EECS 442 – Computer vision

Epipolar Geometry

• Why is stereo useful?
• Epipolar constraints
• Essential and fundamental matrix
• Estimating F
• Examples

Reading: [AZ] Chapters: 4, 9, 11
[FP] Chapters: 10
## Recovering structure from a single view

<table>
<thead>
<tr>
<th>From calibration rig</th>
<th>→ location/pose of the rig, ( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>From points and lines at infinity + orthogonal lines and planes</td>
<td>→ structure of the scene, ( K )</td>
</tr>
<tr>
<td>Knowledge about scene (point correspondences, geometry of lines &amp; planes, etc...)</td>
<td></td>
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</tbody>
</table>
Recovering structure from a single view

Why is it so difficult?

Intrinsic ambiguity of the mapping from 3D to image (2D)
Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)

Courtesy slide S. Lazebnik
Two eyes help!
Two eyes help!

This is called **triangulation**
Triangulation

• Find $X$ that minimizes

$$d^2(x_1, P_1X) + d^2(x_2, P_2X)$$
Stereo-view geometry

- Scene geometry: Find coordinates of 3D point from its projection into 2 or more images.

- Correspondence: Given a point in one image, how can I find the corresponding point $x'$ in another one?

- Camera geometry: Given corresponding points in two images, find camera matrices, position, and pose.
**Epipolar geometry**

- **Epipolar Plane**
- **Baseline**
- **Epipolar Lines**
- **Epipoles** $e_1, e_2$
  - intersections of baseline with image planes
  - projections of the other camera center
  - vanishing points of camera motion direction
Example: Converging image planes
Example: Parallel image planes

- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to x axis
Example: Parallel image planes
Example: Forward translation

- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)
Epipolar Constraint

- Two views of the same object
- Suppose I know the camera positions and camera matrices
- Given a point on left image, how can I find the corresponding point on right image?
Epipolar Constraint

• Potential matches for $p$ have to lie on the corresponding epipolar line $l'$.

• Potential matches for $p'$ have to lie on the corresponding epipolar line $l$. 
Epipolar Constraint

\[ \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \]

\[
\begin{align*}
P & \rightarrow \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} \\
M & = K \begin{bmatrix} I & 0 \end{bmatrix} \\
M' & = K \begin{bmatrix} R & T \end{bmatrix}
\end{align*}
\]
Epipolar Constraint

\[ p^T \cdot [T \times (R \ p')] = 0 \]

K₁ and K₂ are known (calibrated cameras)

Perpendicular to epipolar plane
Cross product as matrix multiplication

\[ \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b} \]
Epipolar Constraint

\[ p^T \cdot [T \times (R \ p')] = 0 \rightarrow p^T \cdot [T_x] \cdot R \ p' = 0 \]

\textbf{E} = \text{essential matrix}

(Longuet-Higgins, 1981)
Epipolar Constraint

- $E x_2$ is the epipolar line associated with $x_2$ ($l_1 = E x_2$)
- $E^T x_1$ is the epipolar line associated with $x_1$ ($l_2 = E^T x_1$)
- $E$ is singular (rank two)
- $E e_2 = 0$ and $E^T e_1 = 0$
- $E$ is 3x3 matrix; 5 DOF
Epipolar Constraint

P → M P → p = \begin{bmatrix} u \\ v \end{bmatrix}

M = \begin{bmatrix} K & I & 0 \\ \text{unknown} \end{bmatrix}
Epipolar Constraint

\[ p \rightarrow K p \]

\[ p' \rightarrow K' p' \]

\[ p^T \cdot [T_x] \cdot R \ p' = 0 \rightarrow (K^{-1} p)^T \cdot [T_x] \cdot R \ K'^{-1} \ p' = 0 \]

\[ p^T K^{-T} \cdot [T_x] \cdot R \ K'^{-1} \ p' = 0 \rightarrow p^T [F] p' = 0 \]
Epipolar Constraint

\[ p \rightarrow K p \]

\[ p' \rightarrow K' p' \]

\[ p^T F p' = 0 \]

\[ F = \text{Fundamental Matrix} \]

(Faugeras and Luong, 1992)
Epipolar Constraint

- $Fx_2$ is the epipolar line associated with $x_2$ ($l_1 = Fx_2$)
- $F^Tx_1$ is the epipolar line associated with $x_1$ ($l_2 = F^Tx_1$)
- $F$ is singular (rank two)
- $Fe_2 = 0$ and $F^Te_1 = 0$
- $F$ is 3x3 matrix; 7 DOF
Why $F$ is useful?

- Suppose $F$ is known
- No additional information about the scene and camera is given
- Given a point on left image, how can I find the corresponding point on right image?

\[ l' = F^T x \]
Why F is useful?

• F captures information about the epipolar geometry of 2 views + camera parameters

• **MORE IMPORTANTLY:** F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

• Powerful tool in:
  • 3D reconstruction
  • Multi-view object/scene matching
Estimating F

The Eight-Point Algorithm

(Longuet-Higgins, 1981)
(Hartley, 1995)

\[
P \rightarrow p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad P \rightarrow p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}
\]

\[p^T F p' = 0\]
Estimating $F$

$$p^T F p' = 0$$

$$(u, v, 1) \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

Let's take 8 corresponding points
Estimating F
Estimating $F$

- Homogeneous system $Wf = 0$
- Rank 8 → A non-zero solution exists (unique)
- If $N > 8$ → Lsq. solution by SVD! → $\hat{F}$
  $\|f\| = 1$
Rank-2 constraint

\[ p^T \hat{F} p' = 0 \]

The estimated F may have full rank \((\det(F) \neq 0)\)
\((F \text{ should have rank}=2 \text{ instead})\)

Find F that minimizes \[ \|F - \hat{F}\| = 0 \]
\(\text{Frobenius norm}\)

Subject to \(\det(F)=0\)

SVD (again!) can be used to solve this problem
Normalization

Is the accuracy in estimating $F$ function of the ref. system in the image plane?

E.g. under similarity transformation ($T = \text{scale} + \text{translation}$):

$$q_i = T_i \ p_i \quad q'_i = T'_i \ p'_i$$

Does the accuracy in estimating $F$ change if a transformation $T$ is applied?
Data courtesy of R. Mohr and B. Boufama.
With transformation

Without transformation

Mean errors:
10.0 pixel
9.1 pixel

Mean errors:
1.0 pixel
0.9 pixel
The accuracy in estimating $F$ does change if a transformation $T$ is applied.

\[ W f = 0, \quad \|f\| = 1 \rightarrow F \]

The constrain under which $|W f|$ is minimized is not invariant under similarity transformation.
Same issue for the DLT algorithm

\[ x_i' = H x_i \]  

[Section 4.4 in AZ]
Normalization

Transform image coordinate system \((T = \text{translation} + \text{scaling})\) such that:

- Origin = centroid of image points
- Mean square distance of the data points from origin is 2 pixels

\[
q_i = T_i \ p_i \quad q_i' = T'_i \ p'_i \quad \text{(normalization)}
\]
The Normalized Eight-Point Algorithm

0. Compute $T_i$ and $T_i'$

1. Normalize coordinates:
   \[ q_i = T_i \, p_i \quad q'_i = T'_i \, p'_i \]

2. Use the eight-point algorithm to compute $F'_q$ from the points $q_i$ and $q'_i$.

1. Enforce the rank-2 constraint. \[ \rightarrow F_q \quad \begin{cases} q^T F_q \, q' = 0 \\ \det( F_q ) = 0 \end{cases} \]

2. De-normalize $F_q$: \[ F = T'^T \, F_q \, T \]
Example: Parallel image planes

$K_1 = K_2 = \text{known}$

$x$ parallel to $O_1O_2$

$E = ?$

Hint:

$R = I \quad t = (T, 0, 0)$
Example: Parallel image planes

$x_1$ parallel to $O_1O_2$

$K_1 = K_2 = \text{known}$

$E = ?$

$E = [t_x]R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix}$
Example: Parallel image planes

Rectification: making two images “parallel”

Why it is useful? Epipolar constraint $\rightarrow y = y'$
Application: view morphing

Morphing without using geometry
Rectification
From its reflection!
Next lecture:

Reconstruction using stereo systems