_2)C_0}{\lambda_1\mu_2(\lambda_1+\lambda_2+\mu_1+\mu_2)\tau}e^{KN\tau}, \\
P_{N,01} &= 0, \\
P_{N,00} &= \tfrac{(\lambda_1+\lambda_2)C_0}{\mu_2(\lambda_1+\lambda_2+\mu_1+\mu_2)\tau}e^{KN\tau}.
\end{aligned} \tag{14}$$

To evaluate the free constant, substitute (13) and (14) into the total probability formula,

$$\sum_{i=0}^{1}\sum_{j=0}^{1}\int_{0^+}^{N^-} f_{H,I,J}(h,i,j)dh + \sum_{i=0}^{1}\sum_{j=0}^{1}P_{0,ij} + \sum_{i=0}^{1}\sum_{j=0}^{1}P_{N,ij} = 1, \tag{15}$$

leading to

$$C_0 = \frac{1}{D_2 + D_3 + D_4}. \tag{16}$$

Thus the steady state distributions of buffer occupancy are as follows:

$$\begin{aligned}
f_{H,I,J}(h,1,1) &= \tfrac{D_1\tau}{D_2+D_3+D_4}e^{Kh\tau}, \\
f_{H,I,J}(h,1,0) &= \tfrac{\tau}{D_2+D_3+D_4}e^{Kh\tau}, \\
f_{H,I,J}(h,0,1) &= \tfrac{\tau}{D_2+D_3+D_4}e^{Kh\tau}, \\
f_{H,I,J}(h,0,0) &= \tfrac{\tau}{D_1(D_2+D_3+D_4)}e^{Kh\tau}, \\
P_{0,11} &= \tfrac{\mu_1+\mu_2}{\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)(D_2+D_3+D_4)\tau}, \\
P_{0,10} &= 0, \\
P_{0,01} &= \tfrac{(\lambda_1+\lambda_2)(\mu_1+\mu_2)}{\lambda_2\mu_1(\lambda_1+\lambda_2+\mu_1+\mu_2)(D_2+D_3+D_4)\tau}, \\
P_{0,00} &= \tfrac{\lambda_1+\lambda_2}{\mu_1(\lambda_1+\lambda_2+\mu_1+\mu_2)(D_2+D_3+D_4)\tau},
\end{aligned} \tag{17}$$

$$
\begin{aligned}
P_{N,11} &= \tfrac{\mu_1+\mu_2}{\lambda_1(\lambda_1+\lambda_2+\mu_1+\mu_2)(D_2+D_3+D_4)\tau} e^{KN\tau}, \\
P_{N,01} &= 0, \\
P_{N,10} &= \tfrac{(\lambda_1+\lambda_2)(\mu_1+\mu_2)}{\lambda_1\mu_2(\lambda_1+\lambda_2+\mu_1+\mu_2)(D_2+D_3+D_4)\tau} e^{KN\tau}, \\
P_{N,00} &= \tfrac{\lambda_1+\lambda_2}{\mu_2(\lambda_1+\lambda_2+\mu_1+\mu_2)(D_2+D_3+D_4)\tau} e^{KN\tau},
\end{aligned}
$$

The marginal steady state pdf of the buffer is given by

$$
\begin{aligned}
f_H(h) = \sum_{i=0}^{1}\sum_{j=0}^{1} f_{H,I,J}(h,i,j) + \sum_{i=0}^{1}\sum_{j=0}^{1} P_{0,ij}\delta(h) \\
+ \sum_{i=0}^{1}\sum_{j=0}^{1} P_{N,ij}\delta(h-N).
\end{aligned} \tag{18}
$$

Thus we have,

$$
f_H(h) = A_1 e^{Kh\tau} + A_2\delta(h) + A_3\delta(h-N), \qquad h \in [0,N], \tag{19}
$$

where

$$
\begin{aligned}
A_1 &= \frac{\left(2 + D_1 + \frac{1}{D_1}\right)\tau}{D_2(\tau) + D_3(\tau) + D_4(\tau)}, \\
A_2 &= \frac{D_3(\tau)}{D_2(\tau) + D_3(\tau) + D_4(\tau)}, \\
A_3 &= \frac{D_4(\tau)}{D_2(\tau) + D_3(\tau) + D_4(\tau)},
\end{aligned}
$$

and D_i, $= 1, 2, 3, 4$, are introduced in the formulation of the theorem.

Using the above distribution, the system performance metrics can be expressed as follows:

$$
\begin{aligned}
PR &= P[s_2 = 1](1 - P[h = 0, s_1 = 0 | s_2 = 1]) \\
&= P[s_1 = 1](1 - P[h = N, s_2 = 0 | s_1 = 1]) \\
&= e_2\left(1 - \frac{P_0, 01}{e_2}\right) = e_1\left(1 - \frac{P_N, 10}{e_1}\right) \\
WIP &= \int_0^N f_H(h)h\,dh = \sum_{i=0}^{1}\sum_{j=0}^{1}\int_{0^+}^{N^-} f_{H,I,J}(h,i,j)h\,dh + \sum_{i=0}^{1}\sum_{j=0}^{1} P_{N,ij}N, \\
BL_1 &= P[h = N, s_1 = 1, s_2 = 0] = P_{N,10}, \\
ST_2 &= P[h = 0, s_1 = 0, s_2 = 1] = P_{0,01}.
\end{aligned}
$$

Using (17), and substituting τ with $\frac{w_i}{W_i}$, it can be shown that

$$
\begin{aligned}
\frac{P_{0,01}}{e_2} &= Q\left(\lambda_1, \mu_1, \lambda_2, \mu_2, N\frac{w_i}{W_i}\right), \\
\frac{P_{N,10}}{e_1} &= Q\left(\lambda_2, \mu_2, \lambda_1, \mu_1, N\frac{w_i}{W_i}\right).
\end{aligned}
$$

Thus, above equations can be rewritten as

$$\begin{aligned} PR\left(\frac{w_i}{W_i}\right) &= e_2\left(1 - Q\left(\lambda_1, \mu_1, \lambda_2, \mu_2, N\frac{w_i}{W_i}\right)\right) \\ &= e_1\left(1 - Q\left(\lambda_2, \mu_2, \lambda_1, \mu_1, N\frac{w_i}{W_i}\right)\right), \end{aligned}$$

$$TP\left(\frac{w_i}{W_i}\right) = \frac{60W_i}{w_i}PR\left(\frac{w_i}{W_i}\right),$$

$$WIP\left(\frac{w_i}{W_i}\right) = \begin{cases} \dfrac{D_5\left(\frac{w_i}{W_i}\right)}{D_2\left(\frac{w_i}{W_i}\right)+D_3\left(\frac{w_i}{W_i}\right)+D_4\left(\frac{w_i}{W_i}\right)}, & \text{if } e_1 \neq e_2, \\ \dfrac{\left(\frac{D_2\left(\frac{w_i}{W_i}\right)}{2}+D_4\left(\frac{w_i}{W_i}\right)\right)N\frac{w_i}{W_i}}{D_2\left(\frac{w_i}{W_i}\right)+D_3\left(\frac{w_i}{W_i}\right)+D_4\left(\frac{w_i}{W_i}\right)}, & \text{if } e_1 = e_2, \end{cases}$$

$$ST_2\left(\frac{w_i}{W_i}\right) = e_2Q\left(\lambda_1, \mu_1, \lambda_2, \mu_2, N\frac{w_i}{W_i}\right),$$

$$BL_1\left(\frac{w_i}{W_i}\right) = e_1Q\left(\lambda_2, \mu_2, \lambda_1, \mu_1, N\frac{w_i}{W_i}\right).$$

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