The (α, β)-Precise Estimates of MTBF and MTTR: Definitions, Calculations, and Effect on Machine Efficiency and Throughput Evaluation in Serial Production Lines*

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Abstract

This paper is intended to explore how many measurements of machine up- and downtimes are necessary to calculate reliable estimates of MTBF and MTTR, and what would be the resulting effect on the machine efficiency evaluation. This issue is addressed by introducing the notion of (α, β)-precise estimate, where α characterizes the accuracy of the estimate and β its likelihood (probability). Based on this notion, the smallest number of measurements, $n^*(\alpha, \beta)$, which leads to the desired estimate, is evaluated and utilized for investigation of the induced machine efficiency estimate.

Keywords: MTBF and MTTR estimates, factory floor measurements, induced machine efficiency estimate, critical number of measurements for desired estimates accuracy.

1 Introduction

The mean time between failures (MTBF) and mean time to repair (MTTR) are of fundamental importance for production systems analysis, continuous improvement, and design. Indeed, MTBF and MTTR are used in practically every method for evaluating throughput and other performance metrics of production systems analytically (see, for instance, [1, 3–5, 9, 11, 12]) and by discrete event

*This work was supported by the DGIST R&D Program of the Ministry of Science, ICT and Future Planning (18-EE-01).

simulations (see [2, 7, 8]). In this situation, it is remarkable that the literature offers very little guidance on how many measurements of up- and downtime occurrences are necessary before reliable estimates of MTBF and MTTR can be calculated. In fact, we were able to identify only two papers discussing this issue. The first one, reporting on Ford's experience (see [13]), lists questions to be asked before MTTR can be evaluated. The second, based on GM's research (see [6]), mentions the number of up- and downtime occurrences, which has been used to estimate up- and downtime probability distributions, without going into specifics of why one or another number has been selected.

The current paper is intended to provide guidance for selecting the number of measurements necessary for calculating reliable estimates of MTBF and MTTR. The term "reliable estimate" is used here to indicate an estimate, which has the desired accuracy with the desired probability. Denoting the accuracy by α and the probability by β (see Section 2 for formalization), the goal of this paper is two-fold:

- Evaluate how many realizations of machine up- and downtimes are necessary and sufficient to obtain reliable estimates of MTBF and MTTR.
- Investigate how the uncertainty of these estimates (i.e., α and β) propagates into uncertainty of the machine efficiency evaluation.

Accordingly, the outline of this paper is as follows: Section 2 presents the definition of (α, β)-precise estimates and formulates problems addressed in the paper. In Section 3, a method for calculating the smallest number of up- and downtime measurements necessary and sufficient for the desired precision of MTBF and MTTR estimates is developed. The induced precision of machine efficiency estimate is analyzed in Section 4. Based on these results, recommendations for selecting the "right" number of up- and downtime measurements are discussed in Section 5. The conclusions and topics for future research are included in Section 6. All proofs are given in Appendices A-E.

2 Definitions and Problems Formulation

Consider an unreliable machine with up- and downtime being random variables with expected values T_{up} and T_{down}, respectively. Obviously, these expected values are the exact values of MTBF and MTTR; we use these two types of notations interchangeably $-$ depending on the issue at hand.

Let $t_{up,i}$ and $t_{down,i}$ be the durations of the i-th occurrence of up- and downtime, $i = 1, 2, \ldots$. Then, the estimates of MTBF and MTTR, based on n observations (measurements), are the following random variables:

$$\widehat{T}_{up}(n) := \frac{\sum_{i=1}^{n} t_{up,i}}{n}, \quad \widehat{T}_{down}(n) := \frac{\sum_{i=1}^{n} t_{down,i}}{n}. \tag{1}$$

Definition 1. *The estimates* $\widehat{T}_{up}(n)$ *and* $\widehat{T}_{down}(n)$ *are referred to as* (α,β)*-precise if*

$$\begin{aligned} &P\left\{\tfrac{|T_{up}-\widehat{T}_{up}(n)|}{T_{up}}\le\alpha\right\}\ge\beta,\\ &P\left\{\tfrac{|T_{down}-\widehat{T}_{down}(n)|}{T_{down}}\le\alpha\right\}\ge\beta,\end{aligned}\tag{2}$$

or, equivalently,

$$\begin{aligned} &P\left\{(1-\alpha)T_{up}\le\widehat{T}_{up}(n)\le(1+\alpha)T_{up}\right\}\ge\beta,\\ &P\left\{(1-\alpha)T_{down}\le\widehat{T}_{down}(n)\le(1+\alpha)T_{down}\right\}\ge\beta.\end{aligned}\tag{3}$$

Clearly, this definition implies that the accuracy of the estimates is quantified by α and their likelihood by β. For instance, if $\alpha = 0.1$ and $\beta = 0.9$, the appropriately selected value of n guarantees that $\widehat{T}_{up}(n)$ and $\widehat{T}_{down}(n)$ are within $\pm 10\%$ of T_{up} and T_{down}, respectively, and this event takes place with the probability at least 0.9.

Definition 2. *The smallest integer* n*, for which* (2) *takes place, is called the critical number* $n^*(\alpha,\beta)$*.*

The first problem addressed in this paper consists of two parts:

Problem 1a: Evaluate $n^*(\alpha,\beta)$ for machines with exponential reliability model (i.e., with up- and downtime distributed exponentially with parameters λ and μ, respectively). Note that this problem can be viewed as an inverse of the confidence interval problem (see, for instance, [10]), where, for a given n and β, the value of α is calculated.

Problem 1b: Generalize the results of Problem 1a to machines with non-exponential reliability models, having the coefficient of variation (CV) less than 1. Note that, as it is shown in [9], if the machine breakdown rate (respectively, repair rate) is an increasing function of time, the resulting distribution of uptime (respectively, downtime) has $CV < 1$. Empirical evidence supporting this conclusion can be found in [6].

Consider now the efficiency of an unreliable machine defined by

$$e = \frac{T_{up}}{T_{up}+T_{down}},\tag{4}$$

and its estimate

$$\widehat{e}(n) = \frac{\widehat{T}_{up}(n)}{\widehat{T}_{up}(n)+\widehat{T}_{down}(n)},\tag{5}$$

where $\widehat{T}_{up}(n)$ and $\widehat{T}_{down}(n)$ are (α,β)-precise estimates of T_{up} and T_{down}. The precision of $\widehat{e}(n)$ is induced by the precision of $\widehat{T}_{up}(n)$ and $\widehat{T}_{down}(n)$. The former can be specified as

$$P\left\{\frac{|e-\widehat{e}(n)|}{e}\le\alpha_e\right\}\ge\beta_e\tag{6}$$

or, equivalently,

$$P\{(1-\alpha_e)e \le \widehat{e}(n) \le (1+\alpha_e)e\} \ge \beta_e, \tag{7}$$

where α_e and β_e are functions of α and β.

The second problem addressed in this paper is:

Problem 2: Calculate α_e and β_e for exponential machines and generalize the results obtained for non-exponential machines with $CV < 1$.

3 Evaluation of Critical Number

3.1 Exponential Machines

Theorem 1. *The critical number, $n^*(\alpha, \beta)$, for the case of machines with exponential reliability model is the smallest integer n, which satisfies the following inequality:*

$$\beta \le \quad \sum_{i=0}^{n-1} \tfrac{1}{i!} e^{-(1-\alpha)n}((1-\alpha)n)^i - \sum_{i=0}^{n-1} \tfrac{1}{i!} e^{-(1+\alpha)n}((1+\alpha)n)^i. \tag{8}$$

Proof. See Appendix A. □

Note that since the right-hand side of (8) does not depend on the parameter of the exponential distribution, the critical number $n^*(\alpha, \beta)$ is the same for both MTBF and MTTR.

The value of $n^*(\alpha, \beta)$ can be obtained by monotonically increasing n in (8) until the inequality is satisfied. Based on this calculation, the behavior of $n^*(\alpha, \beta)$ is illustrated in Fig. 1. As expected, this function is monotonically increasing in β and monotonically decreasing in α.

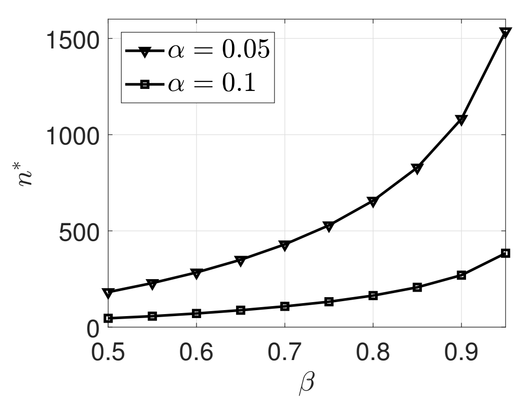


Figure 1: Critical number n^* as a function of β and α

Example 1. Let $\alpha = 0.1$ and $\beta = 0.9$. Then, as it follows from Fig. 1, $n^* = 270$. If one wants to decrease α to 0.05 (keeping the same β), $n^* = 1082$. On the other hand, if one wants to increase β to 0.95 (keeping the same α), $n^* = 385$.

Along with (8), it would be desirable to have an analytical expression for $n^*(\alpha, \beta)$. Such an expression can be derived using the fact that the numerators of the expressions in (1), being sums of iid exponential random variables, have the Erlang distribution of order n, which can be approximated by Gaussian distribution when n is sufficiently large. Based on this approximation, the following is obtained:

Theorem 2. *The Gaussian approximation of the critical number, $n_G^*(\alpha, \beta)$, is given by:*

$$n_G^*(\alpha, \beta) = \left\lceil 2 \left(\frac{erf^{-1}(\beta)}{\alpha} \right)^2 \right\rceil, \tag{9}$$

where $\lceil x \rceil$ denotes the smallest integer larger than x and $erf^{-1}(y)$ is the inverse of the error function, $erf(y) = \frac{1}{\sqrt{\pi}} \int_{-y}^{y} e^{-t^2} dt$.

Proof. See Appendix B. □

A comparison of $n^*(\alpha, \beta)$ and $n_G^*(\alpha, \beta)$ is given in Tables 1 and 2, indicating that they are practically the same for all α and β considered. Note that (9) allows for orders of magnitude faster evaluation of critical numbers than (8).

Table 1: Critical number $n^*(\alpha, \beta)$

α \ β	**0.7**	**0.8**	**0.85**	**0.9**	**0.95**
0.02	2686	4106	5181	6764	9604
0.04	672	1026	1295	1691	2401
0.06	299	456	576	751	1067
0.08	168	257	324	423	600
0.10	108	164	207	270	384

Table 2: Critical number $n_G^*(\alpha, \beta)$

α \ β	**0.7**	**0.8**	**0.85**	**0.9**	**0.95**
0.02	2686	4106	5181	6764	9604
0.04	672	1027	1296	1691	2401
0.06	299	457	576	752	1068
0.08	168	257	324	423	601
0.10	108	165	208	271	385

Using (9), the contour lines for critical number $n_G^*(\alpha, \beta)$, have been calculated and shown in Fig. 2. These contour lines offer guidance for selecting n_G^* for the desired α and β.

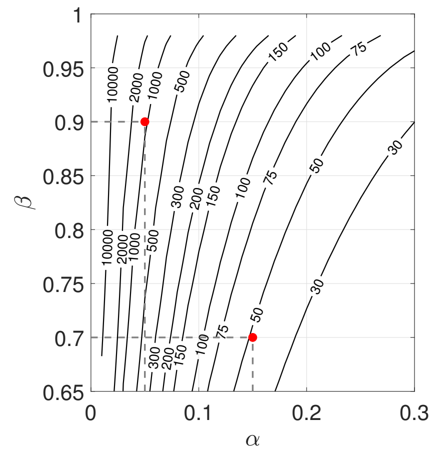


Figure 2: Contour plot of n^* as a function of α and β

Example 2. If $\alpha = 0.05$ and $\beta = 0.9$, from Fig. 2 we obtain $n^*_G(\alpha, \beta) \approx 1000$. On the other hand, if $\alpha = 0.15$ and $\beta = 0.7$, $n^*_G(\alpha, \beta) \approx 50$. In some cases, practical number of observations that one can collect during two weeks of measurements on the factory floor is about 50. Thus, in such situations only relatively inaccurate estimates of MTBF and MTTR could be obtained.

3.2 Non-exponential Machines

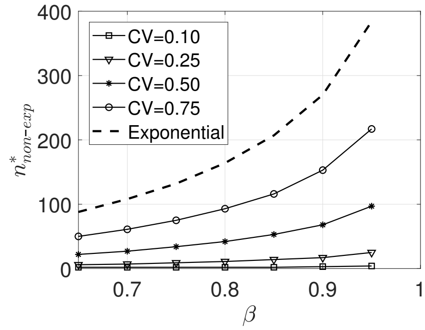
To investigate the critical number in the non-exponential case, we considered machines obeying Weibull, gamma and log-normal reliability models with $MTBF = 10$ and $CV \in \{0.1, 0.25, 0.5, 0.75\}$ and evaluated by simulations $n^*_{\text{non-exp}}(\alpha, \beta)$ for $\alpha = 0.1$ and $\beta \in \{0.65, 0.70, \ldots, 0.95\}$. The results are summarized in Fig. 3, where $n^*(\alpha, \beta)$ for exponential distribution is shown as well for comparison. From this figure, we conclude:

Observation 1. *For all non-exponential machines analyzed:*

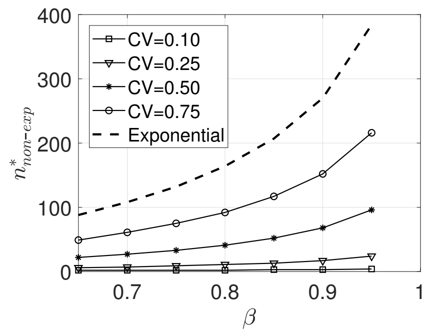
- $n^*_{non\text{-}exp}(\alpha, \beta) < n^*(\alpha, \beta)$*;*
- $n^*_{non\text{-}exp}(\alpha, \beta)$ *is independent of the machine up- and downtime distribution as long as CV is the same;*
- $n^*_{non\text{-}exp}(\alpha, \beta)$ *approaches* $n^*(\alpha, \beta)$ *when* $CV \to 1$*.*

Thus, the number of measurements, selected based on exponential assumption, can be used for non-exponential machines as well, provided $CV < 1$.

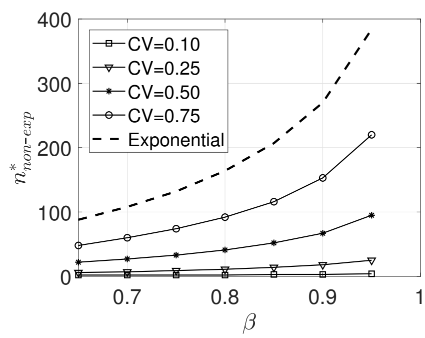
We hypothesize that Observation 1 holds not only for the distributions analyzed, but for any unimodal distribution of up- and downtime.



(a) Weibull reliability model



(b) gamma reliability model



(c) log-normal reliability model

Figure 3: Critical numbers $n^*_{\text{non-exp}}(\alpha,\beta)$ and $n^*(\alpha,\beta)$ as functions of β

4 Precision of Machine Efficiency Estimate

4.1 Exponential Machines

Consider an exponential machine with its efficiency estimate (5) and induced precision (6), characterized by α_e and β_e. The relationship between (α,β)-precise estimates of T_{up} and T_{down} and (α_e,β_e)-precise estimate of e is investigated next.

Lemma 1. *Given $\frac{|T_{up}-\widehat{T}_{up}(n)|}{T_{up}} \leq \alpha$ and $\frac{|T_{down}-\widehat{T}_{down}(n)|}{T_{down}} \leq \alpha$, the smallest α_e, which satisfies $\frac{|e-\widehat{e}(n^*(\alpha,\beta))|}{e} \leq \alpha_e$, is given by*

$$\alpha_e^{induced} = 2\alpha(1-\widehat{e}(n^*(\alpha,\beta))) + O(\alpha^2). \tag{10}$$

Proof. See Appendix C. □

Thus, $\alpha_e^{induced}$ depends explicitly on α and implicitly on β (through $n^*(\alpha,\beta)$); it also depends on $\widehat{T}_{up}$ and $\widehat{T}_{down}$ (through $\widehat{e}$). Note that $\alpha_e < \alpha$ if $\widehat{e} > 0.5$.

Below, we neglect the $O(\alpha^2)$ term in (10) and consider

$$\alpha_e = 2\alpha(1-\widehat{e}(n^*(\alpha,\beta))). \tag{11}$$

Theorem 3. *If the machine obeys the exponential reliability model and α_e in* (6) *is selected as* (11)*, the resulting β_e is given by*

$$\begin{aligned}\beta_e = \quad & \textstyle\sum_{i=0}^{n^*-1} \frac{(2n^*-2-i)!}{(n^*-1-i)!(n^*-1)!}\big[(1+2\alpha)^{n^*} \times \\ & (2+2\alpha)^{-2n^*+i+1} - (1-2\alpha)^{n^*}(2-2\alpha)^{-2n^*+i+1}\big],\end{aligned} \tag{12}$$

where n^* denotes $n^*(\alpha,\beta)$.

Proof. See Appendix D. □

Thus, β_e depends explicitly on α, implicitly on β (through $n^*(\alpha,\beta)$), and does not depend on T_{up} and T_{down} and, therefore, on e.

The values of β_e for various pairs (α,β) are illustrated in Table 3. As one can see, for all (α,β) investigated, $\beta_e > \beta$. Thus, α_e in (6) is smaller than α in (2) (if $\widehat{e} > 0.5$) and β_e in (6) is larger than β in (2). In other words, (α_e,β_e)-precise estimate of e is better than (α,β)-precise estimates of T_{up} and T_{down}.

Example 3. Assume $\alpha = 0.1$ and $\beta = 0.9$. Assume also that $\widehat{e}(n^*(\alpha,\beta)) = 0.8$. Then, according to (11) and (12), $\alpha_e = 0.044$ and $\beta_e = 0.9782$. Thus, the structure of (5) induces a significantly more precise estimate of e than that of T_{up} and T_{down}.

Similar to β, the value of β_e can be calculated using Gaussian approximation.

Proposition 1. *The Gaussian approximation of β_e, denoted as $\beta_{e,G}$, is given by*

$$\beta_{e,G} = erf(\alpha\sqrt{n^*(\alpha,\beta)}). \tag{13}$$

Justification. See Appendix E. □

A comparison of β_e and $\beta_{e,G}$ is given in Tables 3 and 4. As one can see, these values are almost always the same.

Table 3: Values of β_e as a function of α and β

α \ β	**0.7**	**0.8**	**0.85**	**0.9**	**0.95**
0.02	0.8573	0.9300	0.9582	0.9799	0.9944
0.04	0.8575	0.9299	0.9580	0.9797	0.9942
0.06	0.8578	0.9298	0.9576	0.9794	0.9940
0.08	0.8576	0.9294	0.9571	0.9788	0.9937
0.10	0.8585	0.9294	0.9568	0.9782	0.9933

Table 4: Values of $\beta_{e,G}$ as a function of α and β

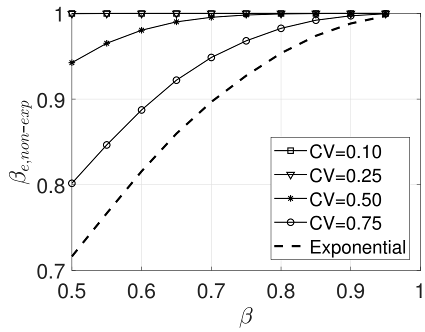
α \ β	**0.7**	**0.8**	**0.85**	**0.9**	**0.95**
0.02	0.8573	0.9301	0.9582	0.9800	0.9944
0.04	0.8575	0.9300	0.9582	0.9800	0.9944
0.06	0.8577	0.9300	0.9583	0.9799	0.9944
0.08	0.8575	0.9303	0.9583	0.9800	0.9944
0.10	0.8584	0.9299	0.9581	0.9799	0.9944

4.2 Non-exponential Machines

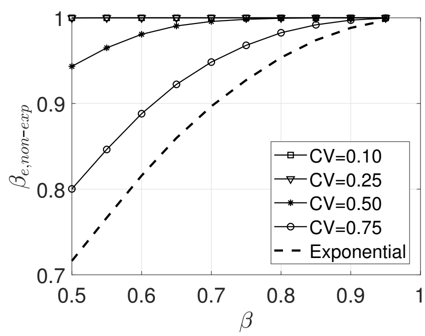
To investigate the behavior of β_e in (6) for non-exponential machines (with α_e given in (11)), we use simulations similar to those of Subsection 3.2. Specifically, consider machines obeying either Weibull or gamma or log-normal reliability model with $T_{down} = 2$, $e \in \{0.7, 0.75, \ldots, 0.9\}$, $\alpha = 0.1$ and $CV \in \{0.1, 0.25, 0.5, 0.75\}$. For each of these cases, we evaluated $\beta_{e,\text{non-exp}}(n^*(\alpha,\beta))$ by simulations as a function of β. The results are summarized in Fig. 4, where β_e is shown as well for comparison. From this figure, we conclude:

Observation 2. *For all non-exponential machines analyzed:*

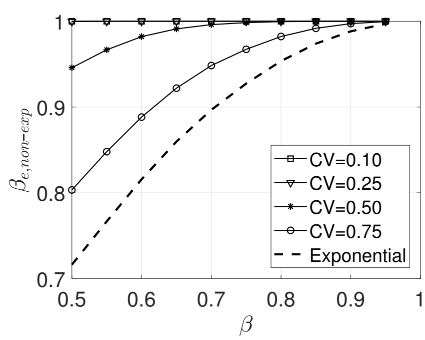
- $\beta_{e,non\text{-}exp}(n^*(\alpha,\beta)) > \beta_e(n^*(\alpha,\beta))$;
- $\beta_{e,non\text{-}exp}(n^*(\alpha,\beta))$ *is independent of machine efficiency;*
- $\beta_{e,non\text{-}exp}(n^*(\alpha,\beta))$ *is independent of machine reliability model as long as* CV *is the same;*
- $\beta_{e,non\text{-}exp}(n^*(\alpha,\beta))$ *approaches* $\beta_e(n^*(\alpha,\beta))$ *when* $CV \to 1$.



(a) Weibull reliability model



(b) gamma reliability model



(c) log-normal reliability model

Figure 4: $\beta_{e,\text{non-exp}}(n^*(\alpha,\beta))$ and $\beta_e(n^*(\alpha,\beta))$ as functions of β

Therefore, $\beta_{e,\text{non-exp}}(n^*(\alpha,\beta))$ is at least as large as $\beta_e(n^*(\alpha,\beta))$, if CV of up- and downtime is less than 1.

5 Discussion and Recommendations

The purpose of this section is to pose and answer several questions concerning practical issues of MTBF and MTTR evaluation, as well as evaluation of e.

- How many measurements of $t_{up,i}$ and $t_{down,i}$ are necessary to obtain reliable estimates of T_{up} and T_{down}, e.g., with $\alpha = 0.04$ and $\beta = 0.9$? As it follows from Section 3 (see Table 1), the answer is $n^*(0.04, 0.9) = 1691$.

- How many measurements of $t_{up,i}$ and $t_{down,i}$ are necessary to obtain an estimate of e at least as precise as those of T_{up} and T_{down}? Assuming $\widehat{e} \geq 0.8$ and utilizing (11) with $\alpha_e = 0.04$, we obtain $\alpha \geq 0.1$. Then, from Table 3 with $\alpha = 0.1$, we obtain that $\beta = 0.8$ results in $\beta_e > 0.9$. Finally, from Table 1 with $\alpha = 0.1$ and $\beta = 0.8$, we conclude that $n^* = 164$.

Thus, sufficiently accurate evaluation of e may require, somewhat surprisingly, a much smaller number of measurements than evaluation of T_{up} and T_{down}; in this example, order of magnitude smaller.

Another practical issue of importance is the time necessary to collect the required number of measurements. We define this time as:

$$T_{eval} = (T_{up} + T_{down})n^*(\alpha, \beta),$$

where T_{up} and T_{down} are the average values of up- and downtime, and $n^*(\alpha, \beta)$ is the number of measurements necessary to obtain (α, β)-precise estimates of T_{up} and T_{down}. Obviously, T_{eval} depends on the duration of what can be called the "up/downtime cycle", $T_{up}+T_{down}$. If this cycle is very long, say, one month, the observation period will be quite long for any number of required measurements. Such breakdowns are referred to as *catastrophic*. They are not considered in the standard methods of Production Systems Engineering. What is considered, are up/downtime cycles, which occur several times during a shift, e.g., 5-10 times per eight hours. In this case, to collect a reasonable number of measurements, one would have to wait 1-2 weeks. Such breakdowns are referred to as *regular*, and this is what Production Systems Engineering is based on.

Thus, this work shows that, in practical terms, one would have to wait one or two weeks before relatively reliable estimates of T_{up} and T_{down} can be obtained. This conclusion coincides with our practical experience, according to which reliable evaluation of system's "health" can be obtained using a two-week observation period.

6 Conclusions and Future Work

This paper provides a method for calculating the smallest number of measurements necessary and sufficient for evaluating MTBF and MTTR of unreliable machines with the desired accuracy. It turns out that, in most cases, this number is quite large (in practical scenarios, ranging in hundreds). Collecting such a large number of measurements in a relatively short period time when they would be useful for decision-making, is often impossible. This leads to the conclusion that MTBF and MTTR are hardly available in practice with a high precision.

On the other hand, it is shown that induced estimates of the machines efficiency are, in many cases, much higher than the accuracy of MTBF and MTTR used for their calculation. This is a fortunate situation, where a performance index may be calculated with a higher precision than the data used in its evaluation. Based on this phenomenon, this paper shows how a smaller number of up- and downtime measurements can be determined to obtain the desired accuracy of machine efficiency estimate. As in turns out, in practical cases this number may be in dozens rather than in hundreds.

Several problems in this area remain, however, open. The main one is to evaluate the induced precision of the throughput and other performance metrics of production systems with unreliable machines. Solving this problem for serial lines and assembly operations will provide a relatively complete methodology for collecting data on the factory floor necessary and sufficient for performance analysis of production systems using either analytical or numerical tools.

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Appendix A Proof of Theorem 1

First, we prove the statement of this theorem for $\widehat{T}_{up}$, and then address the issue for $\widehat{T}_{down}$.

Rewrite the first expression in (3) as follows:

$$\begin{aligned} & P\left\{(1-\alpha)T_{up} \le \widehat{T}_{up}(n) \le (1+\alpha)T_{up}\right\} \\ = \; & P\left\{\widehat{T}_{up}(n) \le (1+\alpha)T_{up}\right\} \\ & -P\left\{\widehat{T}_{up}(n) \le (1-\alpha)T_{up}\right\} \ge \beta. \end{aligned} \tag{A1}$$

To evaluate this probability, observe that the numerator of the first expression in (1) for a machine with exponential reliability model is a sum of iid exponential random variables, and, thus, obeys Erlang distribution with shape parameter n and scale parameter $T_{up} = \frac{1}{\lambda}$. Therefore, the cdf of $Y(n) = \sum_{i=1}^{n} t_{up,i}$ is

$$F_{Y(n)}(y) = P\{Y(n) \le y\} = 1 - \sum_{i=0}^{n-1} \frac{1}{i!} e^{-\lambda y} (\lambda y)^i. \tag{A2}$$

Substituting this expression in (A1), we obtain

$$\begin{aligned} & P\left\{\widehat{T}_{up}(n) \le (1+\alpha)T_{up}\right\} - P\left\{\widehat{T}_{up}(n) \le (1-\alpha)T_{up}\right\} \\ =& P\left\{Y(n) \le (1+\alpha)nT_{up}\right\} - P\left\{Y(n) \le (1-\alpha)nT_{up}\right\} \\ =& F_{Y(n)}\left((1+\alpha)nT_{up}\right) - F_{Y(n)}\left((1-\alpha)nT_{up}\right) \\ =& 1 - \sum_{i=0}^{n-1} \frac{1}{i!} e^{-\lambda(1+\alpha)nT_{up}} \left(\lambda(1+\alpha)nT_{up}\right)^i \\ & -1 + \sum_{i=0}^{n-1} \frac{1}{i!} e^{-\lambda(1-\alpha)nT_{up}} \left(\lambda(1-\alpha)nT_{up}\right)^i \\ =& \sum_{i=0}^{n-1} \frac{1}{i!} e^{-(1-\alpha)n} ((1-\alpha)n)^i - \sum_{i=0}^{n-1} \frac{1}{i!} e^{-(1+\alpha)n} ((1+\alpha)n)^i \ge \beta. \end{aligned} \tag{A3}$$

Thus, the critical number $n^*(\alpha, \beta)$ is the smallest integer n that satisfies the following inequality:

$$\beta \leq \sum_{i=0}^{n-1} \frac{1}{i!} e^{-(1-\alpha)n}((1-\alpha)n)^i - \sum_{i=0}^{n-1} \frac{1}{i!} e^{-(1+\alpha)n}((1+\alpha)n)^i. \qquad \text{(A4)}$$

Since the last expression is independent of λ, it takes place for any exponential distribution, i.e., for downtime as well. □

Appendix B Proof of Theorem 2

For large n, $Y(n)$ can be approximated by the Gaussian distribution, with mean $M = n/\lambda = nT_{up}$ and variance $V = n/\lambda^2 = nT_{up}^2$.

Therefore we have:

$$\begin{aligned} & P\{(1-\alpha)nT_{up} \leq Y(n) \leq (1+\alpha)nT_{up}\} \\ = \; & P\left\{-\alpha\sqrt{n} \leq \frac{Y(n)-nT_{up}}{\sqrt{n}T_{up}} \leq \alpha\sqrt{n}\right\} \\ \approx \; & P\{|Z| \leq \alpha\sqrt{n}\}, \end{aligned} \qquad \text{(B1)}$$

where $Z \sim N(0,1)$ denotes a Gaussian random variable with mean 0 and variance 1. Therefore, the Gaussian approximation of β, denoted as β_G, is:

$$\beta \approx \beta_G \leq \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} f_Z(z)dz = \operatorname{erf}\left(\frac{\alpha\sqrt{n}}{\sqrt{2}}\right). \qquad \text{(B2)}$$

Thus, $n_G^* = \left\lceil 2\left(\frac{\operatorname{erf}^{-1}(\beta)}{\alpha}\right)^2 \right\rceil$. □

Appendix C Proof of Lemma 1

Denote $\rho = \frac{T_{down}}{T_{up}}$ and $\widehat{\rho} = \frac{\widehat{T}_{down}}{\widehat{T}_{up}}$. Since $(1-\alpha)T_{up} \leq \widehat{T}_{up} \leq (1+\alpha)T_{up}$, and $0 < (1-\alpha)T_{down} \leq \widehat{T}_{down} \leq (1+\alpha)T_{down}$, we have:

$$\begin{aligned} & \frac{(1-\alpha)T_{up}}{(1+\alpha)T_{down}} \leq \frac{\widehat{T}_{up}}{\widehat{T}_{down}} \leq \frac{(1+\alpha)T_{up}}{(1-\alpha)T_{down}} \\ \Leftrightarrow \; & \frac{1-\alpha}{1+\alpha}\frac{1}{\rho} \leq \frac{1}{\widehat{\rho}} \leq \frac{1+\alpha}{1-\alpha}\frac{1}{\rho} \\ \Leftrightarrow \; & \frac{1-\alpha}{1+\alpha}\widehat{\rho} \leq \rho \leq \frac{1+\alpha}{1-\alpha}\widehat{\rho} \\ \Leftrightarrow \; & \frac{-2\alpha}{1+\alpha}\widehat{\rho} \leq \rho - \widehat{\rho} \leq \frac{2\alpha}{1-\alpha}\widehat{\rho}. \end{aligned} \qquad \text{(C1)}$$

Dividing (C1) by $(1+\rho)(1+\widehat{\rho})$, and substituting $\frac{1}{1+\rho}$ with e, and $\frac{1}{1+\widehat{\rho}}$ with $\widehat{e}$, we obtain:

$$\frac{-2\alpha}{1+\alpha}e(1-\widehat{e}) \leq \widehat{e} - e \leq \frac{2\alpha}{1-\alpha}e(1-\widehat{e}). \qquad \text{(C2)}$$

For small α, using Taylor expansion we have:

$$-2\alpha e(1-\widehat{e}) + O(\alpha^2) \leq \widehat{e} - e \leq 2\alpha e(1-\widehat{e}) + O(\alpha^2). \qquad \text{(C3)}$$

Therefore, we obtain:

$$\frac{|e-\widehat{e}|}{e} \leq \alpha_e = 2(1-\widehat{e})\alpha + O(\alpha^2). \tag{C4}$$

$\square$

Appendix D Proof of Theorem 3

As discussed in Appendix A, $Y(n) = \sum_{i=1}^{n} t_{up,i}$ obeys Erlang distribution with shape parameter n and scale parameter T_{up}. Therefore, the pdf of $Y(n)$ is

$$f_{Y(n)}(x) = (\frac{1}{T_{up}})^n \cdot \frac{x^{n-1}e^{-\frac{x}{T_{up}}}}{(n-1)!},$$

and the pdf of $\widehat{T}_{up}(n) = \frac{Y(n)}{n}$ is

$$f_{\widehat{T}_{up}(n)}(x) = n \cdot (\frac{1}{T_{up}})^n \cdot \frac{(n\cdot x)^{n-1}e^{-\frac{n\cdot x}{T_{up}}}}{(n-1)!}.$$

Similarly,

$$f_{\widehat{T}_{down}(n)}(x) = n \cdot (\frac{1}{T_{down}})^n \cdot \frac{(n\cdot x)^{n-1}e^{-\frac{n\cdot x}{T_{down}}}}{(n-1)!}.$$

Since $\widehat{T}_{up}(n)$ and $\widehat{T}_{down}(n)$ are independent,

$$f_{\widehat{T}_{up}(n),\widehat{T}_{down}(n)}(x,y) = f_{\widehat{T}_{up}(n)}(x) \cdot f_{\widehat{T}_{down}(n)}(y).$$

Using the notation in Appendix C, $\rho = \frac{T_{down}}{T_{up}}$ and $\widehat{\rho} = \frac{\widehat{T}_{down}(n)}{\widehat{T}_{up}(n)}$, we can write:

$$\begin{aligned}
& P\left\{\tfrac{|e-\widehat{e}(n^*)|}{e} \leq \alpha_e\right\} \\
= \;& P\left\{-2\alpha\big(1-\widehat{e}(n^*)\big)e \leq e-\widehat{e}(n^*) \leq 2\alpha\big(1-\widehat{e}(n^*)\big)e\right\} \\
= \;& P\left\{2\alpha\tfrac{\widehat{\rho}}{1+\widehat{\rho}}\tfrac{1}{1+\rho} \leq \tfrac{1}{1+\rho} - \tfrac{1}{1+\widehat{\rho}} \leq 2\alpha\tfrac{\widehat{\rho}}{1+\widehat{\rho}}\tfrac{1}{1+\rho}\right\} \\
= \;& P\left\{\tfrac{1}{1+2\alpha}\rho \leq \widehat{\rho} \leq \tfrac{1}{1-2\alpha}\rho\right\}.
\end{aligned} \tag{D1}$$

Denote $\rho_1 = \frac{1}{1+2\alpha}\rho$, and $\rho_2 = \frac{1}{1-2\alpha}\rho$, we derive:

$$\begin{aligned}
& P\left\{\tfrac{|e-\widehat{e}(n^*)|}{e} \leq \alpha_e\right\} \\
= \;& \int_0^{+\infty}\int_{\rho_1 x}^{\rho_2 x} f_{\widehat{T}_{up}(n^*)}(x) f_{\widehat{T}_{down}(n^*)}(y)dydx \\
= \;& (\tfrac{n^{*2}}{T_{up}T_{down}})^{n^*}(\tfrac{1}{(n^*-1)!})^2 \int_0^{+\infty} x^{n^*-1}e^{-\frac{n^*}{T_{up}}x}\int_{\rho_1 x}^{\rho_2 x} y^{n^*-1}e^{-\frac{n^*}{T_{down}}y}dydx,
\end{aligned} \tag{D2}$$

where

$$
\begin{aligned}
& \int_{\rho_1 x}^{\rho_2 x} y^{n^*-1} e^{-\frac{n^*}{T_{down}} y} dy \\
= \;& e^{-\frac{n^*}{T_{down}} y} \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!}{(n^*-1-i)!} (\tfrac{T_{down}}{n^*})^{i+1} y^{n^*-1-i} |_{\rho_1 x}^{\rho_2 x} \\
= \;& e^{-\frac{n^*}{T_{down}} \rho_1 x} \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!}{(n^*-1-i)!} (\tfrac{T_{down}}{n^*})^{i+1} (\rho_1 x)^{n^*-1-i} - \\
& e^{-\frac{n^*}{T_{down}} \rho_2 x} \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!}{(n^*-1-i)!} (\tfrac{T_{down}}{n^*})^{i+1} (\rho_2 x)^{n^*-1-i}.
\end{aligned}
\tag{D3}
$$

Therefore,

$$
\begin{aligned}
& P\left\{ \tfrac{|e-\widehat{e}(n^*)|}{e} \le \alpha_e \right\} \\
= \;& (\tfrac{n^{*2}}{T_{up} T_{down}})^{n^*} (\tfrac{1}{(n^*-1)!})^2 \int_0^{+\infty} x^{n^*-1} e^{-\frac{n^*}{T_{up}} x} \\
& \Big(e^{-\frac{n^*}{T_{down}} \rho_1 x} \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!}{(n^*-1-i)!} (\tfrac{T_{down}}{n^*})^{i+1} (\rho_1 x)^{n^*-1-i} - \\
& e^{-\frac{n^*}{T_{down}} \rho_2 x} \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!}{(n^*-1-i)!} (\tfrac{T_{down}}{n^*})^{i+1} (\rho_2 x)^{n^*-1-i} \Big) dx.
\end{aligned}
\tag{D4}
$$

Denote

$$
\begin{aligned}
\mathcal{A} = \;& \int_0^{+\infty} x^{n^*-1} e^{-\frac{n^*}{T_{up}} x} \Big(e^{-\frac{n^*}{T_{down}} \rho_1 x} \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!}{(n^*-1-i)!} \times \\
& (\tfrac{T_{down}}{n^*})^{i+1} (\rho_1 x)^{n^*-1-i} \Big) dx,
\end{aligned}
\tag{D5}
$$

and

$$
\begin{aligned}
\mathcal{B} = \;& \int_0^{+\infty} x^{n^*-1} e^{-\frac{n^*}{T_{up}} x} \Big(e^{-\frac{n^*}{T_{down}} \rho_2 x} \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!}{(n^*-1-i)!} \times \\
& (\tfrac{T_{down}}{n^*})^{i+1} (\rho_2 x)^{n^*-1-i} \Big) dx,
\end{aligned}
\tag{D6}
$$

we can write:

$$
P\left\{ \frac{|e-\widehat{e}(n^*)|}{e} \le \alpha_e \right\} = (\frac{n^{*2}}{T_{up} T_{down}})^{n^*} (\frac{1}{(n^*-1)!})^2 \big(\mathcal{A} - \mathcal{B}\big). \tag{D7}
$$

We obtain:

$$
\begin{aligned}
\mathcal{A} = \;& \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!}{(n^*-1-i)!} (\tfrac{T_{down}}{n^*})^{i+1} \rho_1^{n^*-1-i} \int_0^{+\infty} e^{-(\frac{n^*}{T_{up}} + \frac{n^*}{T_{down}} \rho_1) x} x^{2n^*-2-i} dx \\
= \;& \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!}{(n^*-1-i)!} (\tfrac{T_{down}}{n^*})^{i+1} \rho_1^{n^*-1-i} \Bigg(e^{-(\frac{n^*}{T_{up}} + \frac{n^*}{T_{down}} \rho_1) x} \times \\
& \sum_{j=0}^{a} (-1)^j \frac{a!}{(a-j)! \Big[-(\frac{n^*}{T_{up}} + \frac{n^*}{T_{down}} \rho_1) \Big]^{j+1}} x^{a-j} |_0^{+\infty} \Bigg) \\
= \;& n^{*-2n^*} \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!(2n^*-2-i)!}{(n^*-1-i)!} T_{up}^{n^*} T_{down}^{n^*} (1+2\alpha)^{n^*} (2+2\alpha)^{-2n^*+i+1}.
\end{aligned}
\tag{D8}
$$

Similarly,

$$
\mathcal{B} = \; n^{*-2n^*} \sum_{i=0}^{n^*-1} \tfrac{(n^*-1)!(2n^*-2-i)!}{(n^*-1-i)!} T_{up}^{n^*} T_{down}^{n^*} (1-2\alpha)^{n^*} (2-2\alpha)^{-2n^*+i+1}. \tag{D9}
$$

Substitute values of $\mathcal{A}$ and $\mathcal{B}$ in (D7), we get:

$$\begin{aligned} & P\left\{\frac{|e-\widehat{e}(n^*)|}{e} \leq \alpha_e\right\} \\ = & \textstyle\sum_{i=0}^{n^*-1} \frac{(2n^*-2-i)!}{(n^*-1-i)!(n^*-1)!}\left[(1+2\alpha)^{n^*}(2+2\alpha)^{-2n^*+i+1}\right. \\ & \left.-(1-2\alpha)^{n^*}(2-2\alpha)^{-2n^*+i+1}\right]. \end{aligned} \tag{D10}$$

Thus, by (6), we have:

$$\begin{aligned} \beta = & \textstyle\sum_{i=0}^{n^*-1} \frac{(2n^*-2-i)!}{(n^*-1-i)!(n^*-1)!}\left[(1+2\alpha)^{n^*}(2+2\alpha)^{-2n^*+i+1}\right. \\ & \left.-(1-2\alpha)^{n^*}(2-2\alpha)^{-2n^*+i+1}\right]. \end{aligned} \tag{D11}$$

$\square$

Appendix E Justification of Proposition 1

As mentioned in Appendix D, the pdfs of $\widehat{T}_{up}(n)$ and $\widehat{T}_{down}(n)$ are as follow:

$$f_{\widehat{T}_{up}(n)}(x) = \frac{1}{(n-1)!}(\frac{n}{T_{up}})^n x^{n-1} e^{-\frac{n}{T_{up}}x},$$

$$f_{\widehat{T}_{down}(n)}(x) = \frac{1}{(n-1)!}(\frac{n}{T_{down}})^n x^{n-1} e^{-\frac{n}{T_{down}}x}.$$

Therefore, the pdf of $\widehat{\rho} = \frac{\widehat{T}_{down}(n)}{\widehat{T}_{up}(n)}$ is:

$$\begin{aligned} f_{\widehat{\rho}}(z) &= \textstyle\int_{-\infty}^{\infty} |x| f_{\widehat{T}_{down}(n)}(xz) f_{\widehat{T}_{up}(n)}(x)dx \\ &= \textstyle\int_0^{\infty} x f_{\widehat{T}_{down}(n)}(xz) f_{\widehat{T}_{up}(n)}(x)dx \\ &= (\frac{n^2}{T_{up}T_{down}})^n (\frac{1}{(n-1)!})^2 \textstyle\int_0^{\infty} x(xz)^{n-1} e^{-\frac{nzx}{T_{down}}} x^{n-1} e^{-\frac{nx}{T_{up}}} dx \\ &= (\frac{n^2}{T_{up}T_{down}})^n \frac{(2n-1)!}{(n-1)!^2} z^{n-1} (\frac{n}{T_{down}}z + \frac{n}{T_{up}})^{-2n}. \end{aligned} \tag{E1}$$

As mentioned in Appendix C, denote $\rho = \frac{T_{down}}{T_{up}}$. From (E1), we derive the pdf of $\widehat{e}(n) = \frac{1}{1+\widehat{\rho}}$:

$$f_{\widehat{e}(n)}(x) = \frac{(2n-1)!}{(n-1)!^2} \frac{1}{x(1-x)} \left[\rho^{-1}(\frac{1}{x}-1) + 2 + \rho(\frac{1}{x}-1)^{-1}\right]^{-n}, x \in (0,1). \tag{E2}$$

We plot the pdf of $\widehat{e}(n)$ (see Figure E1 as an example). The figure shows it is a unimodal distribution, and we approximate it with a Gaussian distribution. We use $e = \frac{T_{up}}{T_{up}+T_{down}}$ as the mean of the Gaussian distribution, and use the error propagation formula to compute the approximated variance:

$$var\left(\widehat{e}(n)\right) = \left(\frac{\partial \widehat{e}(n)}{\partial \widehat{T}_{up}(n)}\right)^2 \cdot var\left(\widehat{T}_{up}(n)\right) + \left(\frac{\partial \widehat{e}(n)}{\partial \widehat{T}_{down}(n)}\right)^2 \cdot var\left(\widehat{T}_{down}(n)\right). \tag{E3}$$

For large n, $\widehat{T}_{up}(n)$ and $\widehat{T}_{down}(n)$ can be approximated by the Gaussian distribution, with mean T_{up} and T_{down}, and variance $var\left(\widehat{T}_{up}(n)\right) = \frac{T_{up}^2}{n}$ and $var\left(\widehat{T}_{down}(n)\right) = \frac{T_{down}^2}{n}$. Substitute the variances of $\widehat{T}_{up}$ and $\widehat{T}_{down}$ into (E3) , we have:

$$var\left(\widehat{e}\right) = \frac{\widehat{T}_{down}^2 T_{up}^2 + \widehat{T}_{up}^2 T_{down}^2}{n(\widehat{T}_{up}+\widehat{T}_{down})^4}. \tag{E4}$$

Thus, the Gaussian approximated distribution of $\widehat{e}(n)$ has mean e and variance $\frac{\widehat{T}_{down}^2 T_{up}^2 + \widehat{T}_{up}^2 T_{down}^2}{n(\widehat{T}_{up}+\widehat{T}_{down})^4}$. The exact pdfs of $\widehat{e}(n)$ and its Gaussian approximated pdfs, for $e = 0.8$, are shown in Figure E1 as an example. We can see that the Gaussian approximated distribution approaches to the exact distribution of $\widehat{e}(n)$ as n increases.

From (6), for observing n^* number of realizations of up- and downtimes, we can write:

$$\begin{aligned} & Pr\left\{(1-\alpha_e)e \le \widehat{e}(n^*) \le (1+\alpha_e)e\right\} \\ = \; & Pr\left\{\widehat{e}(n^*) \le (1+\alpha_e)e\right\} - Pr\left\{\widehat{e}(n^*) \le (1-\alpha_e)e\right\} \\ = \; & F_{\widehat{e}(n^*)}\big((1+\alpha_e)e\big) - F_{\widehat{e}(n^*)}\big((1-\alpha_e)e\big). \end{aligned} \tag{E5}$$

Using the Gaussian approximation of $\widehat{e}(n^*)$, we obtain:

$$\begin{aligned} & Pr\left\{(1-\alpha_e)e \le \widehat{e}(n^*) \le (1+\alpha_e)e\right\} \\ \approx \; & \tfrac{1}{2}\big[1 + erf\big(\tfrac{(1+\alpha_e)e-e}{\sqrt{2var(\widehat{e}(n^*))}}\big)\big] - \tfrac{1}{2}\big[1 + erf\big(\tfrac{(1-\alpha_e)e-e}{\sqrt{2var(\widehat{e}(n^*))}}\big)\big]. \end{aligned} \tag{E6}$$

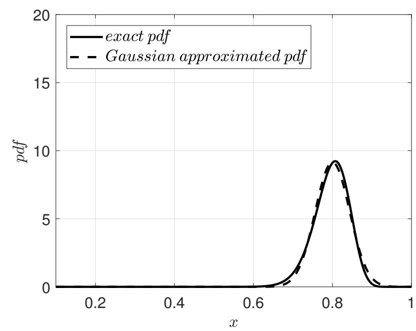
Under the assumption that for large n^*, $\widehat{T}_{up}(n^*) \approx T_{up}$, and $\widehat{T}_{down}(n^*) \approx T_{down}$, and take $\alpha_e = 2\widehat{e}(1-\widehat{e})$ in Lemma 1, we derive:

$$\begin{aligned} & Pr\left\{(1-\alpha_e)e \le \widehat{e}(n^*) \le (1+\alpha_e)e\right\} \\ \approx \; & \tfrac{1}{2}\left[erf(\alpha\sqrt{n^*}) - erf(-\alpha\sqrt{n^*})\right] \\ = \; & erf(\alpha\sqrt{n^*}). \end{aligned} \tag{E7}$$

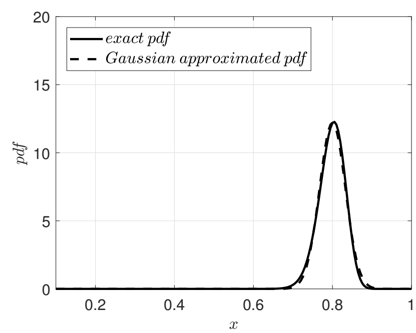
Therefore, by (6), we have:

$$\beta_e \approx erf(\alpha\sqrt{n^*}). \tag{E8}$$

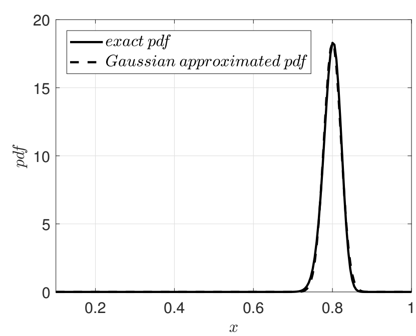
□



(a) $e = 0.8$, $n = 27$



(b) $e = 0.8$, $n = 48$



(c) $e = 0.8$, $n = 108$

Figure E1: Plots of the exact and the Gaussian approximated probability density functions of $\widehat{e}(n)$, for $n = 27, 48, 108$, and $e = 0.8$.