Robust protocols for securely expanding randomness and distributing keys using untrusted quantum devices

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Abstract

Randomness is a vital resource for modern day information processing, especially for cryptography. A wide range of applications critically rely on abundant, high quality random numbers generated securely. Here we show how to expand a random seed at an exponential rate without trusting the underlying quantum devices. Our approach is secure against the most general adversaries, and has the following new features: cryptographic quality output security, tolerating a constant level of implementation imprecision, requiring only a constant size quantum memory for the honest implementation, unbounded output length on a fixed input length, and allowing a large natural class of constructions. When adapted for distributing cryptographic keys, our method achieves (for the first time) exponential expansion combined with cryptographic security and noise tolerance. The proof proceeds by showing that the Rényi divergence of the outputs of the protocol (for a specific bounding operator) decreases linearly as the protocol iterates. At the heart of the proof are a new uncertainty principle on quantum measurements, and a method for simulating trusted measurements with untrusted devices.
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1 Background and Summary of Results

The problem and its motivations. Randomness is an indispensable resource for modern day information processing. Without randomness, there would be no fast randomized algorithms, accurate statistical scientific simulations, fair gaming, or secure cryptography. A wide range of applications rely on methods for generating randomness with high quality and in a large quantity. Consider, for example, all the computers and handheld devices that connect to the Internet using public key cryptography like RSA and DSA for authentication and encryption, and that use secret key cryptography for secure connections. It is probably conservative to estimate that the number of random bits used each day for cryptography is in the order of trillions.

While randomness seems to be abundant in everyday life, its efficient and secure generation is a difficult problem. A typical random number generator such as the /dev/random/ generator in Linux kernel, would start with random “seeds”, including the thermal noise of the hardware (e.g. from Intel’s Ivy Bridge processors), system boot time in nanoseconds, user inputs, etc., and apply a deterministic function to produce required random bits. Those methods suffer from at least three fundamental vulnerabilities.

The first is due to the fact that no deterministic procedure can increase randomness. Thus when there is not enough randomness to start with, the output randomness is not sufficient to guarantee security. In particular, if the internal state of the pseudorandom generator is correctly guessed or is exposed for other reasons, the output would become completely predictable to the adversary. The peril of the lack of entropy has been demonstrated repeatedly [20, 35, 21], most strikingly by Heninger et al. [21]. They were able to break the DSA secret keys of over 1% of the SSH hosts that they scanned on the Internet, by exploiting the insufficient randomness used to generate the keys.

The second vulnerability is that the security of current pseudorandom generators are not only based on unproven assumptions, such as the hardness of factoring the product of two large primes, but also assume that their adversaries have limited computational capability. Therefore, they will fail necessarily if the hardness assumptions turn out to be completely false, or the adversaries gain dramatic increase in computational power, such as through developing quantum computers.

Finally, all those methods rely on trusting the correctness and truthfulness of the generator. The dynamics of market economy leads to a small number of vendors supplying the hardware for random number generation. The demand for platform compatibility results in a small number of generating methods. Thus the risk of the generators containing exploitable vulnerabilities or secret backdoors is necessarily significant. Recent evidence suggest that this is in fact the reality [29]. Thus for users demanding highest level of security with the minimum amount of trust, no current solution is satisfactory.

Quantum mechanics postulates true randomness, and thus provides a promising approach for mitigating those drawbacks. Applying a sequence of quantum operations can increase entropy even when the operations are applied deterministically, as some quantum operations are inherently unpredictable. Indeed, commercial random number generators based on quantum technology have started to emerge (e.g. by ID Quantique SA). Furthermore, the randomness produced can be unconditionally secure, i.e. without assumptions on the computational power of the adversary.

However, as classical beings, users of quantum random number generators can not be certain that the quantum device — the quantum state inside and the quantum operations applied — is running according to the specification. How can a classical user ensure that a possibly malicious quantum device is working properly?
Figure 1: A three-part device playing the GHZ game. Each part $D_1$, $D_2$, and $D_3$, receives a single bit and outputs a single bit. The input $(x, y, z)$ is drawn uniformly from \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}. The device wins if $a \oplus b \oplus c = x \lor y \lor z$. No communication among the parts is allowed when the game starts. An optimal classical strategy is for each part to output 1, winning with $3/4$ probability. An optimal quantum strategy is for the three parts to share the GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and for each part to measure $\sigma_x$ on input 0, and measure $\sigma_y$ on input 1. This strategy wins with certainty.

Special relativity and non-local games provide such a possibility. Consider, for example, the celebrated GHZ game \cite{16} illustrated in Fig. 1. It is now known \cite{23, 24, 33} that any quantum strategy achieving close to the optimal quantum winning probability must be very close to the optimal strategy itself. Consequently, the output of each component is near perfectly random. Intuitively, one needs only to run the game multiple times (using some initial randomness to choose the input string for each round) and if the observed winning average is close to the optimal quantum winning probability, then the output should be sufficiently random. Therefore, the trust on the quantum device can now be replaced by the (classically verifiable) condition of spatial separation of the different components. The security of the protocol would eventually rest on special relativity, in addition to quantum mechanics.

Colbeck and Kent \cite{6, 7} formulated the above framework for untrusted-device randomness expansion. Turning the intuition into rigorous proofs turns out to be rather challenging. Classical security was proved in \cite{30, 15, 31}. While useful, classical security does not guard against quantum adversaries, and thus is inadequate as quantum computation is becoming a reality. Furthermore, an expansion protocol without quantum security cannot be safely composed with other quantum protocols. Vazirani and Vidick \cite{41} were the first to prove quantum security. Furthermore, their protocol expands randomness exponentially. Coudron et al. \cite{9} broadened the Vazirani-Vidick protocol to allow much more flexibility for the underlying non-local game, but unfortunately they only proved classical security.

To put this in a larger context, the thread of untrusted-device randomness expansion is part of the broader area of untrusted-device, or “device-independent,” quantum cryptography. This is an important and intensively researched paradigm for studying the power and limitations of quantum cryptography when the underlying quantum devices may be adversarial. This area includes two tasks related to randomness expansion. The first is randomness amplification \cite{8}, where one wants to obtain near-perfect randomness from a weak random source using untrusted quantum devices (and without any additional randomness). Whereas the current paper focuses on randomness expansion, a companion paper by Chung, Shi (one of us), and Wu studies the amplification problem \cite{5}. The second related task is untrusted-device quantum key distribution (e.g. \cite{40}), where two parties attempt to establish a secure secret key using untrusted devices and openly accessible classical communication.
Overview of our results. In this work, we analyze a simple exponentially expanding untrusted-device randomness expansion protocol (referred to as the one-shot protocol). We prove that its output has exponentially (in the input length) small errors, and is secure against the most general quantum adversaries. More importantly, the protocol accomplishes all of the following additional features, none of which has been accomplished by previous works.

The first is cryptographic security in the output. The error parameters are not only exponentially small in the input length, but are also negligible (i.e. smaller than any inverse polynomial function) in the running time of the protocol (which is asymptotically the number of uses of the device.) This is precisely the condition that the output is suitable for cryptographic applications — it means that for the adversary to defeat the substitution of uniform bits by the protocol output, his cost will be dramatically larger than that for generating the output.

Secondly, the protocol is robust, i.e. tolerating a constant level of “noise”, or implementation imprecision. Thus any honest implementation that performs below the optimal level by a small constant amount on average will still pass our test with overwhelming probability. For example, we show that any device which wins the GHZ game with probability at least 0.985 will achieve exponential randomness expansion with probability approaching 1. This feature greatly reduces the technological requirements for practical implementation, especially at this moment of time when even small scale accurate quantum information processing is still being developed. We model noise as the average deviation of a device interaction from an ideal implementation. This “black-box” noise model does not refer to the inner-working of the devices thus is fairly general.

Third, our protocol requires only a constant size quantum memory for an honest implementation. In between two rounds of interactions, the different components of the device are allowed to interact arbitrarily. Thus an honest device could establish its entanglement on the fly, and needs only to maintain the entanglement (with only a constant level of fidelity) for the duration of a single game. Given the challenge of maintaining coherent quantum states, this feature greatly reduces implementation complexity.

Fourth, relying on a powerful observation of Chung, Shi and Wu [5], we show that one can sequentially compose instances of our one-shot protocol, alternating between two untrusted devices and achieve unbounded length expansion starting with a fixed length seed. The additively accumulating error parameters remain almost identical to the one-shot errors, since they decrease geometrically.

Finally, our protocol allows a large natural class of games to be used. The class consists of all binary XOR games — games whose inputs and outputs are binary and whose scoring function depends on the inputs and the XOR of the outputs — that are strongly self-testing. The latter property says that any strategy that is \( \epsilon \)-close to optimal in its winning probability must be \( O(\sqrt{\epsilon}) \) close to a unique optimal strategy in its both its state and its measurements. (We call this “strongly self-testing” because this error relationship is the best possible.) The class of strong self-tests includes the CHSH game and the GHZ game, two commonly used games in quantum information. Broadening the class of usable games has the benefit of enabling greater design space, as different implementation technologies may favor different games. For example, the highly accurate topological quantum computing approach using Majorana fermions is known not to be quantum universal [27]. In particular, Deng and Duan [13] showed that for randomness expansion using Majorana fermions, three qubits are required. Our proof allows the use of Majorana fermions for randomness expansion through the GHZ game.

We include two applications of our expansion protocols. Our one-shot protocol can be used as a building block in the Chung-Shi-Wu protocol for randomness amplification. In addition, composing their amplification protocol and our concatenated expansion protocol gives a robust, untrusted-device quantum protocol that converts an arbitrary weak random source into near-
perfect output randomness of an arbitrary large length. This composed protocol is rapid and secure even against quantum attack. It opens the possibility for unconditionally secure cryptography with the minimum trust on the randomness source and the implementing device.

The second application is to adapt our protocol for untrusted-device quantum key distributions, resulting in a robust and secure protocol that at the same time exponentially expands the input randomness.

Related works. Prior to our work, the Vazirani-Vidick randomness expansion protocol [41] is the first and only work achieving simultaneous exponential expansion and quantum security. However, as far as we know from their analysis, the protocol achieves only inverse polynomial security, thus is not appropriate for cryptographic applications. It is not noise-tolerant, requiring that as the number of games grows, each strategy played by an honest device must approach rapidly to the optimal quantum strategy. It also requires a growing amount of quantum memory as the output length grows. Those drawbacks limit the practical feasibility of the protocol.

The possibility of concatenating expansion protocols to yield a larger expansion factor was first suggested in Colbeck’s Thesis [6]. The Equivalence Lemma of Chung-Shi-Wu [5] actually implies that any untrusted-device protocols taking uniform input and producing almost uniform output can be composed securely, including those of Vazirani-Vidick [41, 40]. Note, however, an unbounded expansion based on Vazirani-Vidick [41] would remain non-robust. Our robustness, constant quantum memory requirement, and flexibility in choosing games make such concatenation meaningful.

The Vazirani-Vidick protocol for untrusted-device key distribution [40] can be used as a randomness expansion protocol with a linear rate of expansion. One could easily use ingredients from the same authors’ expansion protocol [41] to reduce the length of the initial randomness used. It is not clear though if the analysis can be correspondingly strengthened to apply to such an extension. Similarly, as presented and analyzed the protocol does not allow inter-component communication in-between rounds. It is not clear (when used as a linear expansion protocol) whether their analysis can be strengthened to allow some in-between-round communications.

In an earlier work [24], the present authors characterized the class of all binary XOR games that are strong self-tests. The characterization is used critically in the current work. While the Vazirani-Vidick protocol for key distribution protocol uses a specific game, the analysis remains valid for a much broader class of games, such as those "randomness generating" games defined in [9], as pointed out to us by Vidick [42]. Since all strong self-testing binary XOR games are randomness generating, the class of games allowed for untrusted-device key distribution is broader in [40] than ours. The class of games allowed in [41] require more special properties and the class of games allowed there appear to be incomparable with ours. Note that it is not clear if this latter comparison is meaningful as our protocol achieves several additional features not in [41].

Noise model. We now move to more technical discussions, starting with the noise model. Different implementation approaches may have their own appropriate model of noise. In this work, we model noise in a “black-box” manner, quantifying noise in terms of deviations of the implementation from an ideal implementation. More specifically, consider one interaction with a device component on an input $x$. Let $Y(x)$, or $Y'(x)$, respectively, be the response from the current device, or an ideal device, respectively. The deviation of the device with respect to the ideal device in this interaction is simply the largest, over all $x$, statistical distance $\frac{1}{2} \| Y(x) - Y'(x) \|_1$. We make the following definition of the noise level. We refer the interested readers to [5] for a formal formulation of untrusted-device protocols.

**Definition 1.** Let $\epsilon \in [0, 1]$. An implementation $\tilde{\Pi}$ of a untrusted device protocol is said to have a noise level $\leq \epsilon$ if there exists an ideal implementation $\Pi$ such that on any input to the protocol, the average
deviation of each device-interaction in $\tilde{\Pi}$ from that of $\Pi$ is $\leq \epsilon$.

Our definition facilitates the application of Azuma-Hoeffding inequality in establishing our noise-tolerance results.

The one-shot protocol and technical statements of results. Our main protocol (Figure 2) is essentially the same as the one used by Coudron, Vidick, and Yuen [9] (which is in turn closely related to that of Vazirani-Vidick). The main differences in our version of the protocol are the class of games used, and most importantly, the allowance of in-between-rounds quantum communications. The games we use involve $n$ parties, with $n \geq 2$. Such a game is played by a single device, which consists of $n$ components, where each component has a classical input/output interface. For any nonlocal game $G$, let $w_G$ denote the highest probability with which a quantum strategy can win the game, and let $f_G = 1 - w_G$ denote the smallest possible failure probability that can be achieved by a quantum strategy.

Arguments:

- $N$: a positive integer (the output length.)
- $\eta$: A real $\in (0, \frac{1}{2})$. (The error tolerance.)
- $q$: A real $\in (0, 1)$. (The test probability.)
- $G$: An $n$-player nonlocal game that is a strong self-test [24].
- $D$: An untrusted device (with $n$ components) that can play $G$ repeatedly and cannot receive any additional information. In a single use the different components cannot communicate; in between uses, there is no restriction.

Protocol $R$:

1. A bit $g \in \{0, 1\}$ is chosen according to a biased $(1 - q, q)$ distribution.
2. If $g = 1$ ("game round"), then an input string is chosen at random from $\{0, 1\}^n$ (according a probability distribution specified by $G$) and given to $D$. Depending on the outputs, a "P" (pass) or an "F" (fail) is recorded according to the rules of the game $G$.
3. If $g = 0$ ("generation round"), then the input string $00 \ldots 0$ is given to the device, and the output of the first component $D_1$ is determined. If the output of the first component is 0, the event $H$ ("heads") is recorded; otherwise the event $T$ ("tails") is recorded.
4. Steps 1 – 3 are repeated $N - 1$ (more) times.
5. If the total number of failures exceeds $(1 - w_G + \eta)qN$, the protocol aborts. Otherwise, the protocol succeeds. If the protocol succeeds, the output consists of an $N$-length sequence from the alphabet $\{P, F, H, T\}$ representing the outcomes of each round.

Figure 2: The Central Protocol $R$

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1We note that the literature on this subject has some differences in terminology. Some authors would use the word “device” in the way that we have used the word “component.”
To produce near perfect random output, we need to apply quantum-proof randomness extractors to the outputs of protocol R. Those are deterministic functions Ext(X, S) on two arguments, X being the source, which in this case is the output of the protocol, and S being the perfectly random seed. It is known that there are quantum-proof randomness extractors Ext(X, S) that will convert any N bits X that have min-entropy Ω(N) to near perfect output randomness of length Θ(N).

Here are some quantities that describe important properties of a randomness expansion protocol. Let y be a positive integer. We call a Classical-Quantum (C-Q) state ideal with y extractable bits if the first bit of the classical part indicates “Success” or “Abort”, and the (unnormalized) C-Q state corresponding to the “Success” event has conditional min-entropy ≥ y. Let ε, η, λ be reals in (0, 1). A randomness expansion protocol is said to have a yield y with soundness error εs if for any device D, the output is always within trace distance εs of an ideal state with y extractable bits. It is said to tolerate a noise level λ with completeness error εc if any device that plays each game independently with an expected score no less than wG − λ will abort with probability at most εc. If both the soundness and the completeness errors are ≤ ε, we simply say the protocol has an error ε.

Our main result is the following.

**Theorem 1.1 (Main Theorem).** For any strong self-test G, and any δ > 0, there exist positive constants q0, η0, K, b, c, such that the following hold when Protocol R is executed with parameters q ≤ q0, η ≤ η0.

1. (Soundness.) The yield is at least (1 − δ)N extractable bits with a soundness error εs = K exp(−bqN).

2. (Completeness.) For any constant η′, 0 < η′ < η, the protocol tolerates η′ noise level with a completeness error εc = exp(−c(η − η′)2qN).

The difficult part of this result is the soundness claim, which follows from the results of section I in the appendix (see Corollary I.5). The completeness claim follows from the Azuma-Hoeffding inequality.

Note that the bits g1, . . . , gN can be generated by O(Nh(q)) uniformly random bits with an error exp(−Ω(qN)), where h denotes the Shannon entropy function. Therefore, when q is chosen to be small, the protocol needs only ω(log N) initial bits and one device to achieve Ω(N) extractable bits with negligible error.

**Corollary 1.2 (One-shot Min-entropy Expansion).** For any real ω ∈ (0, 1), setting q = Θ(kω / 2k1−ω) in Theorem 1.1 Protocol R converts any k uniform bits to 2k1−ω extractable bits with exp(−Ω(kω)) soundness and completeness errors.

To obtain near perfect random bits, we apply a quantum-proof strong randomness extractor, in particular one that extract a source of a linear amount of conditional quantum min-entropy. The parameters of our protocols depend critically on the seed length of such extractors, thus we introduce the following definition.

**Definition 2 (Seed Length Index).** We call a real v a seed length index if there exists a quantum-proof strong extractor extracting Θ(N) bits from a (N, Θ(N)) source with error parameter ε using log1/v(N/ε) bits of seed. Denote by μ the supremum of all seed length indices.

Such extractors exist with v ≥ 1/2, e.g., Trevisan’s extractors [32] shown to be quantum-proof by De et al. [11]. Thus μ ≥ 1/2. The definition of soundness error for producing y bits of perfect randomness is the same for producing extractable random bits, except that the ideal C-Q state conditioned on Success is the product state of y perfectly random bits and a quantum state. The following corollary follows directly by composing protocol R and an extractor with v close to μ. In accounting the running time, one round of interaction with the device is considered a unit time.
Corollary 1.3 (One-shot Randomness Expansion). For any \( \omega \in (0, \mu) \), setting \( q = \Theta(k^\omega / 2^{k^{1-\omega}}) \) in Theorem 1.1, Protocol R composed with an appropriate quantum-proof strong extractor converts \( k \) bits to \( 2^{k^{1-\omega}} \) almost uniform bits with soundness and completeness errors \( \exp(-\Omega(k^\omega)) \).

Corollary 1.4 (Cryptographic Security). With the parameters in Corollary 1.3, the running time of the protocol is \( T := \Theta(2^{k^{1-\omega}}) \). Thus for any \( \lambda > 1 \), setting \( \omega = \frac{\lambda}{1 + \lambda} \mu \), the errors are \( \exp(-\Omega(\log^\lambda T)) \), which are negligible in \( T \). That is, the protocol with those parameters achieves cryptographic quality of security (while still exponentially expanding.)

Once we have near perfect randomness as output, we can use it as the input to another instance of the protocol, thus expanding further with an accumulating error parameter. As the error parameters decrease at an exponential rate, they are dominated by the first set of errors.

Corollary 1.5 (Unbounded Randomness Expansion). For all integers \( N \) and \( k \), and any real \( \omega \in (0, \mu) \), \( k \) uniformly random bits can be expanded to \( N \) output bits with \( \exp(-\Omega(k^\omega)) \) error under a constant level of noise. The procedure uses \( O(\log^* N) \) iterations of Protocol R on \( 2 \) multi-part devices.

That the \( O(\log^* N) \) iterations of Protocol R require only \( 2 \) devices follows from a powerful insight (Equivalence Lemma) of Chung, Shi and Wu [3]. In Section 7 we will prove a general result, the Concatenation Lemma, which shows how their insight leads to the striking conclusion that any untrusted-device protocol for randomness generation, including ours and those of Vazirani and Vidick [41, 40], can be composed sequentially (and with each other) using only two untrusted devices with additive soundness errors. Here we point out the key observation of the proof. Let \( \rho_{XDE} \) be a Classical-Quantum-Quantum state where \( X \) is classical, \( D \) and \( E \) represent the subsystems of a untrusted quantum device and the adversary, respectively. We call \( \rho \) device-uniform or adversary-uniform if conditioned on passing (i.e. non-aborting), \( X \) is uniform with respect to \( D \), or \( E \), respectively. Recall that the soundness of a protocol on an input is the minimum distance of the output to an adversary-uniform state. The Equivalence Lemma states that for any untrusted-device protocol (not necessarily expanding), the worst soundness error on uniform inputs (with respect to both the device and the adversary) is precisely the same as the worst soundness error on device-uniform inputs.

Consequently, suppose that \( D_0 \) and \( D_1 \) are two untrusted quantum devices and \( D_0 \) is first used, with an soundness error \( \epsilon_0 \), on an input state \( \rho \), which is \( \delta \)-close to device-uniform. The resulting state \( \rho' \) is then \( \epsilon_0 + \delta \)-close to adversary-uniform. Since \( D_1 \) was part of the adversary system when \( D_0 \) was used, \( \rho' \) is \( \epsilon_0 + \delta \)-close to a device-uniform state for \( D_1 \), ensuring that the output is \( \epsilon_0 + \epsilon_1 + \delta \)-close to a device-uniform for \( D_0 \) to use, where \( \epsilon_1 \) is the soundness error when \( D_1 \) is used. Continuing this argument leads to our conclusion.

Application: randomness amplification. To apply our protocol to randomness amplification, we choose \( q = \Theta(1) \) for those uses of our protocol inside the Chung-Shi-Wu randomness amplification protocol, which converts a \( n \)-bit, min-entropy \( \geq k \) weak source to a near perfectly random output of \( \Theta(k) \) bits. Then we apply Corollary 1.5 to expand to an arbitrarily long near perfect randomness.

While the QKD protocol of Vazirani and Vidick [41] can also serve as a building block for the Chung-Shi-Wu protocol, the use of our protocol has two advantages for in-between-rounds communications and flexibility of the non-local game used.

Corollary 1.6 (of [1,3,1.5, and 5] — Randomness Amplification). Let \( v \in [1/2, \mu] \) be a seed length index. For all sufficiently large integer \( k \), any integer \( n = \exp(O(k^v)) \), any real \( \epsilon = \exp(-O(k^v)) \),
any \((n,k)\) source can be converted to an arbitrarily long near perfect randomness with \(\epsilon\) soundness and completeness errors under a (universal) constant level of noise. The number of devices used is \(2^{O(\log^{1/\epsilon}(n/\epsilon))}\).

We point out that the number of devices used as a function of the weak source length \(n\) and the error parameter \(\epsilon\) grows super-polynomially (if \(\mu < 1\)) or polynomially (if \(\mu = 1\)). It remains a major open problem if this number can be substantially reduced or even be made a constant.

**Application: exponentially expanding key distribution.** Suppose that Alice and Bob would like to establish a secret string in an environment that trusted randomness is a scarce resource, and consequently their initial randomness is much shorter than the desired output length. As with other studies on qkd, we will sidestep the authentication issue, assuming that the man-in-the-middle attack is already dealt with. An obvious application of our randomness expansion protocol for untrusted device QKD is for Alice to expand her initial randomness, then use the expanded, secure randomness to execute the untrusted device QKD protocol of Vazirani and Vidick [40]. The end result is an exponentially expanding key distribution protocol.

An alternative approach, which is the focus of our new contribution, is to directly adapt our expansion protocol to achieve simultaneously randomness expansion and key distribution. The benefits of doing so compared with the first approach are two-folds. First, it would reduce the number of untrusted devices from 2 to 1. When such devices are valuable, this could be an important saving. Second, the Vazirani-Vidick protocol relies on a specific game, while we will show that any strong self-test suffices.

We now describe in high level the adaption, the challenges, and our solution. To apply Protocol R (Fig. 2) to untrusted-device QKD between two parties Alice and Bob, we have Alice interact with the first component of the device, while Bob interacts with all the other components. They share randomness for executing our protocol, as well as that for later classical post-processing stages. For those game rounds that the bit \(g\) is 1, Alice and Bob use the public channel to compare their device outputs. Once Protocol R succeeds, they apply an information reconciliation (IR) protocol for agreeing on a common string. To obtain the final key, they then apply the randomness extraction on this common string. The full protocol \(R_{qkd}\) is provided in Section K. We separate the randomness extraction stage from our protocol as it is standard.

To make the above straightforward idea work, the main technical challenge is in the IR stage. First, we need to make sure that the amount of leaked information does not reduce the asymptotic quantity of the smooth min-entropy. Furthermore, we cannot use too much randomness. This latter problem is new as in previous studies, Alice and Bob are assumed to have access to a linear (in the output length) amount of shared randomness (which does not need to be secure against the adversary).

The first problem is tackled by applying an Azuma-Hoeffding Inequality to argue that other than a chance of the same order as the expansion soundness error, Alice and Bob’s strings differ by at most a \(1/2 - \lambda\) fraction, for some constant \(\lambda\). We then make use of several known results to construct a randomized protocol through which Bob recovers Alice’s string without consuming too much randomness and efficiently in term of time complexity. The details are provided in Section K. We state our main result on QKD below. The notion of soundness and completeness errors are similarly defined: the soundness error is the distance of the output distribution to a mixture of aborting and an output randomness of a desired smooth min-entropy, and the completeness error is the probability of aborting for an honest (possibly noisy) implementation.

**Corollary 1.7 (Untrusted-Device Quantum Key Distribution).** For any strong self-test \(G\), there exists a positive constant \(s\) such that the following holds. For any \(\delta > 0\), there are positive constants \(q_0, \eta_0, B_0, b, c, 8...
such that the following assertions hold when Protocol $R_{qkd}$ (Fig. 6) is executed with parameters $q \leq q_0$, $\eta \leq \eta_0$, and $qN \geq B_0$.

1. (Soundness.) The protocol obtains a key of $(s - \delta)N$ extractable bits with a soundness error $\epsilon_s = \exp(-bqN)$.

2. (Completeness.) For any constant $\eta'$, $0 < \eta' < \eta$, the protocol tolerates $\eta'$ noise level with a completeness error $\epsilon_c = \exp(-c(\eta - \eta')^2qN)$.

The number of initial random bits is $O(Nh(q))$.

A final key of length $\Omega(N)$ can be obtained by applying a quantum-proof randomness extractor with additional $O(\log^\omega(N/\epsilon))$ bits as the seed for the extractor, where $\omega$ can be chosen $\leq 2$, and $\epsilon$ is added to the soundness error.

2 Central Concepts and Proofs

While proving classical security of randomness expansion protocols is mainly appropriate applications of Azuma-Hoeffding inequality, proving quantum security is much more challenging. The proof for the Vazirani-Vidick protocol [41] relies on a characterization of quantum smooth min-entropy based on the quantum-security of Trevisan’s extractors [11]. We take a completely different approach, without any reference to extractors (for our min-entropy expansion.)

Achieving the results above required developing multiple new techniques. We are hopeful that some of the ideas in this paper will also find applications elsewhere. We now sketch the main insights and technical contributions from our proofs. (The proofs are written out in full detail in the appendix.)

The application of quantum Rényi entropies. A critical challenge in constructing our proofs was finding the right measure of randomness. We use the quantum Rényi entropies which were first defined in [22] and further studied in [25, 44]. For any $\alpha > 1$, and any density matrix $\rho$ and any positive semidefinite operator $\sigma$, let

$$d_{\alpha}^{\rho}(\rho \parallel \sigma) = \frac{1}{\alpha - 1} \log_2 \left[ \left( \sigma^{\frac{1}{\alpha}} \rho \sigma^{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha - 1}} \right].$$

and let $D_{\alpha}^{\rho}(\rho \parallel \sigma) = \log_2 d_{\alpha}^{\rho}(\rho \parallel \sigma)$. (The quantity $D_{\alpha}^{\rho}(\rho \parallel \sigma)$ is called the Rényi divergence.) The $\alpha$-Rényi entropy $H_{\alpha}(A|E)$ of a bipartite quantum system $(A, E)$ is computed by the maximum of the value $[-D_{\alpha}^{\rho}(\rho \parallel \sigma)]$ over all operators $\sigma$ that are of the form the form $\sigma = \mathbb{I}_A \otimes \sigma'$, where $\text{Tr}(\sigma') = 1$. (See Definition 7 in the appendix.) The quantum Rényi entropies have a number of interesting properties (see [25]). For our purposes, they are interesting because if $(A, E)$ is a classical quantum system, any lower bound on the Rényi entropy of $(A, E)$ provides a lower bound on the number of random bits than can be extracted from $A$. (See Corollary D.5)

The use of quantum Rényi entropies is delicate because the parameter $\alpha$ must be chosen appropriately. If $\alpha$ is too large, the quantum Rényi entropy is not sensitive enough to detect the effect of the game rounds in Protocol R. On the other hand, if $\alpha$ is too close to 1, the bound on extractable bits obtained from Corollary D.5 is not useful. As we will discuss later in this section, it turns out that it is ideal to choose $\alpha$ such that $(\alpha - 1)$ is proportional to the parameter $q$ from Protocol R.

An uncertainty principle for Rényi entropy. Suppose that $Q$ is a qubit, and $E$ is a quantum system that is entangled with $Q$. Let $\rho$ be a density operator which represents the state of $E$. Let $\{\rho_0, \rho_1\}$
and \( \{\rho_+, \rho_-\} \) represent the states that arise when \( Q \) is measured along the \( \{0,1\}\)-basis and the \( \{+,-\}\)-basis. We prove the following. (See Theorem E.2.)

**Theorem 2.1.** There is a continuous function \( \Pi : (0,1] \times [0,1] \to \mathbb{R} \) satisfying

\[
\lim_{(x,y) \to (0,0)} \Pi(x,y) = 1
\]

(2.2)

such that the following holds. For any operators \( \rho_0, \rho_1, \rho_+, \rho_- \) representing states arising from anti-commutative measurements, if \( \delta = \frac{\text{Tr}(\rho_1^{1+\epsilon})}{\text{Tr} (\rho^{1+\epsilon})} \), then

\[
\frac{\text{Tr}(\rho_1^{1+\epsilon} + \rho_-^{1+\epsilon})}{\text{Tr}(\rho^{1+\epsilon})} \leq 2^{-\epsilon \Pi(\epsilon, \delta)}.
\]

(2.3)

This theorem asserts that if the quantity \( \delta \) determined by the \( \{0,1\} \) measurement is small, then the outcome of the \( \{+, -\}\)-measurement must be uncertain (as measured by the \((1 + \epsilon)\)-Rényi divergence). This parallels other uncertainty principles that have been used in quantum cryptography [38].

Our proof of this result is based on a known matrix inequality for the \((2 + 2\epsilon)\)-Schatten norm.

**Certifying randomness from a device with trusted measurements.** Let us say that a device with trusted measurements \( D \) is a single-part input-output device which receives a single bit as an input, and, depending on the value of the bit, performs one of two perfectly anti-commutative binary measurements on a quantum system. The measurements of the device are trusted, but the state is unknown.

Suppose that we make the following modifications to the procedure that defines Protocol R.

1. Instead of a multi-part binary device, we use a single-part binary device with trusted measurements.
2. Instead of playing a nonlocal game, at each round we simply use the bit \( g \) as input to the device and record the output.

Let us refer to this modified version as “Protocol A.” (See section G.1 for a full description.)

Note that Protocols A and R both involve conditioning on a “success” event. One of the central difficulties we have found in establish quantum security is in determining the impact that this conditioning has on the randomness of the device \( D \). In the classical security context, one can show that once we conditioning on the success event, “most” uses of the device \( D \) (in an appropriate sense) generate random outputs. By elementary arguments, the outputs therefore accumulate min-entropy linearly over multiple iterations, and randomness expansion is achieved. Yet carrying this approach over to the quantum context—even with the assumption of trusted measurements—seemed to us to be quite difficult.

We have found a successful way to interpret the success/abort events in the quantum context. It involves two adjustments to the classical approach outlined above. First, we use the quantum Rényi entropy in place of the smooth min-entropy. (The quantum Rényi entropies have elegant arithmetic properties which make them more amenable to induction.) Secondly, rather than directly considering “success” and “abort” as discrete events, we consider a graded measurement of performance which interpolates between the two.
Suppose that \( E \) is a quantum system which is initially entangled with \( D \). For the purposes of this discussion, let us assume that \( E \) and \( D \) are maximally entangled and the state \( \rho = \rho_E \) is totally mixed. Then, the state of \( E \) after one iteration can be expressed as

\[
\rho := (1 - q)\rho_+ \oplus (1 - q)\rho_- \oplus q\rho_0 \oplus q\rho_1. \tag{2.4}
\]

Suppose that we are measuring the randomness of this state with respect to a second party who knows the value of the bit \( g \). Then, an appropriate measure of randomness would be the Rényi divergence \( D_\alpha(\rho) \) with respect to the operator \( \sigma := (1 - q)I \oplus (1 - q)I \oplus qI \oplus qI \). For the parameter \( \alpha \), it turns out that simply taking \( \alpha = 1 + q \) is useful.

Then,

\[
d_{1+q}(\rho) = \text{Tr}_E \left[ (1 - q)\rho_+^{1+q} + (1 - q)\rho_-^{1+q} + q\rho_0^{1+q} + q\rho_1^{1+q} \right]^{1/q}. \tag{2.5}
\]

One could hope that this quantity is strictly smaller than \( d_\alpha(\rho) \), but this is not always so (for example, for measurements on a maximally entangled Bell state). But consider instead the modified expression

\[
\text{Tr}_E \left[ (1 - q)\rho_+^{1+q} + (1 - q)\rho_-^{1+q} + q\rho_0^{1+q} + \left( \frac{1}{2} \right) q\rho_1^{1+q} \right]^{1/q}. \tag{2.6}
\]

Theorem 2.1 implies that this quantity is always less than \( C^{-1}d_{1+q}(\rho) \), where \( C > 1 \) is a fixed constant. (Essentially, this is because if the quantity \( \delta \) is large, then the introduction of the \( (1/2) \) coefficient lowers the value of the expression significantly, and if \( \delta \) is small, then (2.3) implies the desired bound.)

If we let

\[
\sigma := (1 - q)I \oplus (1 - q)I \oplus qI \oplus 2^{1/3}qI, \tag{2.7}
\]

then \( d_{1+q}(\rho) \) is equal to expression (2.6). One can think of the function \( d_{1+q}(\cdot) \) as an error-tolerant measure of performance. The presence of the coefficient \( 2^{1/3} \) compensates for the loss of randomness when the device-failure quantity \( \text{Tr}[\rho_1^{1+q}] \) is large.

Now let \( B \) denote the output register of Protocol \( R \), and let \( \Lambda_{BE} \) denote the joint state of \( E \) and \( B \) at the conclusion of the protocol. Let \( \Sigma \) be an operator on \( BE \) defined by

\[
\Sigma = \sum_{b \in \{H,T,P,F\}^N} (1 - q)^{\text{# of gen. rounds}} q^{\text{# of game rounds}} 2^{\text{(1/3)} \cdot \text{# of failures}}} |b\rangle \langle b| \otimes I. \tag{2.8}
\]

An inductive argument proves that \( d_{1+q}(\Gamma \Sigma) \leq C^{-N} \). This inequality is sufficient to deduce that the Rényi entropy of the “success” state \( \Gamma^5 \) grows linearly in \( N \). One can therefore deduce that (for appropriate parameters) the outputs of Protocol \( A \) contain a linear number of extractable quantum proof bits.

The reasoning sketched above is given in full detail in Appendix A. In that section, we actually prove something stronger: if Protocol \( A \) is executed with a partially trusted measurement device (i.e., a measurement device whose measurements are anticommutative only with a certain positive probability) then it produces a linear amount of randomness. (See Theorem H.7) This generalization, as we will see, is what allows to carry over our results into the fully device-independent setting.
Simulation results for partially trusted devices. The second central insight that enables our results is that nonlocal games simulate partially trusted devices. When certain nonlocal games are played—even with a device that is completely untrusted—their outcomes match the behavior of a device that is partially trusted.

Let us begin by formalizing a class of devices. (Our formalism is a variation on that which has appeared in other papers on untrusted devices, such as [33].)

**Definition 3.** Let $n$ be a positive integer. A *binary quantum device with $n$ components* $D = (D_1, \ldots, D_n)$ consists of the following.

1. Quantum systems $Q_1, \ldots, Q_n$, and a density operator $\Phi$ on $Q_1 \otimes \ldots \otimes Q_n$ which defines the initial state of the systems.

2. For any $k \geq 0$, and any “transcript” $T$ (representing the collective previous inputs and outputs during previous rounds) a unitary operator $U_T: \bigotimes_i Q_i \rightarrow \bigotimes_i Q_i$ and a collection of Hermitian operators $M_{T,i}^{b}$ on $Q_i$ satisfying $\|M_{T,i}^{b}\| \leq 1$.

The behavior of the device $D$ is as follows: at round $i$, the devices first collectively perform the unitary operation $U_T$, and then, according to their inputs $b_i$, each performs binary measurement specified by the operators $M_{T,i}^{b}$. (This is a device model that allows communication in between rounds.)

Now we define a somewhat more specific type of device. Suppose that $E$ is a single-part binary quantum device. Then let us say that $E$ is a *partially trusted device with parameters $(v, h)$* if the measurement operators $N_{T}^{(1)}$ that $E$ uses on input 1 decompose as

$$N_{T}^{(1)} = (v)P_T + (1 - v - h)Q_T,$$

where $P_T$ is perfectly anti-commutative with the other measurement $N_{T}^{(0)}$, and $Q_T$ satisfies $\|Q_T\| \leq 1$ (and is otherwise unspecified). Essentially, the device behaves as follows. On input 0, it performs a perfect measurement. On input 1, it does one of the following at random: it performs a perfectly anti-commuting measurement (probability = $v$), or it performs an unknown measurement (probability = $1 - v - h$), or it ignores its quantum system and merely outputs a perfect coin flip (probability = $h$). (The second possibility is what we call a “dishonest mistake,” and the third is what we call an “honest mistake.”)

We wish to prove that untrusted devices can be simulated by partially trusted devices. This again is an example of a task that is fairly easy in the classical security context but difficult in the quantum context. For example, if one knows that a quantum device performs at a superclassical level at a particular nonlocal game, then one knows that its outcomes are at least partly random, and thus can be “simulated” by a biased coin flip (or a “partially trusted” coin flip). But to prove quantum security one needs a stronger notion of simulation—one that allows for the possibility quantum side information.

The basis for our simulation result is previous work by the present authors on quantum self-testing [24]. We consider games from the class of strong self-tests which we referred to in section [1]. In [24] we proved a criterion for strong self-tests. This criterion is a building block in the following theorem.

**Theorem 2.2.** Let $G$ be a strong self-test, and let $D$ be an (untrusted) binary device with $n$ components. Then, the behavior of $D$ in Protocol $R$ can be simulated by a partially trusted device.
This result is proved in the Appendix as Theorem G.2. We briefly sketch the proof. We can reduce to the case where \( \dim Q_i = 2 \) and each measurement operator is projective. After an appropriate choice of basis, we have

\[
M_j^{(0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad M_j^{(1)} = \begin{bmatrix} 0 & \alpha_j \\ \bar{\alpha}_j & 0 \end{bmatrix}
\]

with \( |\alpha_j| = 1 \). The output of \( D \) during a generation round is derived from the measurement operator \( M_j^{(0)} \otimes I \otimes \ldots \otimes I \), which, under an appropriate basis, can be represented as the block matrix \( \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \) on \( \mathbb{C}^{2^n} \). The behavior of \( D \) during a game round can be given represented by a reverse diagonal Hermitian matrix \( M \) on \( \mathbb{C}^{2^n} \) with entries

\[
P_1(\alpha_1, \ldots, \alpha_n), P_2(\alpha_1, \ldots, \alpha_n), \ldots, P_2(\alpha_1, \ldots, \alpha_n), P_1(\alpha_1, \ldots, \alpha_n),
\]

where \( \{P_i\} \) are rational functions depending on the game. Using the strong self-testing condition, we show the existence of another reverse diagonal matrix \( R \) with entries \( \beta_1, \ldots, \beta_{2^n-1}, \beta_1 \) which anti-commutes with \( M_j^{(0)} \) and which satisfies \( \|M - R\| + \|R\| = \|M\| \). This implies that \( M \) satisfies the decomposition (2.9) which defines a partially trusted device.

Proving the existence of the sequence \( \beta_1, \ldots, \beta_{2^n-1} \) is matter of manipulations of complex numbers. One surprising aspect of this proof is that depends critically on the fact that \( G \) is not only a self-test, but a strong self-test. (See Corollary F.6, which is used in the proof of Theorem F.8.)

The proof of security of Protocol R. We define a third protocol, Protocol A’ (Fig. 4), which is the same as Protocol A except that a partially trusted measurement device is used. We show that Protocol A’ produces a linear amount of entropy (Theorem H.7). Protocol R can be simulated by Protocol A’ for an appropriately chosen partially trusted device. This means not only that the probability distributions of the outputs of the two protocols are exactly the same, but also that there is a simulation of the behavior of any external quantum environment that may be possessed by an adversary. Since Protocol A’ produces a linear amount of min-entropy, the same is true of Protocol R. This completes the proof. (See Theorem I.1.)

Numerical bounds. The proof methods we have discussed are sufficient to give actual numerical bounds for the amount of randomness generated by Protocol R. In subsection I.3 we offer a preliminary result showing how this is done. Let \( \Pi \) be the actual function that is used in the proof of Theorem E.2. Then, the limiting function

\[
\pi(y) := \lim_{x \to 0} \Pi(x, y)
\]

is given by the following expression:

\[
\pi(y) = 1 - 2y \log y - 2(1 - y) \log(1 - y).
\]

That is,

\[
\pi(y) = 1 - 2h(y),
\]

where \( h(y) \) denotes the Shannon entropy of the vector \((y, 1 - y)\).

If \( G \) is a strong self-test, let \( v_\varepsilon \) denote the trust coefficient of \( G \) — that is, the largest real number \( v \) such that \( G \) simulates a partially trusted device with parameters \((v, h)\). Corollary I.3 asserts that
the linear output of Protocol R is lower bounded by the quantity $\pi(\eta/v_G)$. In particular, a positive rate is achieved provided that $\pi(\eta/v_G) > 0$. Using (2.13), a positive rate is therefore achieved if $\eta < 0.11 \cdot v_G$.

In subsection 13, we show that $v_{GHZ} \geq 0.14$. Therefore, the GHZ game achieves a positive linear rate provided that $\eta < 0.11 \cdot 0.14 = 0.0154$.

### 3 Further directions

A natural goal at this point is to improve the certified rate of Protocol R. This is important for the practical realization of our protocols. By the discussion above, this reduces to two simple questions:

1. What techniques are there for computing the trust coefficient $v_G$ of a binary XOR game?
2. Is it possible to reprove Theorem 2.1 in such a way that the limiting function

$$\pi(y) = \lim_{x \to 0} \Pi(x, y) \quad (3.1)$$

becomes larger?

A related question is to improve the key rate of Protocol $R_{qkd}$. The “hybrid” technique of Vazirani and Vidick [40] for mixing the CHSH game with a trivial game with unit quantum winning strategy may extend to general binary XOR games.

It would also be interesting to explore whether Theorem 1.1 could be extended to nonlocal games outside the class of strong self-tests. The strong self-testing condition is useful because it simplifies many of the proof sequences in the paper, but with some additional work it may be possible to carry off similar proofs in a more general class of games. While enlarging the set of games capable of generating randomness may facilitate the realization of those protocols, another important reason is that such generalization will further identify the essential feature of quantum information enabling those protocols. As the characterization of strong self-tests is critical for our proof, developing a theory of robust self-testing beyond binary XOR games may be essential for our question.

A security proof based on physical principles more general than quantum mechanics may also shed light on the essence of the source for the security. Non-signaling principle, or information causality [28] are interesting candidates for such general principles.

Our protocols require some initial perfect randomness to start with. The Chung-Shi-Wu protocol [5] relaxes this requirement to an arbitrary min-entropy source and tolerates a universal constant level of noise. However, those were achieved at a great cost on the number of non-communicating devices. A major open problem is our protocol can be modified to handle non-uniform input.

Randomness extraction can be thought of as a “seeded” extractions of randomness from untrusted quantum devices, as pointed out by Chung, Shi and Wu [5] in their “Physical Randomness Extractors” framework. Our one-shot and unbounded expansion results demonstrate a tradeoff between the seed length and the output length different from that in classical extractors. Recall that a classical extractor with output length $N$ and error parameter $\epsilon$ requires $\Omega(\log N / \epsilon)$ seed length, while our unbounded expansion protocol can have a fixed seed length (which determines the error parameter). What is the maximum amount of randomness one can extract from a device of a given amount of entanglement (i.e. is the exponential rate optimal for one device)? What can one say about the tradeoff between expansion rate and some proper quantity describing the communication restrictions?


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A Organization of Appendix

Section B defines some notation, establishes the model of quantum devices that we are using, and also states the full version of the central protocol. Section C proves a few mathematical statements (pertaining to operators and real-valued functions) which are used in the later proofs.

Section D introduces the quantum Rényi entropies \( H_\alpha (\cdot \| \cdot) \) and the Rényi divergences \( D_\alpha (\cdot \| \cdot) \). We give a definition of smooth min-entropy (see Definition 8) which is the standard measurement of security in the quantum context. The importance of Rényi entropies for our purposes is illustrated by Corollary D.5, which asserts that any lower bound for Rényi entropy implies a lower bound for smooth min-entropy. We prove Corollary D.5 by an easy derivative of a known proof.

In Section E we prove the uncertainty principle (Theorem E.2). The proof is based on a matrix inequality for the Schatten norm. Subsection F reviews some theory for nonlocal games which comes from [44] and [24], and derives some new consequences. Section G formalizes the notion of partially trusted measurement devices, and proves (using theory from section F) that strong self-tests can be used to simulate the behavior of such devices.

Section H contains the most technical part of the security proof for randomness expansion. Section H proves that Protocol A’ (see Figure 4) is secure. The proof rests mainly on the uncertainty principle (Theorem E.2). Section I uses the results of section H to deduce the security of Protocol R (using simulation). Our central result for randomness expansion is stated as Theorem I.1.

Sections J and K provide the supporting proofs for the results on unbounded expansion, randomness amplification, and key distribution.

B Notation and Definitions

B.1 Notation

When a sequence is defined, we will use Roman font to refer to individual terms (e.g., \( h_1, \ldots, h_n \)) and boldface font to refer to the sequence as a whole (e.g., \( h \)). For any bit \( b \), let \( \overline{b} = 1 - b \). For any sequence of bits \( b = (b_1, \ldots, b_n) \), let \( \overline{b} = (\overline{b_1}, \ldots, \overline{b_n}) \).

We will use capital letters (e.g., \( X \)) to denote quantum systems. We use the same letter to denote both the system itself and the complex Hilbert space which represents it.

An \( n \)-player binary nonlocal XOR game \( G \) consists of a probability distribution

\[
\{ p_i \mid i \in \{0,1\}^n \}
\]

on the set \( \{0,1\}^n \), together with an indexed set

\[
\{ \eta_i \mid i \in \{0,1\}^n \}.
\]

The quantities \( \{ p_i \} \) specify the probability distribution that is used to choose input strings for the game. The set \( \{ \eta_i \} \) specifies the score: if the output bits satisfy \( \bigoplus \eta_i = 0 \), then a score of \( \eta_i \) is awarded; otherwise a score of \( -\eta_i \) is awarded. (See the beginning of section F for a more detailed discussion of this definition.) When the score awarded is +1, we say that the players have passed. When the score is −1, we say that the players have failed.

We will write \( q_G \) to denote the maximum (expected) score that can be achieved by quantum strategies for the game \( G \). Note that this is different from the maximum passing probability for quantum strategies, which we denote by \( w_G \). The two are related by \( w_G = \left( \frac{1+q_G}{2} \right) \). We will also write \( f_G \) for the minimum failing probability, which is given by \( f_G = 1 - w_G \).
We write the expression \( f(x)^y \) (where \( f \) is a function) to mean \((f(x))^y\). Thus, for example, in the expression
\[
\text{Tr}[Z]^{1/q}
\]
the \((1/q)\)th power map is applied after the trace function, not before it.

If \( \rho_1 : X_1 \rightarrow Y_1 \) and \( \rho_2 : X_2 \rightarrow Y_2 \) are two linear operators, then we denote by \( \rho_1 \oplus \rho_2 \) the operator from \( X_1 \oplus X_2 \) to \( Y_1 \oplus Y_2 \) which maps \((x_1, x_2)\) to \((\rho_1(x_1), \rho_2(x_2))\).

If \((B, E)\) is a bipartite system, and \( \rho \) is a density operator on \( B \otimes E \) representing a classical-quantum state, then we may express \( \rho \) as a diagonal-block operator
\[
\rho = \begin{bmatrix}
\rho_1 \\
\rho_2 \\
\rho_3 \\
\ddots \\
\rho_m
\end{bmatrix},
\]
where \( \rho_1, \ldots, \rho_m \) denote the subnormalized operators on \( E \) corresponding to the basis states of the classical register \( B \). Alternatively, we may express \( \rho \) as \( \rho = \rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_m \).

We write \((\log x)\) to denote the logarithm with base 2, and we write \((\ln x)\) to denote the logarithm with base \( e \).

### B.2 Untrusted Quantum Devices

Let us formalize some terminology and notation for describing quantum devices. (Our formalism is a variation on that which has appeared in other papers on untrusted devices, such as [33].)

**Definition 4.** Let \( n \) be a positive integer. A *binary quantum device with \( n \) components* \( D = (D_1, \ldots, D_n) \) consists of the following.

1. Quantum systems \( Q_1, \ldots, Q_n \) whose initial state is specified by a density operator,
\[
\Phi : (Q_1 \otimes \ldots \otimes Q_n) \rightarrow (Q_1 \otimes \ldots \otimes Q_n)
\]  
(B.5)

2. For any \( k \geq 0 \), and any function
\[
T : \{0,1\} \times \{1,2,\ldots,k\} \times \{1,2,\ldots,n\} \rightarrow \{0,1\}
\]  
(B.6)
a unitary operator
\[
U_T : (Q_1 \otimes \ldots \otimes Q_n) \rightarrow (Q_1 \otimes \ldots \otimes Q_n).
\]  
(B.7)

and a collection of Hermitian operators
\[
\left\{ M_{T,j}^{(b)} : Q_j \rightarrow Q_j \right\}_{b \in \{0,1\}, 1 \leq j \leq n}
\]  
(B.8)
satisfying \( \|M_{T,j}^{(b)}\| \leq 1 \).
The device $D$ behaves as follows. Suppose that $k$ iterations of the device have already taken place, and suppose that $T$ is such that $T(0,i,j) \in \{0,1\}$ and $T(1,i,j) \in \{0,1\}$ represent the input bit and output bit, respectively, for the $j$th player on the $i$th round ($i \leq k$). ($T$ is the transcript function.) Then,

1. The components $D_1, \ldots, D_n$ collectively perform the unitary operation $U_T$ on $Q_1 \otimes \ldots \otimes Q_n$.

2. Each component $D_j$ receives its input bit $b_j$, then applies the binary nondestructive measurement on $Q_i$ given by

\[
X \mapsto \begin{pmatrix} \sqrt{\frac{1 + M_{T,j}(b_j)}{2}} & 0 \\ 0 & \sqrt{\frac{1 - M_{T,j}(b_j)}{2}} \end{pmatrix} X \begin{pmatrix} \sqrt{\frac{1 + M_{T,j}(b_j)}{2}} & 0 \\ 0 & \sqrt{\frac{1 - M_{T,j}(b_j)}{2}} \end{pmatrix},
\]

and then outputs the result.

B.3 Simulation

Let us say that one binary quantum device $D'$ simulates another binary quantum device $D$ if, for any purifying systems $E'$ and $E$ (for $D$ and $D'$, respectively), and any input sequence $i_1, \ldots, i_k \in \{0,1\}^n$, the joint state of the outputs of $D$ together with $E$ is isomorphic to the joint state of the outputs of $D'$ together with $E'$ on the same input sequence. Similarly, let us say that a protocol $X$ simulates another protocol $Y$ if, for any purifying systems $E$ and $E'$ for the quantum devices used by $X$ and $Y$, respectively, the joint state of $E$ together with the outputs of $X$ is isomorphic to the joint state of $E'$ together with the outputs of $Y$.

Definition 5. Let us say that a binary quantum device $D$ is in canonical form if each of its quantum systems $Q_j$ is such that $Q_j = C^{2m_j}$ for some $m_j \geq 1$, and each measurement operator pair $(M^{(0)}_j, M^{(1)}_j)$ has the following $2 \times 2$ diagonal block form:

\[
M^{(0)}_j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ & 0 & 1 & 1 \\ & & \ddots & \vdots \\ & & & 0 & 1 \\ & & & & 1 & 0 \\ & & & & & \ddots \\ & & & & & & \ddots \end{bmatrix},
\]

\[
M^{(1)}_j = \begin{bmatrix} 0 & \zeta_1 \\ \frac{\zeta_1}{\bar{\zeta}_1} & 0 \\ & 0 & \zeta_2 \\ & \frac{\zeta_2}{\bar{\zeta}_2} & 0 \\ & & \ddots \\ & & & 0 & \zeta_{m_j} \\ & & & \frac{\zeta_{m_j}}{\bar{\zeta}_{m_j}} & 0 \end{bmatrix},
\]

and
where the complex numbers $\zeta_\ell$ satisfy
\[
|\zeta_\ell| = 1 \text{ and } \text{Im}(\zeta_\ell) \geq 0. \quad \text{(B.11)}
\]
(Note that the complex numbers $\zeta_\ell$ may be different for each transcript $H$ and each player $j$.)

When we discuss quantum devices that are in canonical form, we will frequently make use of the isomorphism $\mathbb{C}^{2m} \cong \mathbb{C}^2 \otimes \mathbb{C}^m$ given by $e_{2k-1} \mapsto e_1 \otimes e_k, e_{2k} \mapsto e_2 \otimes e_k$. (Here, $e_1, \ldots, e_r$ denote the standard basis vectors for $\mathbb{C}^r$.)

The following proposition is easy to prove. (See Lemma 3.4 in the supplementary information of [24].)

**Proposition B.1.** Any binary quantum device can be simulated by a device that is in canonical form. □

## C Mathematical Preliminaries

In this section we state some purely mathematical results that will be used in later sections.

**Proposition C.1.** Let $\gamma \in [0, 1]$, and let $Z, W$ denote positive semidefinite operators on $\mathbb{C}^n$.

(a) If $Z \leq W$, then $Z^\gamma \leq W^\gamma$.

(b) If $Z \leq W$ and $X = W - Z$, then
\[
\text{Tr}(X^{1+\gamma}) + \text{Tr}(Z^{1+\gamma}) \leq \text{Tr}(W^{1+\gamma}). \quad \text{(C.1)}
\]

**Proof.** Part (a) is given by Theorem 2.6 in [4]. Part (b) follows from part (a) by the following reasoning:

\[
\text{Tr}(W^{1+\gamma}) = \text{Tr}(W \cdot W^\gamma) \quad \text{(C.2)}
\]
\[
= \text{Tr}(X \cdot W^\gamma) + \text{Tr}(Z \cdot W^\gamma) \quad \text{(C.3)}
\]
\[
\geq \text{Tr}(X \cdot X^\gamma) + \text{Tr}(Z \cdot Z^\gamma) \quad \text{(C.4)}
\]
\[
= \text{Tr}(X^{1+\gamma}) + \text{Tr}(Z^{1+\gamma}). \quad \text{(C.5)}
\]

This completes the proof. □

**Proposition C.2.** Let $U \subseteq \mathbb{R}^n$, let $z \in \mathbb{R}^n$ be an element in the closure of $U$, and let $f, g$ be continuous functions from $U$ to $\mathbb{R}$. Suppose that
\[
\lim_{x \to z} f(x) = 0 \quad \text{(C.6)}
\]
and
\[
\lim_{x \to z} \frac{f(x)}{g(x)} = c. \quad \text{(C.7)}
\]

Then,
\[
\lim_{x \to z} (1 + f(x))^{1/g(x)} = e^c. \quad \text{(C.8)}
\]

**Proof.** This can be proved easily by taking the natural logarithm of both sides of (C.8). □
Proposition C.3. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$, and assume that $V$ is compact. Let $f : U \times V \to \mathbb{R}$ be a continuous function. Let $z \in \mathbb{R}^n$ be an element in the closure of $U$, and assume that $\lim_{x \to z} f(x, y)$ exists for every $y \in V$. Then,

$$\lim_{x \to z} \min_{y \in V} f(x, y) = \min_{y \in V} \lim_{x \to z} f(x, y).$$

(C.9)

Proof. Let $\overline{f}$ be the continuous extension of $f$ to $(U \cup \{z\}) \times V$. Let $h(x, y) = \overline{f}(x, y) - \overline{f}(z, y)$.

Let $\delta > 0$. For any $y \in V$, since $h(z, y) = 0$ and $h$ is continuous at $(z, y)$, we can find an $\epsilon_y > 0$ such that the values of $h$ on the cylinder

$$\{(x, y') \mid |x - z| < \epsilon_y, |y' - y| < \epsilon_y\}$$

are confined to $[-\delta, \delta]$. Since $V$ is compact, we can choose a finite set $S \subseteq V$ such that the $\epsilon_y$-cylinders for $y \in S$ cover $V$. Letting $\epsilon = \min_{y \in S} \epsilon_y$, we find that the values of $h$ on the $\epsilon$-neighborhood of $V$ are confined to $[-\delta, \delta]$. Therefore, the minimum of $f(x, y)$ on the $\epsilon$-neighborhood of $V$ is within $\delta$ of $\min_{y \in V} \overline{f}(z, y)$. The desired equality (C.9) follows.

D Quantum Rényi Entropies

We will use the notion of quantum Rényi divergence which is defined in [25].

Definition 6 (25). Let $\rho$ be a density matrix on $\mathbb{C}^n$. Let $\sigma$ be a positive semidefinite matrix on $\mathbb{C}^n$ whose support contains the support of $\rho$. Let $\alpha > 1$ be a real number. Then,

$$d_\alpha(\rho \| \sigma) = \log \left( \frac{1}{\text{Tr}[\rho]} \left( \sigma^{\frac{1}{\alpha-1}} \rho^{\frac{1}{\alpha}} \sigma^{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha-1}} \right).$$

(D.1)

More generally, for any positive semidefinite matrix $\rho'$ whose support is contained in $\text{Supp} \ \sigma$, let

$$d_\alpha(\rho' \| \sigma) := \log d_\alpha(\rho' \| \sigma).$$

(D.2)

Let

$$D_\alpha(\rho' \| \sigma) = d_\alpha(\rho' \| \sigma).$$

(D.3)

Definition 7. Let $AB$ be a classical-quantum system, and let $\rho_{AB}$ be a density operator. Let $\alpha > 1$ be a real number. Then,

$$H_\alpha(A \mid B)_{\rho | \sigma} := -D_\alpha(\rho_{AB} \| \mathbb{1}_A \otimes \sigma).$$

(D.4)

Equivalently,

$$H_\alpha(A \mid B)_{\rho | \sigma} = \frac{1}{\alpha - 1} \log \left( \sum_\sigma \text{Tr} \left[ \left( (\sigma)^{\frac{1}{\alpha-1}} (\rho_B^\sigma)^{\frac{1}{\alpha}} (\sigma)^{\frac{1}{\alpha}} \right)^{\frac{1}{\alpha-1}} \right] \right).$$

(D.5)

Define (as in [25]),

$$H_\alpha(A \mid B)_{\rho} = \max_{\sigma \geq 0, \text{Tr}(\sigma) = 1} H_\alpha(A \mid B)_{\rho | \sigma}.$$
Additionally, we will need a definition of smooth min-entropy. There are multiple definitions of smooth min-entropy that are essentially equivalent. The definition that we will use is not the most up-to-date (see [37]) but it is good for our purposes because it is simple.

**Definition 8.** Let AB be a classical-quantum system, and let \( \rho_{AB} \) be a positive semidefinite operator. Let \( \epsilon > 0 \) be a real number. Then,

\[
H_{\min}^{\epsilon}(A | B)_{\rho_{AB}} = \max_{\|\rho' - \rho\|_1 \leq \epsilon} \max_{\sigma : B \rightarrow B \quad I_A \otimes \sigma \geq \rho'} \log(\text{Tr}(B)),
\]

(D.7)

where the minimization is taken over positive semidefinite operators \( \sigma \) on \( B \).

The smooth min-entropy measures the number of random bits that can be extracted from a classical source in the presence of quantum information [34]. When it is convenient, we will use the notation \( H_{\min}^{\epsilon}(\rho_{AB}|B) \) instead of \( H_{\min}^{\epsilon}(A|B)_{\rho_{AB}} \).

Following [10], let us define the relative smooth max-entropy of two operators.

**Definition 9.** Let \( \rho, \sigma \) be positive semidefinite operators on \( \mathbb{C}^n \) such that the support of \( \sigma \) contains the support of \( \rho \). Then,

\[
D_{\max}(\rho \| \sigma) = \log \min_{\lambda \in \mathbb{R}} (\lambda).
\]

For any \( \epsilon \geq 0 \),

\[
D_{\max}^{\epsilon}(\rho \| \sigma) = \inf_{\|\rho' - \rho\|_1 \leq \epsilon} D_{\max}(\rho' \| \sigma).
\]

The quantity \( D_{\max}^{\epsilon} \) is convenient for computing lower bounds on \( H_{\min}^{\epsilon} \). Note that if \( \psi \) is any density matrix on \( B \),

\[
H_{\min}^{\epsilon}(\rho_{AB}|B) \geq \log \min_{\lambda \in \mathbb{R}} (\lambda).
\]

Now we will prove some results about Rényi entropy and smooth min-entropy. (These are easy derivatives of known results for \( \alpha = 2 \) ([14], [36]).)

**Proposition D.1.** Let \( \alpha \in (1, 2] \). Let \( \rho, \sigma \) be positive semidefinite operators on a finite-dimensional Hilbert space \( V \) such that \( \text{Supp} \sigma \supseteq \text{Supp} \rho \). Then, there exists a positive semidefinite operator \( \rho' \) such that \( \rho' \leq \sigma \) and

\[
\log \|\rho - \rho'\|_1 \leq \frac{\alpha - 1}{2} D_{\alpha}(\rho \| \sigma) + \frac{1}{2}.
\]

(D.11)

**Proof.** Our proof is based on the proof of Lemma 19 in [14] (which, in turn, is based on [36]). For any Hermitian operator \( H \), let \( P_H^+ \) denote projection on the subspace spanned by the positive eigenvectors of \( H \), and let \( \text{Tr}^+(H) = \text{Tr}(P_H^+ HP_H^+) \). Let

\[
\delta = \text{Tr}^+(\rho - \sigma).
\]

(D.12)

Note that, by the construction from the proof of Lemma 15 in [36], there must exist an operator \( \rho' \) such that \( \rho' \leq \sigma \) and \( \|\rho' - \rho\|_1 \leq \sqrt{2\delta} \).
Let $P = P_{ho_{-\rho}}^\perp$, and let $P^\perp$ denote the complement of $P$. Note that by applying the data processing inequality for $D_a$ (see Theorem 5 in [25]) to the quantum operation $X \mapsto |0\rangle \langle 0| \otimes PXP + |1\rangle \langle 1| \otimes P^\perp XP^\perp$, we have

\[
D_a(\rho || \sigma) \geq \frac{1}{\alpha - 1} \log \left( \text{Tr} \left[ \left( \rho^{\frac{1}{\alpha}} \sigma^{\frac{1}{\alpha}} \right)^a \left( \rho^{1/a} \sigma^{1/a} \right)^{a-1} \right] \right) \quad (D.13)
\]

\[
D_a(\rho || \sigma) \geq \frac{1}{\alpha - 1} \log \left( \text{Tr} \left[ \left( \sigma^{\frac{1}{\alpha}} \rho^{\frac{1}{\alpha}} \right)^a \left( \sigma^{1/a} \rho^{1/a} \right)^{a-1} \right] \right) \quad (D.14)
\]

\[
D_a(\rho || \sigma) \geq \frac{1}{\alpha - 1} \log \left( \text{Tr} \left[ \left( \sigma^{\frac{1}{\alpha}} \rho^{\frac{1}{\alpha}} \right)^a \right] \right) \quad (D.15)
\]

Let $\sigma = P\sigma P$ and $\rho = P\rho P$. We have the following.

\[
D_a(\rho || \sigma) \geq \frac{1}{\alpha - 1} \log \left( \text{Tr} \left[ \left( \sigma^{\frac{1}{\alpha}} \rho^{\frac{1}{\alpha}} \right)^a \left( \sigma^{1/a} \rho^{1/a} \right)^{a-1} \right] \right) \quad (D.16)
\]

Note that $\rho \geq \sigma$ by construction, and $Z \mapsto Z^{a-1}$ is a monotone function (see part (a) of Proposition C.1). Therefore we have the following.

\[
D_a(\rho || \sigma) \geq \frac{1}{\alpha - 1} \log \left( \text{Tr} \left[ \left( \sigma^{\frac{1}{\alpha}} \rho^{\frac{1}{\alpha}} \right)^a \left( \sigma^{1/a} \rho^{1/a} \right)^{a-1} \right] \right) \quad (D.17)
\]

\[
D_a(\rho || \sigma) \geq \frac{1}{\alpha - 1} \log \left( \text{Tr} \left[ \left( \sigma^{\frac{1}{\alpha}} \rho^{\frac{1}{\alpha}} \right)^a \right] \right) \quad (D.18)
\]

\[
D_a(\rho || \sigma) \geq \frac{1}{\alpha - 1} \log \left( \text{Tr} \left[ \left( \sigma^{\frac{1}{\alpha}} \rho^{\frac{1}{\alpha}} \right)^a \right] \right) \quad (D.19)
\]

\[
D_a(\rho || \sigma) \geq \frac{1}{\alpha - 1} \log \delta \quad (D.20)
\]

where in the last line we used the fact that $\text{Tr}(\sigma) \geq \text{Tr}(\rho) = \delta$. Let $\rho'$ be a positive semidefinite operator satisfying $\rho' \leq \sigma$ and $\|\rho' - \rho\|_1 \leq \sqrt{\delta}$. Then we have

\[
D_a(\rho || \sigma) \geq \frac{1}{\alpha - 1} \log \left( \|\rho' - \rho\|_1^2 / 2 \right), \quad (D.21)
\]

which implies the desired result. \hfill \Box

**Proposition D.2.** Suppose that in Proposition D.1, $V$ is the state space of a bipartite quantum system $AB$, and $\rho, \sigma$ are classical-quantum operators.\footnote{That is, $A$ is a classical register and $\rho, \sigma$ have the form $\rho = \sum_i |a_i\rangle \langle a_i| \otimes \rho_i$ and $\sigma = \sum_i |a_i\rangle \langle a_i| \otimes \sigma_i$ where $\{a_1, \ldots, a_n\}$ is a standard basis for $A$.} Then, there exists an operator $\rho'$ satisfying the conditions of Proposition D.1 such that $\rho'$ itself is a classical-quantum operator.

**Proof.** This is an easy consequence of the construction for $\rho'$ (from the proof of Lemma 15 in [36]) which was used in the proof of Proposition D.1. \hfill \Box

**Corollary D.3.** Let $\rho, \sigma$ be as in Proposition D.1, and let $\epsilon > 0$. Then,

\[
D^\epsilon_{\text{max}}(\rho || \sigma) \leq D_a(\rho || \sigma) + \frac{2\log(1/\epsilon) + 1}{\alpha - 1}. \quad (D.22)
\]

Additionally, if $\rho$ is a classical-quantum operator on a bipartite state, then there exists a classical-quantum operator $\rho'$ with $\|\rho' - \rho\|_1 \leq \epsilon$ and $\rho' \geq 0$ such that $D^\epsilon_{\text{max}}(\rho' || \sigma)$ satisfies the above bound.
Proof. Let \( \lambda \) be the quantity on the right side of inequality (D.22). We have
\[
D_\alpha(\rho \| 2^{-\lambda} I \otimes \sigma) = \frac{2 \log \epsilon - 1}{\alpha - 1}.
\]
(D.23)

By Proposition [D.1], we can find a positive semidefinite operator \( \rho' \leq 2^{-\lambda} I \otimes \sigma \) such that
\[
\|\rho' - \rho\|_1 \leq \epsilon.
\]
(D.24)

The result follows from the definition of \( D_{\max}^\epsilon \).
\( \square \)

The next corollaries can be similarly proved.

Corollary D.4. Let \( AB \) be a classical-quantum bipartite system, and let \( \rho_{AB} \) be a density operator. Let \( \sigma \) be a density operator on \( B \). Let \( \epsilon > 0 \) and \( \alpha \in (1, 2] \) be real numbers. Then, for any \( \epsilon > 0 \),
\[
H_{\min}^\epsilon(A \mid B) \rho \geq -D_\alpha(\rho \| I_A \otimes \sigma) - \frac{2 \log(1/\epsilon) + 1}{\alpha - 1}.
\]
(D.25)

Corollary D.5. Let \( AB \) be a classical-quantum bipartite system, and let \( \rho_{AB} \) be a density operator. Let \( \epsilon > 0 \) and \( \alpha \in (1, 2] \) be real numbers. Then, for any \( \epsilon > 0 \),
\[
H_{\min}^\epsilon(A \mid B) \rho \geq H_\alpha(A \mid B) \rho - \frac{2 \log(1/\epsilon) + 1}{\alpha - 1}.
\]
(D.26)

E An Uncertainty Principle

In this section, we consider the behavior of the map \( \rho \mapsto \text{Tr}[\rho^{1+\epsilon}] \) when measurements are applied to a qubit and the operator \( \rho \) represents the state of a system that is entangled with the qubit.

For any linear operator \( A \), and any \( p > 0 \), let \( \|A\|_p \) denote the Schatten \( p \)-norm, which is defined by
\[
\|A\|_p = \text{Tr} \left( (A^* A)^{p/2} \right)^{1/p}.
\]
(E.1)

We begin by quoting the following theorem, which appears as part of Theorem 5.1 in [32].

Theorem E.1. Let \( X, Y : \mathbb{C}^m \to \mathbb{C}^n \) be linear operators. Let \( p \geq 2 \) be a real number, and let \( p' = 1/(1 - 1/p) \). Then,
\[
\left[ \frac{1}{2} \left( \|X + Y\|_p^p + \|X - Y\|_p^p \right) \right]^{1/p} \leq \left( \|X\|_{p'}^{p'} + \|Y\|_{p'}^{p'} \right)^{1/p'}.
\]
(E.2)

Inequality (E.2) may alternatively be expressed as
\[
\left[ \|X + Y\|_p^p + \|X - Y\|_p^p \right]^{1/p} \leq 2^{1/p - 1/2} \left( \|X\|_{p'}^{p'} + \|Y\|_{p'}^{p'} \right)^{1/p'}.
\]
(E.3)

or,
\[
\left[ \frac{X + Y}{\sqrt{2}} \right]_p^p + \left[ \frac{X - Y}{\sqrt{2}} \right]_p^p \leq 2^{1 - p/2} \left( \|X\|_{p'}^{p'} + \|Y\|_{p'}^{p'} \right)^{p'/p'}.
\]
(E.4)
Let us use the following notation: if
\[ Z : V \rightarrow W \otimes C^2, \]  
where \( V \) and \( W \) are finite dimensional \( C \)-vector spaces, and
\[ \rho = Z^* Z, \]  
then for any unit vector \( v \in C^2 \), let
\[ \rho_v = Z^* (I_W \otimes vv^*) Z. \]  
We will apply this notation in particular using the vectors \( |0\rangle, |1\rangle, |+\rangle, |−\rangle \in C^2 \).

**Theorem E.2.** There exists a continuous function \( \Pi : (0, 1] \times [0,1] \rightarrow \mathbb{R} \) such that the following conditions hold.

1. For any \((\epsilon, \delta) \in (0, 1] \times [0,1]\), and any linear operators \( Z : V \rightarrow W \otimes C^2 \) and \( \rho = Z^* Z \) such that
\[ \text{Tr}(\rho_{1+\epsilon}^1) = \delta \cdot \text{Tr}(\rho^{1+\epsilon}), \]  
we must have
\[ \frac{\text{Tr}(\rho_1^{1+\epsilon} + \rho_1^{1+\epsilon})}{\text{Tr}(\rho^{1+\epsilon})} \leq \left( \frac{1}{2} \right)^{e \cdot \Pi(\epsilon, \delta)}. \]  
2. The function \( \Pi \) satisfies \( \lim_{(x,y) \to (0,0)} = 1 \).
3. The function \( \pi : [0,1] \rightarrow \mathbb{R} \) defined by
\[ \pi(y) = \lim_{x \to 0} \Pi(x, y) \]  
is convex, differentiable on \((0,1)\), and strictly decreasing on \((0,1/2)\).

**Remark E.3.** Our proof will show that, in particular, we may assume that the function \( \pi \) is given by
\[ \pi(y) = 1 - 2(1-y) \log \left( \frac{1}{1-y} \right) - 2y \log \left( \frac{1}{y} \right) \]  
where \( h \) denotes the Shannon entropy of the vector \((y, 1-y)\). (We are stating this fact separately in order to make Theorem E.2 more portable.)

**Proof of Theorem E.2** We begin by constructing the function \( \Pi \) over the smaller domain \((0,1] \times [0, \frac{1}{2}]\). Suppose \( \epsilon \in (0, 1] \) and \( \delta \in [0, \frac{1}{2}] \), and that \( Z \) and \( \rho \) are operators satisfying (E.8). Decompose the operator \( Z \) as
\[ Z = X \otimes |0\rangle + Y \otimes |1\rangle. \]  
where \( X, Y : V \rightarrow W \) are linear operators. Then we have \( \rho_0 = X^* X, \rho_1 = Y^* Y, \) and
\[ \rho_+ = \left( \frac{X+Y}{\sqrt{2}} \right)^* \left( \frac{X+Y}{\sqrt{2}} \right), \]  
\[ \rho_- = \left( \frac{X-Y}{\sqrt{2}} \right)^* \left( \frac{X-Y}{\sqrt{2}} \right), \]  
24
Applying (E.4) with $p = 2 + 2\epsilon$, and $p' = 1/(1 - 1/p)$ we have the following:

\[
\text{Tr}(\rho_{+}^{1+\epsilon} + \rho_{-}^{1+\epsilon}) = \left\| \frac{X + Y}{\sqrt{2}} \right\|^{2+2\epsilon} + \left\| \frac{X - Y}{\sqrt{2}} \right\|^{2+2\epsilon} \leq 2^{1-p/2} \left( \| X \|_{p'}^{p'} + \| Y \|_{p'}^{p'} \right)^{p'/p'}
\]

(E.16)

Letting

\[
t = \frac{\text{Tr}(\rho_{0}^{1+\epsilon})}{\text{Tr}(\rho^{1+\epsilon})} \quad \text{and} \quad s = \frac{\text{Tr}(\rho_{1}^{1+\epsilon})}{\text{Tr}(\rho^{1+\epsilon})},
\]

we have

\[
\text{Tr}(\rho_{+}^{1+\epsilon} + \rho_{-}^{1+\epsilon}) \leq 2^{-\epsilon} \left[ t^{1/\tau_{2}} + s^{1/\tau_{2}} \right]^{1+2\epsilon} \text{Tr}(\rho^{1+\epsilon}).
\]

(E.21)

Since $\text{Tr}(\rho_{0}^{1+\epsilon}) + \text{Tr}(\rho_{1}^{1+\epsilon}) \leq \text{Tr}(\rho^{1+\epsilon})$, we have $t + s \leq 1$, and therefore the above bound can be replaced with the following.

\[
\text{Tr}(\rho_{+}^{1+\epsilon} + \rho_{-}^{1+\epsilon}) \leq 2^{-\epsilon} \left[ (1 - s)^{1/\tau_{2}} + s^{1/\tau_{2}} \right]^{1+2\epsilon} \text{Tr}(\rho^{1+\epsilon}).
\]

(E.22)

We have $s \leq \delta \leq 1/2$ by assumption, and the function $s \mapsto (1 - s)^{1/\tau_{2}} + s^{1/\tau_{2}}$ is increasing in $s$ over the interval $[0, 1/2]$, and thus we have

\[
\text{Tr}(\rho_{+}^{1+\epsilon} + \rho_{-}^{1+\epsilon}) \leq 2^{-\epsilon} \left[ (1 - \delta)^{1/\tau_{2}} + \delta^{1/\tau_{2}} \right]^{1+2\epsilon} \text{Tr}(\rho^{1+\epsilon}).
\]

(E.23)

Let $\Pi: (0, 1] \times [0, 1/2] \to \mathbb{R}$ be the following function.

\[
\Pi(\epsilon, \delta) = 1 - \left( \frac{1 + 2\epsilon}{\epsilon} \right) \log \left[ (1 - \delta)^{1/\tau_{2}} + \delta^{1/\tau_{2}} \right]
\]

(E.24)

Inequality (E.23) implies

\[
\text{Tr}(\rho_{+}^{1+\epsilon} + \rho_{-}^{1+\epsilon}) \leq 2^{-\epsilon \Pi(\epsilon, \delta)} \text{Tr}[\rho^{1+\epsilon}],
\]

(E.25)
as desired.

Evaluating the limit of the function using L'Hospital's rule yields

\[
\lim_{x \to 0} \Pi(x, y) = 1 + 2(1 - y) \log(1 - y) + 2y \log y.
\]

(E.26)

for $y \in (0, 1/2]$ (a function which is indeed differentiable and convex) and

\[
\lim_{(x,y) \to (0,0)} \Pi(x, y) = 1.
\]

(E.27)

We have defined $\Pi$ over the limited domain $(0, 1] \times [0, 1/2]$. Extend $\Pi$ to $(0, 1] \times [0, 1]$ by defining $\Pi(x, y) = \Pi(x, 1 - y)$. The desired results follow by symmetry. \(\square\)
F The Self-Testing Property of Binary Nonlocal XOR Games

F.1 Definitions and Basic Results

Definition 10. An $n$-player binary nonlocal XOR game consists of a probability distribution
\[ \{ p_i \mid i \in \{0,1\}^n \} \] (F.1)
on the set $\{0,1\}^n$, together with an indexed set
\[ \{ \eta_i \in \{-1,1\} \mid i \in \{0,1\}^n \}. \] (F.2)

Given any indexed sets $\{p_i\}$ and $\{\eta_i\}$ satisfying the above conditions, we can conduct an $n$-player nonlocal game as follows.

1. A referee chooses a binary vector $c \in \{0,1\}^n$ according to the distribution $\{p_i\}$. For each $k$, he gives the bit $c_k$ as input to the $k$th player.
2. Each player returns an output bit $d_k$ to the referee.
3. The referee calculates the bit
\[ d_1 \oplus d_2 \oplus \ldots \oplus d_n \oplus \left( \frac{1 - \eta_c}{2} \right). \] (F.3)

If this bit is equal to 0, a “pass” has occurred, and a score of $+1$ is awarded. If this bit is equal to 1, a “failure” has occurred, and a score of $-1$ is awarded.

We quote some definitions and results from [24] and [43].

Definition 11. An mixed $n$-player quantum strategy is a pair
\[ \left( \Psi, \{ \{ M_j^{(0)} \}, M_j^{(1)} \} \right) \] (F.4)
where $\Psi$ is a density matrix on an $n$-tensor product space $V_1 \otimes \ldots \otimes V_n$ and $M_j^{(i)}$ denotes a linear operator on $V_j$ whose eigenvalues are contained in $\{-1,1\}$. A pure $n$-player quantum strategy is a pair
\[ \left( \psi, \{ \{ M_j^{(0)} \}, M_j^{(1)} \} \right) \] (F.5)
which satisfies the same conditions, except that $\psi$ is merely a unit vector on $V_1 \otimes \ldots \otimes V_n$. A qubit strategy is a pure quantum strategy in which the spaces $V_i$ are equal to $\mathbb{C}^2$ and the operators $M_j^{(i)}$ are all nonscalar.

The score achieved by a quantum strategy at an $n$-player binary nonlocal XOR game $G = (\{p_i\}, \{\eta_i\})$ is the expected score when the qubit strategy is used to play the game $G$. This quantity can be expressed as follows. Let $M$ denote the scoring operator for $G$, which is given by
\[ M = \sum_{i \in \{0,1\}^n} p_i \eta_i M_{1}^{(i_1)} \otimes M_{2}^{(i_2)} \otimes \cdots \otimes M_{n}^{(i_n)}. \] (F.6)

Then, the score for strategy (F.4) at game $G$ is $\text{Tr}(M \Psi)$. The score for the pure strategy (F.5) is $\psi^* M \psi$.

The optimal score for a nonlocal game is highest score that can be achieved at the game by qubit strategies. We denote this quantity by $q_G$. (As explained in [24], this is also the highest score that can
be achieved by arbitrary quantum strategies.) A game $G$ is a self-test if there is only one qubit strategy (modulo local unitary operations on the $n$ tensor components of $(\mathbb{C}^2)^{\otimes n}$) which achieves the optimal score. A game $G$ is winnable if $q_G = 1$.

For any nonlocal game $G = (\{p_i\}, \{\eta_i\})$, define $P_G : \mathbb{C}^n \to \mathbb{C}$ by

$$P_G(\lambda_1, \ldots, \lambda_n) = \sum_{i \in \{0,1\}^n} p_i \eta_i \lambda_1^{i_1} \lambda_2^{i_2} \ldots \lambda_n^{i_n}. \quad (F.7)$$

Define $Z_G : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$Z_G(\theta_0, \theta_1, \ldots, \theta_n) = \sum_{i \in \{0,1\}^n} p_i \eta_i \cos \left( \theta_0 + \sum_{k=1}^n i_k \theta_k \right). \quad (F.8)$$

The two quantities $Z_f$ and $P_f$ are useful for expressing quantities related to binary XOR games. We note the following relationships.

$$Z_G(\theta_0, \ldots, \theta_n) = \Re \left[ e^{i \theta_0} P(e^{i \theta_1}, e^{i \theta_2}, \ldots, e^{i \theta_n}) \right]. \quad (F.9)$$

$$|P_G(e^{i \theta_1}, \ldots, e^{i \theta_n})| = \max_{\theta_0 \in [-\pi, \pi]} Z_G(\theta_0, \ldots, \theta_n). \quad (F.10)$$

The functions $P_G$ and $Z_G$ can be used to calculate $q_G$. This was observed by Werner and Wolf in [43]. We sketch a proof here. (For a more detailed proof, see Proposition 1 in [24].)

**Proposition F.1.** For any nonlocal binary XOR game $G$, the following equalities hold.

$$q_G = \max_{|\lambda_1| = \ldots = |\lambda_n| = 1} |P_G(\lambda_1, \ldots, \lambda_n)| \quad (F.11)$$

$$q_G = \max_{\theta_0, \ldots, \theta_n \in \mathbb{R}} Z_G(\theta_0, \ldots, \theta_n). \quad (F.12)$$

**Proof sketch.** Let $(\psi, \{M_j^{(i)}\})$ be a qubit strategy for $G$. By an appropriate choice of basis, we may assume that

$$M_j^{(0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad M_j^{(1)} = \begin{bmatrix} 0 & \zeta_j \\ \zeta_j & 0 \end{bmatrix}. \quad (F.13)$$

where $\{\zeta_j\}$ are complex numbers of length 1. The scoring operator $M$ can be expressed as a reverse diagonal matrix whose entries are

$$\left\{ P_G(\zeta_1^{b_1}, \ldots, \zeta_n^{b_n}) \right\}_{(b_1, \ldots, b_n) \in \{-1,1\}^n}. \quad (F.14)$$

The eigenvalues of a reverse diagonal Hermitian matrix whose reverse-diagonal entries are $z_1, z_2, \ldots, z_{2n}$ is simply $\pm |z_1|, \pm |z_2|, \ldots, \pm |z_n|$. Therefore the operator norm of $M$ is the maximum absolute value that occurs in (F.14).

The value $q_f$ is the maximum of the operator norm that occurs among all the scoring operators arising from qubit strategies for $G$. The desired formulas follow.

**Proposition F.2.** Let $G$ be a nonlocal binary XOR game. Then, $G$ is a self-test if and only if the following two conditions are satisfied.

(A) There is a maximum $(\alpha_0, \ldots, \alpha_n)$ for $Z_G$ such that none of $\alpha_1, \ldots, \alpha_n$ is a multiple of $\pi$. 

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(B) Every other maximum of \( Z_G \) is congruent modulo \( 2\pi \) to either \((\alpha_0, \ldots, \alpha_n)\) or \((-\alpha_0, \ldots, -\alpha_n)\).

**Proof.** See Proposition 2 in [24]. \( \Box \)

The following definition will be convenient in later proofs.

**Proposition F.3.** Let \( G \) be a nonlocal game which is a self-test. Then, \( G \) is **positively aligned** if a maximum for \( Z_G(\theta_0, \ldots, \theta_n) \) occurs in the region

\[
\{ (\theta_0, \ldots, \theta_n) \mid 0 < \theta_i < \pi \quad \forall i \geq 1 \}.
\]

For any binary XOR self-test \( G = (\{p_i\}, \{\eta_i\}) \), we can construct a positively aligned self-test \( G' = (\{p'_i\}, \{\eta'_i\}) \) by choosing appropriate values \( b_1, \ldots, b_n \in \{0, 1\} \),

\[
p'_i = p_i \quad \eta'_i = \eta_{(i+b) \mod 2}.
\]

It is easy to see that \( q_{G'} = q_G \).

**Definition 12.** Let \((\psi, \{M^{(i)}_j\})\) and \((\phi, \{N^{(i)}_j\})\) be \( n \)-player qubit strategies. Then the **distance** between these two strategies is the quantity

\[
\max \left( \left\{ \| \psi - \phi \| \right\} \cup \left\{ \| M^{(i)}_j - N^{(i)}_j \| : j \in \{1, 2, \ldots, n\}, i \in \{0, 1\} \right\} \right).
\]

(In this formula, the first norm denotes Euclidean distance and second denotes operator norm.) Let \( G \) be a self-test. Then, \( G \) is a **strong self-test** if there exists a constant \( K \) such that any qubit strategy that achieves a score of \( q_G - \epsilon \) is within distance \( K\sqrt{\epsilon} \) from a qubit strategy that achieves the score \( q_G \).

For any twice differentiable \( m \)-variable function \( F: \mathbb{R}^m \rightarrow \mathbb{R} \), and any \( c = (c_1, \ldots, c_m) \in \mathbb{R}^m \), we can define the Hessian matrix for \( F \) at \( c \), which is the \( m \times m \) matrix formed from the second partial derivatives

\[
\frac{\partial^2 F}{\partial x_i \partial x_j}(c_1, \ldots, c_m)
\]

(for \( i, j \in \{1, 2, \ldots, m\} \)).

**Proposition F.4.** Let \( G \) be an \( n \)-player self-test. Then the following conditions are equivalent.

1. \( G \) is a strong self-test.

2. The function \( Z_G \) has nonzero Hessian matrices at all of its maxima.

3. There exists a constant \( K > 0 \) such that any \((\beta_0, \ldots, \beta_n) \in \mathbb{R}^{n+1} \) which satisfies

\[
Z_G(\beta_0, \ldots, \beta_n) \geq q_G - \epsilon
\]

(with \( \epsilon \geq 0 \)) must be within distance \( K\sqrt{\epsilon} \) from a maximum of \( Z_G \).

**Proof.** See [24] (especially Proposition 6.1 in the supplementary information). \( \Box \)

The following proposition asserts a consequence of the strong self-test condition which will be useful in later proofs.
**Proposition F.5.** Let $G$ be a positively-aligned strong self-test. Let $H$ denote the semicircle \( \{e^{i\beta} \mid 0 \leq \beta \leq \pi \} \subseteq \mathbb{C} \). Then, there exists \( \alpha \in [-\pi, \pi] \) and \( c \geq 0 \) such that the set
\[
P_f(H^n) \subseteq \mathbb{C}
\] (F.20)
is bounded by the polar curve
\[
f : [-\pi, \pi] \rightarrow \mathbb{C}
f(\theta) = (q_G - c(\theta - \alpha)^2)e^{i\theta}.
\] (F.21)

**Proof.** Since $G$ is positively aligned, we may find a maximum \((\alpha_0, \ldots, \alpha_n)\) for $Z_G$ such that $\alpha_1, \ldots, \alpha_n \in (0, \pi)$. Choose $K$ according to condition (3) from Proposition F.4. Let $c = 1/K$ and $\alpha = -\alpha_0$.

Suppose, for the sake of contradiction, that there is a point in the set $P_f(H^n)$ which lies outside of (F.21). Then, there exists $\beta_1, \ldots, \beta_n \in [0, \pi]$ such that
\[
P_f(e^{i\beta_1}, \ldots, e^{i\beta_n}) = re^{i\theta}
\] (F.22)
(with $\theta \in [-\pi, \pi]$) and
\[
r > q_G - c(\theta - \alpha)^2.
\] (F.23)

Let $\epsilon = (1/K^2)(\theta - \alpha)^2$. We have
\[
Z_G(-\theta, \beta_1, \ldots, \beta_n) = r > q_G - c(\theta - \alpha)^2
\] (F.24)
\[
= q_G - \epsilon,
\] (F.25)
and the distance between \((-\theta, \beta_1, \ldots, \beta_n)\) and \((\alpha_0, \ldots, \alpha_n)\) is at least $|\theta - \alpha| = K\sqrt{\epsilon}$. (And, it is easy to see that \((-\theta, \beta_1, \ldots, \beta_n)\) is not any closer to any of the other maxima of $Z_G$ than it is to \((\alpha_0, \ldots, \alpha_n)\).) This contradicts condition (3) of Proposition F.4.

**Corollary F.6.** Let $G$ satisfy the assumptions of Proposition F.5. Then, there exists a complex number $\gamma \neq 0$ such that for all $\xi_1, \ldots, \xi_n \in H$,
\[
|P_G(\xi_1, \ldots, \xi_n) - \gamma| + |\gamma| \leq q_G.
\] (F.26)

**Proof.** Let $R \subseteq \mathbb{C}$ be the region enclosed by the polar curve (F.21). Let $S = \{z \in \mathbb{C} \mid |z| = q_G\}$. We have $S \cap R = \{e^{i\alpha}\}$. Since the curvature of the curve (F.21) at $e^{i\alpha}$ is strictly greater than 1, we can find a circle of radius less than 1 which lies inside of $S$, which is tangent to $S$ at $e^{i\alpha}$, and which encloses the region $R$. Then, if we let $\gamma$ be the center of this circle, we have $|z - \gamma| + |\gamma| \leq q_G$ for all $z \in R$. The desired inequality follows.

### F.2 Decomposition Theorems

For any unit-length complex number $\zeta$, let us write $g_{\zeta}$ for the following modified GHZ state:
\[
g_{\zeta} = \frac{1}{\sqrt{2}}(|00\ldots0\rangle + \zeta|11\ldots1\rangle).
\] (F.27)

The next theorem uses the canonical form for binary measurements from subsection B.3. Note that when a collection of four projections $\{P^{(b,c)}\}$ is in canonical form over a space $\mathbb{C}^{2m}$, we can naturally express them as operators on $\mathbb{C}^2 \otimes \mathbb{C}^m$ via the isomorphism $\mathbb{C}^{2m} \rightarrow \mathbb{C}^2 \otimes \mathbb{C}^m$ given by $e_{2k-1} \mapsto e_1 \otimes e_{2k}, e_{2k} \mapsto e_2 \otimes e_k$. 

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Theorem F.7. Let \( G = (\{p_i\}, \{\eta_i\}) \) be a winnable \( n \)-player self-test which is such that

1. \( G \) is positively aligned, and
2. \( p_{00\ldots0} > 0 \) and \( \eta_{00\ldots0} = 1 \).

Then, there exists a constant \( \delta_G > 0 \) such that the following holds. Let \( (\Phi, \{M'_j\}) \) be a quantum strategy whose measurements are in canonical form with underlying space \( (C^2 \otimes W_1) \otimes \ldots \otimes (C^2 \otimes W_n) \). Then the scoring operator \( M \) can be decomposed as

\[
M = \delta_G M' + (1 - \delta_G) M'', \tag{F.28}
\]

where \( \|M''\| \leq 1 \), and

\[
M' = (g_1 g_1^* - g_{-1} g_{-1}^*) \otimes I_{W_1 \otimes \ldots \otimes W_n}. \tag{F.29}
\]

Proof. Let

\[
T^+ = \left\{ (\theta_0, \ldots, \theta_n) \in \mathbb{R}^{n+1} \mid \theta_i > 0 \ \forall i \geq 1 \right\}, \tag{F.30}
\]

and

\[
T^- = \left\{ (\theta_0, \ldots, \theta_n) \in \mathbb{R}^{n+1} \mid \theta_i < 0 \ \forall i \geq 1 \right\}. \tag{F.31}
\]

Let \( q'_G \) be the maximum value of \( Z_G \) that occurs on the set \( [-\pi, \pi]^{n+1} \setminus (T^+ \cup T^-) \). By the criteria from Proposition [F.2], this set does not include any of the global maxima for the function \( Z_G \), and so \( q'_G \) is strictly smaller than the overall maximum \( q_G = 1 \). Let

\[
\delta = \min \left\{ p_{00\ldots0}, q_G - q'_G \right\}, \tag{F.32}
\]

where \( p_{00\ldots0} \) denotes the probability which \( G \) associates to the input string \( 00\ldots0 \).

First let us address the case where \( \dim W_j = 1 \) for all \( j \). Then

\[
M_j^{(0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{F.33}
\]

\[
M_j^{(1)} = \begin{bmatrix} 0 & \tilde{\zeta}_j \\ \tilde{\zeta}_j & 0 \end{bmatrix}. \tag{F.34}
\]

We can compute the scoring operator \( M \) using formula (F.6). When we write this operator as a matrix, using the computational basis for \( (C^2)^\otimes n \) in lexicographical order, we obtain a reverse diagonal matrix,

\[
M = \begin{bmatrix}
\vdots \\
& \leftarrow & & a_{00\ldots0} \\
& \searrow & \cdot & \cdot \\
& & \searrow & a_{11\ldots0} \\
& & & \leftarrow \\
a_{11\ldots1}
\end{bmatrix} \tag{F.35}
\]

where

\[
a_{b_1,\ldots,b_n} = P_G(\tilde{\xi}_1^{(-1)^{b_1}}, \tilde{\xi}_2^{(-1)^{b_2}}, \ldots, \tilde{\xi}_n^{(-1)^{b_n}}). \tag{F.36}
\]

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By canonical form, we have \( \zeta_j = e^{i\theta_j} \) for some \( \theta_j \in [0, \pi] \). Note that we can write
\[
|a_{b_1,...,b_n}| = \max_{\theta_0 \in \mathbb{R}} Z_G(\theta_0, (-1)^{b_1}\theta_1, (-1)^{b_2}\theta_2, \ldots, (-1)^{b_n}\theta_n).
\] (F.37)

By the definition of \( q'_G \), all of the values \( |a_b| \) are bounded by \( q'_G \) except possibly \( |a_{00...0}| \) and \( |a_{11...1}| \), which are both bounded by \( q_G = 1 \).

We claim that the matrix
\[
N = \begin{pmatrix}
& a_{00...0} - \delta_G \\
& \ddots \\
a_{11...1} - \delta_G & a_{11...0}
\end{pmatrix}
\] (F.38)

which arises from subtracting \( \delta_G \) from the two corner entries of \( M \), has operator norm less than or equal to \( 1 - \delta_G \). Indeed, the operator norm of this Hermitian matrix is the maximum of the absolute values of its entries, and we already know that all of its entries other than its corner entries are bounded by \( q'_G \leq 1 - \delta_G \). To show that the absolute values of the corner entries are bounded by \( 1 - \delta_G \), it suffices to write them out in terms of the parameters of the game \( G \): we have
\[
|a_{00...0} - \delta_G| = |P_G(\xi_1, \ldots, \xi_n) - \delta_G| \leq |p_0 - \delta_G| + \sum_{i \neq 0} |\eta_i p_i \xi_{i1} \xi_{i2} \ldots \xi_{in}| \leq 1 - \delta_G,
\] (F.40)

and likewise for \( |a_{11...1} - \delta_G| \). We conclude that \( N \) has operator norm less than or equal to \( 1 - \delta_G \).

Let \( M' = N/(1 - \delta_G) \) and \( M'' = (M - N')/\delta_G \), and the desired conditions hold.

The proof for the case in which \( W_1, \ldots, W_n \) are of arbitrary dimension follows by similar reasoning.

\[ \square \]

**Theorem F.8.** Let \( G = (\{p_i\}, \{\eta_i\}) \) be a strong self-test which is positively aligned. Then, there exist \( \delta_G > 0 \) and \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \) such that the following holds. Let \( (\Phi, \{M_{ij}^{(i)}\}) \) be a quantum strategy whose measurements are in canonical form with underlying space \( (\mathbb{C}^2 \otimes W_1) \otimes \ldots \otimes (\mathbb{C}^2 \otimes W_n) \). Then the scoring operator \( M \) can be decomposed as
\[
M = \delta_G M' + (q_G - \delta_G) M'',
\] (F.44)

where \( \|M''\| \leq 1 \), and
\[
M' = (g^*_a g^*_a - g_a g^*_a) \otimes I_{W_1 \otimes \ldots \otimes W_n}.
\] (F.45)
Proof. We repeat elements of the proof of Theorem F.7. It suffices to prove our desired decomposition for the case in which \( \dim W_i = 1 \) for all \( i \). Let \( q_G' \) be the maximum value of \( Z_G \) that occurs on the set \( [-\pi, \pi]^{n+1} \setminus (T^+ \cup T^-) \) (where \( T^+ \) and \( T^- \) are defined by (F.30) and (F.31)). Let \( \gamma \neq 0 \) be the constant that is given by Corollary F.6 and let

\[
\delta_G = \min \{ |\gamma|, q_G - q_G' \}. \tag{F.46}
\]

We have

\[
M = \begin{bmatrix}
a_{00...0} & a_{00...1} & \cdots & a_{11...0} \\
a_{00...1} & & & \\
\vdots & & & \\
a_{11...0} & & &
\end{bmatrix} \tag{F.47}
\]

where

\[
a_{b_1,...,b_n} = P_G(\zeta_1^{(-1)^{b_1}}, \zeta_2^{(-1)^{b_2}}, \ldots, \zeta_n^{(-1)^{b_n}}). \tag{F.48}
\]

for some \( \zeta_1, \ldots, \zeta_n \in \mathbb{C} \) such that \( |\zeta_i| = 1 \) and \( \text{Im}(\zeta_i) \geq 0 \). By Corollary F.6

\[
|P_G(\zeta_1, \ldots, \zeta_n) - \gamma| + |\gamma| \leq q_G,
\]

and it is easy to see (by the triangle inequality) that for any \( c \in [0, 1] \),

\[
|P_G(\zeta_1, \ldots, \zeta_n) - c\gamma| + |c\gamma| \leq q_G. \tag{F.50}
\]

Let

\[
N = \begin{bmatrix}
a_{00...0} - \frac{\delta_G}{|\gamma|} \cdot \gamma & a_{00...1} & \cdots & a_{11...0} \\
a_{00...1} & & & \\
\vdots & & & \\
a_{11...0} & & &
\end{bmatrix} \tag{F.51}
\]

The absolute values of the corner entries of this matrix are less than or equal to \( q_G - \delta_G \), and the other entries have absolute values less than or equal to \( q_G' \leq q_G - \delta_G \). Thus when we set

\[
\alpha = \gamma / |\gamma|, \tag{F.52}
\]

\[
M' = (g^*_a - g^{-a} g^*_a) \otimes I_{W_1 \otimes \cdots \otimes W_n}, \tag{F.53}
\]

\[
M'' = (M - \delta_G M') / (q_G - \delta_G), \tag{F.54}
\]

the desired result follows. \( \square \)

The operator \( (g^*_a - g^{-a} g^*_a) \) from the statement of Theorem F.8 does not describe a projective measurement. It is convenient to have a decomposition theorem involving a projective measurement. This motivates the next result.

We introduce some additional notation. Let

\[
b: \{0, 1, 2, \ldots, 2^n - 1\} \rightarrow \{0, 1\}^n \tag{F.55}
\]

be the function which maps \( k \) to its base-2 representation. For any \( \zeta \in \mathbb{C} \) with \( |\zeta| = 1 \), and any \( k \in \{0, 1, 2, \ldots, 2^n - 1\} \), let

\[
g_{\zeta, k} = \frac{1}{\sqrt{2}} \left( |b(k)\rangle \langle b(k)| + \zeta \left| \overline{b(k)} \right\rangle \langle b(k) | \right). \tag{F.56}
\]
Theorem F.9. Let $G = (\{p_i\}, \{\eta_i\})$ be a strong self-test which is positively aligned. Then, there exist $\delta_G > 0$ and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that the following holds. Let $(\Phi, \{M_j^{(i)}\})$ be a quantum strategy whose measurements are in canonical form with underlying space $(\mathbb{C}^2 \otimes W_1) \otimes \ldots \otimes (\mathbb{C}^2 \otimes W_n)$. Let $\alpha_0 = \alpha$ and let $\alpha_1, \ldots, \alpha_{2^n-1-1}$ be any unit-length complex numbers. Then the scoring operator $M$ can be decomposed as

\[ M = \delta_G M' + (q_G - \delta_G) M'', \quad (F.57) \]

where $\|M''\| \leq 1$, and

\[ M' = \left[ \sum_{k=0}^{2^n-1-1} (g_{\alpha_k,k} g_{\alpha_k,k}^* - g_{-\alpha_k,k} g_{-\alpha_k,k}^*) \right] \otimes I_{W_1 \otimes \cdots \otimes W_n}. \quad (F.58) \]

Proof. Again it suffices to prove this result for when $\dim W_i = 1$ for all $i$. Let $q_G'$ be the maximum value of $Z_G$ that occurs on the set $[-\pi, \pi]^{n+1} \setminus (T^+ \cup T^-)$, where $T^+$ and $T^-$ are defined by (F.30) and (F.31). Let $\gamma$ be the constant given by Corollary F.6, let $\alpha = \gamma / |\gamma|$, and let

\[ \delta_G = \min\{|\gamma|, (q_G - q_G') / 2\}. \quad (F.59) \]

Write $M$ as

\[ M = \begin{bmatrix} a_{00...0} \\ & \ddots \\ & & a_{00...1} \\ & & & \ddots \\ & & & & a_{11...0} \\ & & & & & \ddots \\ & & & & & & a_{11...1} \end{bmatrix} \quad (F.60) \]

Let

\[ N = M - \delta_G \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{2^n-1-1} \\ \alpha_{2^n-1-1} \\ \vdots \\ \alpha_0 \end{bmatrix}, \quad (F.61) \]

The corner entries of $N$ have absolute value $\leq q_G - \delta_G$ (by Corollary F.6) and the same holds for the other anti-diagonal entries by the triangle inequality: for any $n \in \{1, 2, \ldots, 2^{N-1} - 1\}$,

\[ |a_{b(n)} - \delta_G a_n| \leq |a_{b(n)}| + \delta_G \leq q_G + (q_G - q_G') / 2 \leq q_G - \delta_G. \quad (F.62) \]

Thus we let $M''/(q_G - \delta_G)$ and the desired statements hold.

G Randomness Expansion with Partially Trusted Measurements

The goals of this section are to define randomness expansion protocols based on partially trusted devices, and then to relate these new protocols to Protocol R.
G.1 Devices with Trusted Measurements

We begin by stating a simple protocol that involves a device with trusted measurements.

Definition 13. A device with trusted measurements consists of the following data.

1. A single quantum system $Q$ in an initial state $\Phi$.

2. For every pair $(i, o)$ of binary strings of equal length, two Hermitian operators $M_{i,o}^{(0)}$, $M_{i,o}^{(1)}$ representing the measurements performed on $Q$ when the input and output histories are $i$ and $o$. These operators are assumed to satisfy

$$
(M_{i,o}^{(0)})^2 = (M_{i,o}^{(1)})^2 = I \quad \text{(G.1)}
$$

and

$$
M_{i,o}^{(0)}M_{i,o}^{(1)} = -M_{i,o}^{(1)}M_{i,o}^{(0)}. \quad \text{(G.2)}
$$

A protocol for trusted measurement devices is given in Figure 3. Essentially this protocol is the same as Protocol R, except that we have skipped the process of generating random inputs for the game rounds, and have instead simply used the biased coin flip $g$ itself as input to the device.

**Protocol A:**

**Arguments:**

- $N =$ positive integer
- $q \in (0, 1)$
- $\eta \in (0, 1/2)$
- $D =$ device with trusted measurements

1. A bit $g \in \{0, 1\}$ is chosen according to a biased $(1 - q, q)$ distribution. The bit $g$ is given to $D$ as input, and an output bit $o$ is recorded.

2. If $g = 1$ and the output given by $D$ is 0, then the event $P$ ("pass") is recorded. If $g = 1$ and the output is 1, the event $F$ ("fail") is recorded.

3. If $g = 0$ and the output given by $D$ is 0, then the event $H$ ("heads") is recorded. If $g = 0$ and the output is 0, the event $T$ ("tails") is recorded.

4. Steps 1 – 3 are repeated $N - 1$ (more) times. Bit sequences $g = (g_1, \ldots, g_N)$ and $o = (o_1, \ldots, o_N)$ are obtained.

5. If the total number of failures is more than $\eta q N$, the protocol aborts. Otherwise, the protocol succeeds. If the protocol succeeds, it outputs the bit sequences $g$ and $o$.

Figure 3: A randomness expansion protocol for a trusted measurement device.
G.2 Devices with Partially Trusted Measurements

Definition 14. Let $v \in (0, 1]$ and $h \in [0, 1]$ be real numbers such that $v + h \leq 1$. Then a partially trusted device with parameters $(v, h)$ consists of the following data.

1. A single quantum system $Q$ in an initial state $\Phi$.

2. For every pair $(i, o)$ of binary strings of equal length, two Hermitian operators $M_{i, o}^{(0)}, M_{i, o}^{(1)}$ on $Q$ (representing measurements) satisfy the following conditions:
   - There exist perfectly anti-commuting measurement pairs $(T_{i, o}^{(0)}, T_{i, o}^{(1)})$ such that $M_{i, o}^{(0)} = T_{i, o}^{(0)}$ for all $i, o$, and
   - The operator $M_{i, o}^{(1)}$ decomposes as
     \[
     M_{i, o}^{(1)} = (v)T_{i, o}^{(1)} + (1 - v - h)N_{i, o}
     \]
     with $\|N_{i, o}\| \leq 1$.

The operators $M_{i, o}^{(0)}, M_{i, o}^{(1)}$ determine the measurements performed by the device on inputs 0 and 1, respectively. Intuitively, a partially trusted device is a device $D$ which always performs a trusted measurement $T^{(0)}$ on input 0, and on input 1, selects one of the three operators $(T^{(1)}, N, 0)$ at random according to the probability distribution $(v, 1 - v - h, h)$.

We will call the parameter $v$ the trust coefficient, and we will call $h$ the coin flip coefficient. The parameter $h$ measures the extent to which the output of $D$ on input 1 is determined by a fair coin flip. Note that when the input to the device $D$ is 1, then the probability that $D$ gives an output of 1 is necessarily between $h/2$ and $(1 - h/2)$.

Figure 4 gives a randomness expansion protocol for partially trusted devices. It is the same as Protocol A, except that the trusted device has been replaced by a partially trusted device.

G.3 Entanglement with a Partially Trusted Measurement Device

Suppose that $D$ is a partially trusted measurement device (see Definition 14) with parameters $(v, h)$. Suppose that $E$ is a quantum system that is entangled with $D$, and let $\rho = \rho_{E}$ denote the initial state of $E$. We will use the following notation: let $\rho_{+}$ and $\rho_{-}$ denote the subnormalized operators which represent the states of $E$ when the input bit is 0 and the output bit is 0 or 1, respectively. Let $\rho_{P}$ and $\rho_{F}$ denote the operators which represent an input of 1 and an output of 0 or 1, respectively. Also (using notation from Definition 14), let us write $\rho_{0}$ and $\rho_{1}$ denote the states of $E$ that would occur if the trusted measurement $T^{(1)}$ was applied to $Q$ (instead of the partially trusted measurement $M^{(1)}$). (Note that $T^{(1)}$ is perfectly anticommuting with $M^{(0)}$)

The following proposition expresses the possible behavior of the system $E$.

Proposition G.1. Let $v \in (0, 1]$ and $h \in [0, 1]$ be such that $v + h \leq 1$. Let $D$ be a partially trusted device with parameters $(v, h)$, let $E$ be a quantum system that is entangled with $D$, and let $\rho = \rho_{E}$. Then,\n
\[
(h/2)\rho + v\rho_{0} \leq \rho_{P} \leq (1 - h/2)\rho - v\rho_{1}
\]

and

\[
(h/2)\rho + v\rho_{1} \leq \rho_{F} \leq (1 - h/2)\rho - v\rho_{0}.
\]
Protocol A’:

Arguments:

\[ v, h = \text{nonnegative real numbers such that } v > 0 \text{ and } v + h \leq 1. \]
\[ N = \text{positive integer} \]
\[ q \in (0, 1) \]
\[ \eta \in (0, v/2) \]
\[ D = \text{partially trusted device with parameters } (v, h). \]

1. A bit \( g \in \{0, 1\} \) is chosen according to a biased \((1 - q, q)\) distribution. The bit \( g \) is given to \( D \) as input, and the output bit \( o \) is recorded.

2. If \( g = 1 \) and the output given by \( D \) is 0, then the event \( P \) (“pass”) is recorded. If \( g = 1 \) and the output is 1, the event \( F \) (“fail”) is recorded.

3. If \( g = 0 \) and the output given by \( D \) is 0, then the event \( H \) (“heads”) is recorded. If \( g = 0 \) and the output is 0, the event \( T \) (“tails”) is recorded.

4. Steps 1 – 3 are repeated \( N - 1 \) (more) times. Bit sequences \( g = (g_1, \ldots, g_N) \) and \( o = (o_1, \ldots, o_N) \) are obtained.

5. If the total number of failures is greater than \((h/2 + \eta)qN\), then the protocol aborts. Otherwise, the protocol succeeds. If the protocol succeeds, it outputs the bit sequences \( g \) and \( o \).

Figure 4: A randomness expansion protocol for a partially trusted device.

Proof. Let \( N \) be the measurement operator from the decomposition of \( M^{(1)} \) given in Definition 14. Let \( \rho' \) be the subnormalized operator on \( E \) which denotes the state that would be produced if \( N \) were applied to \( Q \) and the outcome were 0. Clearly, \( 0 \leq \rho' \leq \rho \). From the decomposition (G.3), \( \rho_P \) is a convex combination of the operators \( \rho_0, \rho' \) and \( (\rho/2) \):

\[
\rho_P = v\rho_0 + (1 - v - h)\rho' + h(\rho/2).
\] (G.6)

Since \( \rho' \leq \rho \), we have

\[
\rho_P \leq v\rho_0 + (1 - v - h)\rho + h(\rho/2)
\] (G.7)
\[
= v\rho_0 + (1 - v - h/2)\rho
\] (G.8)
\[
= (1 - h/2)\rho + v(\rho_0 - \rho)
\] (G.9)
\[
= (1 - h/2)\rho - v\rho_1.
\] (G.10)

The other inequalities follow similarly. \( \square \)

G.4 Simulation

Theorem G.2. For any \( n \)-player strong self-test \( G \) which is positively aligned, there exists \( \delta_G > 0 \) such that the following holds. For any any \( n \)-part binary quantum device \( D \), there exists a partially trusted device \( D' \) with parameters \( q_G, \delta_G \) such that Protocol \( A' \) (with arguments \( \delta_G, 2f_G, N, q, \eta, D' \)) simulates Protocol \( R \) (with arguments \( N, \eta, q, G, D \)).
Proof. Choose $\delta_G$ according to Theorem F.9.

Consider the behavior of the device $D$ in the first round. We may assume that the measurements performed by $D_1, \ldots, D_n$ are in canonical form. Write the underlying space as $(C^2 \otimes W_1) \otimes \cdots \otimes (C^2 \otimes W_n)$. If $g = 0$, the measurement performed by $D_1$ is given by the operator

$$M' = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 \\
\end{bmatrix} \otimes I_{W_1 \otimes \cdots \otimes W_n}$$

(where the matrix on the left is an operator on $(C^2)^n$, with the basis taken in lexicographic order as usual).

If $g = 1$ the measurement performed by $D$ is given by the scoring operator $M$. Theorem F.9 guarantees that for some unit-length complex number $\alpha$, and for any choices of unit-length complex numbers $\alpha_1, \ldots, \alpha_{2^n - 1}$, there is a decomposition for $M$ in the form $M = \delta_G M' + (q_G - \delta_G) M''$ with

$$M'' = \begin{bmatrix}
\alpha & \alpha_1 & \cdots & \alpha \\
\alpha_1 & \alpha_2 & \cdots & \alpha_2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha & \alpha_2 & \cdots & \alpha \\
\end{bmatrix} \otimes I_{W_1 \otimes \cdots \otimes W_n}$$

and $\|M''\| \leq 1$. To simulate the behavior of $D$ with a partially trusted device, we need only choose $\alpha_1, \ldots, \alpha_{2^n - 1}$ so that $M'$ is perfectly anti-commutative with the operator G.11. This can be done, for example, by setting $\alpha_1, \alpha_2, \ldots, \alpha_{2^n - 2}$ to be equal to $\alpha$, and $\alpha_{2^n - 1}$ to be equal to $-\alpha$. Thus the behavior of the device $D$ in the first round of Protocol R can be simulated by a partially trusted device with parameters $(\delta_G, 1 - q_G) = (\delta_G, 2f_G)$. Similar reasoning shows the desired simulation result across all rounds.

The following corollary is easy to prove.

**Corollary G.3.** Theorem G.2 holds true without the assumption that $G$ is positively aligned.

Essentially, the above corollary implies that any security result for Protocol A’ can be converted immediately into an identical security result for Protocol R. This will be the basis for our eventual full proof of randomness expansion.

## H The Proof of Security for Partially Trusted Devices

In this section we provide the proof of security for Protocol A’. Our approach, broadly stated, is as follows: we show the existence of a function $T(v, h, \eta, q, \kappa)$ which provides a lower bound on the
linear rate of entropy of the protocol. The main point of our proofs is that, although $T$ depends on several variables, it does not depend on the particular device used in Protocol $A'$. Thus, we have a uniform security result.

In principle, our proofs could be used to compute an explicit formula for the function $T$, but we have not attempted to do this because the formula might be very complicated. We will instead obtain a simple compact formula for the limit function $\lim_{(q,\kappa) \to (0,0)} T(v, h, \eta, q, \kappa)$. This formula is sufficient for calculating the asymptotic rate of Protocol $A'$ in any application where we can take $q$ to be close to 0.

Our proof involves several parameters. For convenience, we include a table here which assigns a name to each parameter (Figure 5).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>number of rounds</td>
</tr>
<tr>
<td>$q$</td>
<td>test probability</td>
</tr>
<tr>
<td>$t$</td>
<td>failure parameter</td>
</tr>
<tr>
<td>$v$</td>
<td>trust coefficient</td>
</tr>
<tr>
<td>$h$</td>
<td>coin flip coefficient</td>
</tr>
<tr>
<td>$\eta$</td>
<td>error tolerance</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>failure penalty</td>
</tr>
<tr>
<td>$r$</td>
<td>multiplier for Rényi coefficient</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>error parameter for smooth min-entropy</td>
</tr>
</tbody>
</table>

Figure 5: Variables used in Appendix H.

To avoid unnecessary repetition, we will use the following conventions in this section.

• Unless otherwise stated, we will assume that the variables from Figure 5 are always restricted to the domains given. (The reader can assume that all unquantified statements are prefaced by, “for all $q \in (0,1)$, all $\epsilon \in (0,\sqrt{2})$,” etc.) If we say “$F(q,\kappa)$ is a real-valued function,” we mean that it is a real valued function on $(0,1) \times (0,\infty)$. If we say “let $x = \kappa q$,” we mean that $x$ is a real valued function on $(0,1) \times (0,\infty)$ defined by $x(\kappa,q) = \kappa q$. If the domain of one parameter of a function depends on another variable (as can occur, e.g., for the variable $h$) we always include the other variable as a parameter of the function.

• When we discuss a single iteration of Protocol $A'$, will use notation from subsection G.3: If $D$ is a partially trusted measurement device, and $E$ is a purifying system for $D$ with initial state $\rho = \rho_E$, then $\rho = \rho_H + \rho_T$ and $\rho = \rho_P + \rho_F$ denote the decompositions that occur for a single use of the device on input 0 and 1, respectively. We denote by $\rho_+, \rho_-, \rho_0, \rho_1$ the respective states that would occur if the corresponding fully trusted measurements were used instead. (Note that $\rho_H = \rho_+$ and $\rho_T = \rho_-$) Let $\overline{\rho}$ denote the operator on $E \oplus E \oplus E \oplus E$ given by

$$\overline{\rho} = (1-q)\rho_H \oplus (1-q)\rho_T \oplus q\rho_P \oplus q\rho_F.$$  \hfill (H.1)

This operator represents the state of $E$ taken together with the input bit and output bit from the first iteration of Protocol $A'$.

• When we discuss multiple iterations of Protocol $A'$, we will use the following notation: let $G$ and $O$ denote classical registers which consist of the bit sequences $g = (g_1, \ldots, g_N)$ and $o = (o_1, \ldots, o_n)$, respectively. We denote basis states for the joint system $GO$ by $|g_o\rangle$. We denote the joint state of the system $EGO$ at the conclusion of Protocol $A'$ by $\Gamma_{EGO}$. 

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• If $D$ is a partially trusted measurement device, $E$ is a purifying system, and $\alpha > 0$, then we refer to the quantity

$$\frac{\text{Tr}(\rho^q)}{\text{Tr}(\rho^\alpha)} \in [0, 1]$$

as the $\alpha$-failure parameter of $D$. (Note that we used the operator $\rho_1$ in the above expression, not the operator $\rho_F$. This parameter measures “honest” failures only.)

• Let $\Pi(x, y)$ and $\pi(y)$ denote the functions from Theorem E.2.

\section*{H.1 One-Shot Result: A Device with Known Failure Parameters}

We begin by proving a one-shot security result under the assumption that some limited information about the device is available.

Let $D$ be a partially trusted measurement device with parameters $v, h$, and let $E$ be a purifying system with initial state $\rho$. Let $\rho$ be the operator on $E \oplus E \oplus E \oplus E \oplus E$ which represents the joint state of $E$ together with the input and output of a single iteration of Protocol $A'$:

$$\rho = (1 - q)\rho_H \oplus (1 - q)\rho_T \oplus q\rho_P \oplus q\rho_F.$$  \hfill (H.3)

We wish to show that the state $\rho$ is more random than the original state $\rho$. Therefore, we wish to show that the ratio

$$\frac{d_{1+\gamma}(\rho \| \sigma)}{d_{1+\gamma}(\rho \| \sigma')}$$

for some appropriate $\gamma, \sigma, \sigma'$, is significantly smaller than 1. For simplicity, we will for the time being take $\sigma' = \mathbb{I}$ for the initial bounding operator. (Later in this section we will generalize this choice.)

A natural choice of bounding operator for $\rho$ would be

$$ (1 - q)\mathbb{I} \oplus (1 - q)\mathbb{I} \oplus q\mathbb{I} \oplus q\mathbb{I}. $$

Computing $d_{1+\gamma}(\rho \| \cdot)$ with this bounding operator would yield

$$ \left\{ (1 - q)\text{Tr}[\rho^1] + (1 - q)\text{Tr}[\rho^1] + q\text{Tr}[\rho^1] + q\text{Tr}[\rho^1] \right\}^{1/\gamma} $$

Computing this quantity would have the effect, roughly speaking, of measuring the randomness of the output bit of Protocol $A'$ conditioned on $E$ and on the input bit $g$. However this is not adequate for our purposes, since it treats “passing” rounds the same as “failing” rounds, and does not take into account that the device is only allowed a limited number of failures. (And indeed, this measurement of randomness does not work: if $D$ performs anticommuting measurements on a half of a maximally entangled qubit pair, the divergence quantity $d_{1+\gamma}(\rho \| \cdot)$ with bounding operator (H.5) is the same as $d_{1+\gamma}(\rho \| \mathbb{I})$.)

We will use a slightly different expression to measure the output of Protocol $A'$. We introduce a single coefficient $2^{-\kappa}$ (with $\kappa > 0$) into the fourth term of the expression:

$$ \left\{ (1 - q)\text{Tr}[\rho^1] + (1 - q)\text{Tr}[\rho^1] + q\text{Tr}[\rho^1] + q2^{-\kappa}\text{Tr}[\rho^1] \right\}^{1/\gamma} $$

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The reason for the introduction of the coefficient $2^{-x}$ is this: in effect, if a game round occurs and the device fails, we lower our expectation for the amount of randomness produced. The quantity $d_{1+\gamma}(\bar{\rho}||\bar{\sigma})$ is equal to $d_{1+\gamma}(\bar{\rho}||\bar{\sigma})$ where

$$\bar{\sigma} = (1 - q)\mathbb{I} \oplus (1 - q)\mathbb{I} \oplus q\mathbb{I} \oplus q2^{x}/\gamma\mathbb{I}. \quad (H.8)$$

Having chosen the bounding operator $\bar{\sigma}$, we need only to choose the coefficient $\gamma \in (0,1]$. For heuristic reasons, we will take $\gamma$ to be of the form $\gamma = rq\lambda$, where $r \in (0,1/(q\lambda))^{[3]}

**Proposition H.1.** There is a continuous real-valued function $\Lambda(v,h,q,\kappa,r,t)$ such that the following conditions hold.

1. Let $D$ be a partially trusted measurement device with parameters $(v,h)$, and let $E$ be a purifying system for $D$. Let $\gamma = rq\lambda$, and let

$$\bar{\sigma} = (1 - q)\mathbb{I} \oplus (1 - q)\mathbb{I} \oplus q\mathbb{I} \oplus q2^{x}/\gamma\mathbb{I}. \quad (H.9)$$

Then,

$$d_{1+\gamma}(\bar{\rho}||\bar{\sigma}) \leq 2^{-\Lambda(v,h,q,\kappa,r,t)} \cdot d_{1+\gamma}(\rho||\mathbb{I}), \quad (H.10)$$

where $t = \text{Tr}(\rho_{1+\gamma}^{1+\gamma})/\text{Tr}(\rho^{1+\gamma})$ denotes the $(1+\gamma)$-failure parameter of $D$.

2. The following limit condition is satisfied:

$$\lim_{(q,\kappa) \to (0,0)} \Lambda(v,h,q,\kappa,r,t) = \pi(t) + \frac{h/2 + vt}{r}, \quad (H.11)$$

where $\pi$ is the function from Theorem E.2

**Proof.** We have

$$d_{1+\gamma}(\bar{\rho}||\bar{\sigma}) = \left\{ (1 - q)\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] + (1 - q)\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] + q\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] + q2^{-x}\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] \right\}^{1/\gamma} \quad (H.12)$$

We will compute a bound on this quantity by grouping the first and second summands together, and then by grouping the third and fourth summands together. Note that by Theorem E.2, we have

$$\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] + \text{Tr}[\rho_{1+\gamma}^{1+\gamma}] \leq 2^{-\gamma\Pi(\gamma,t)}\text{Tr}[\rho^{1+\gamma}] \quad (H.13)$$

Now consider the sum $\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] + 2^{-x}\text{Tr}[\rho_{1+\gamma}^{1+\gamma}]$. By superadditivity (see Proposition C.1),

$$\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] + 2^{-x}\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] = \text{Tr}\left[2^{-x}(\rho_{1+\gamma}^{1+\gamma} + \rho_{1+\gamma}^{1+\gamma}) + (1 - 2^{-x})\rho_{1+\gamma}^{1+\gamma}\right] \quad (H.14)$$

$$\leq \text{Tr}\left[2^{-x}\rho_{1+\gamma}^{1+\gamma} + (1 - 2^{-x})\rho_{1+\gamma}^{1+\gamma}\right] \quad (H.15)$$

By Proposition C.1,

$$\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] + 2^{-x}\text{Tr}[\rho_{1+\gamma}^{1+\gamma}] \leq \text{Tr}\left\{2^{-x}\rho_{1+\gamma}^{1+\gamma} + (1 - 2^{-x})[\rho - (h/2)\rho - v\rho_{1+\gamma}]^{1+\gamma}\right\}. \quad (H.16)$$

---

*[3]The reason for this choice of interval is that we need $\gamma \leq 1$ for the application of results from section D.*
Applying the rule \((X - Y)^{1+\gamma} \leq X^{1+\gamma} - Y^{1+\gamma}\), followed by the fact that \(\text{Tr}[\rho_1^{1+\gamma}] = t\text{Tr}[\rho^{1+\gamma}]\), we have the following:

\[
\text{Tr}[\rho_p^{1+\gamma}] + 2^{-x}\text{Tr}[\rho_F^{1+\gamma}] \leq \text{Tr}\left\{2^{-x}\rho^{1+\gamma} + (1 - 2^{-x})[\rho^{1+\gamma} - (h/2)^{1+\gamma}\rho^{1+\gamma} - \psi^{1+\gamma}\rho_1^{1+\gamma}]\right\}
\]

\[
\text{Tr}\left\{2^{-x}\rho^{1+\gamma} + (1 - 2^{-x})[\rho^{1+\gamma} - (h/2)^{1+\gamma}\rho^{1+\gamma} - \psi^{1+\gamma}t\rho^{1+\gamma}]\right\}
\]

\[
\text{Tr}[\rho^{1+\gamma}] = \left\{2^{-x} + (1 - 2^{-x})[1 - (h/2)^{1+\gamma} + \psi^{1+\gamma}t]\right\} \text{Tr}[\rho^{1+\gamma}] \quad (H.17)
\]

\[
\text{Tr}[\rho^{1+\gamma}] = \left\{1 - (1 - 2^{-x})[(h/2)^{1+\gamma} + \psi^{1+\gamma}t]\right\} \text{Tr}[\rho^{1+\gamma}] \quad (H.18)
\]

Combining (H.12), (H.13), and (H.18), we find the following: if we set

\[
\lambda(v, h, q, \kappa, r, t) = \left((1 - q)2^{-\gamma}(h/2) - 1 + q(2^{-\gamma} - 1)[(h/2)^{1+\gamma} + \psi^{1+\gamma}t]\right)^{1/\gamma}, \quad (H.19)
\]

then

\[
d_{1+\gamma}(\rho||\overline{\sigma}) \leq \lambda(v, h, q, \kappa, r, t) \cdot d_{1+\gamma}(\rho||I). \quad (H.20)
\]

Therefore setting \(\Lambda = -\log \lambda\) yields (H.10).

It remains for us to evaluate the limiting behavior of \(\Lambda\) as \((q, \kappa) \to (0,0)\). We can rewrite the formula for \(\lambda\) as

\[
\lambda(v, h, q, \kappa, r, t) = \left(1 + \left\{(1 - q)(2^{-\gamma}(h/2) - 1) + q(2^{-\gamma} - 1)[(h/2)^{1+\gamma} + \psi^{1+\gamma}t]\right\}\right)^{1/\gamma}
\]

Applying Proposition C.8 to this expression (with \(g = \gamma\), and \(f\) equal to function enclosed by braces), we have

\[
\ln \left[\lim_{(q, \kappa) \to (0,0)} (1 - q) \left(2^{-\gamma}(h/2) - 1\right) + q(2^{-\gamma} - 1)[(h/2)^{1+\gamma} + \psi^{1+\gamma}t]\right] = (1)(-\ln 2)\pi(t) + (-\ln 2)(r^{-1})[(h/2) + vt],
\]

which implies (H.11) as desired.

\[
\text{H.2 One-Shot Result: Full Strength}
\]

Proposition H.1 is not sufficient for our ultimate proof of security because it assumes that additional information (beyond the trust parameters \(v, h\)) is known about the device \(D\). The next proposition avoids this limitation. (It makes no use of the failure parameters of the device.)

**Proposition H.2.** There is a continuous real-valued function \(\Delta(v, h, q, \kappa, r)\) such that the following conditions hold.

1. Let \(D\) be a partially trusted measurement device with parameters \((v, h)\), and let \(E\) be a purifying system for \(D\). Let \(\gamma = rq\kappa\), and let 

\[
\overline{\sigma} = (1 - q)I \oplus (1 - q)I \oplus qI \oplus q2^\kappa I.
\]

Then, 

\[
d_{1+\gamma}(\overline{\rho}||\overline{\sigma}) \leq 2^{-\Delta(v,h,q,r) \cdot \rho(I)}. \quad (H.25)
\]
2. The following limit condition is satisfied:

\[
\lim_{(q,\kappa) \to (0,0)} \Delta(v, h, q, \kappa, r) = \min_{s \in [0,1]} \left( \pi(s) + \frac{h/2 + vs}{r} \right),
\]

where \(\pi\) is the function from Theorem E.2

**Proof.** Let \(\Lambda\) be the function from Proposition H.1 and let

\[
\Delta(v, h, q, \kappa, r) = \min_{t \in [0,1]} \Lambda(v, h, q, \kappa, r, t).
\]


**H.3 Multi-Shot Results**

The goal of this subsection is to deduce consequences of Proposition H.2 across multiple iterations. Let \(\Gamma_{EGO}\) denote the joint state of the registers \(E, G,\) and \(O\). (Note that \(\Gamma\) is a classical-quantum state with respect to the partition \((GO|E)\).

The following proposition follows immediately from Proposition H.2 by induction.

**Proposition H.3.** Let \(D\) be a partially trusted measurement device with parameters \((v, w)\), and let \(E\) be a purifying system for \(D\). Let \(\gamma = rq\kappa\), and let \(\Phi\) be the operator on \(E \otimes G \otimes O\) given by

\[
\Phi = \mathbb{I}_E \otimes \left( \sum_{g, o \in \{0,1\}^N} (1 - q)^{\sum_{i=1}^N (1-g_i)} q^{\sum_{i=1}^N g_i} 2^{\sum_{i=1}^N g_i o_i}/(qr) |go \rangle \langle go| \right).
\]

Then,

\[
D^{1+\gamma}(\Gamma_{EGO}\|\Phi) \leq D^{1+\gamma}(\Gamma_E\|\mathbb{I}) - N \cdot \Delta(v, h, q, \kappa, r),
\]

where \(\Delta\) denotes the function from Proposition H.2.

We note the significance of the exponents in (H.28): the quantity \(\sum_{i=1}^N (1 - g_i)\) is the number of generation rounds that occurred in Protocol A’, the quantity \(\sum_{i=1}^N g_i\) is the number of game rounds, and the quantity \(\sum_{i=1}^N g_i o_i\) is the number of times the “failure” event occurred during the protocol.

As stated, Proposition H.3 is not useful for bounding the randomness of \(\Gamma_{EGO}\) because the quantity \(D^{1+\gamma}(\Gamma_E\|\mathbb{I})\) could be arbitrarily large. We therefore prove the following alternate version of the proposition. The statement is the same, except that we replace \(\mathbb{I}_E\) in (H.28) with \(\Gamma_E\), and we remove the term \(D^{1+\gamma}(\Gamma_E\|\mathbb{I})\) from (H.29).

**Proposition H.4.** Let \(D\) be a partially trusted measurement device with parameters \((v, w)\), and let \(E\) be a purifying system for \(D\). Let \(\gamma = rq\kappa\), and let \(\Sigma\) be the operator on \(E \otimes G \otimes O\) given by

\[
\Sigma = \Gamma_E \otimes \left( \sum_{g, o \in \{0,1\}^N} (1 - q)^{\sum_{i=1}^N (1-g_i)} q^{\sum_{i=1}^N g_i} 2^{\sum_{i=1}^N g_i o_i}/(qr) |go \rangle \langle go| \right).
\]

Then,

\[
D^{1+\gamma}(\Gamma_{EGO}\|\Sigma) \leq -N \cdot \Delta(v, h, q, \kappa, r),
\]

where \(\Delta\) denotes the function from Proposition H.2.
Proof. Let $\Gamma = \Gamma_E$. Let $(D, E')$ be the device-environment pair that arises from taking the pair $(D, E)$ and applying the stochastic operation

$$X \mapsto \Gamma^{\frac{1}{1+\gamma}} X \Gamma^{\frac{1}{1+\gamma}}$$

(H.32)

to the system $E$. The state $\Gamma_{E'}$ of the resulting system $E'$ satisfies

$$\Gamma_{E'} = \frac{\Gamma^{1/(1+\gamma)}}{K},$$

(H.33)

where $K = \text{Tr}(\Gamma^{1/(1+\gamma)})$.

By directly applying the definition of $D_\alpha$ (see Definition H.6) we can see that certain divergences of $\Gamma_{EGO}$ and $\Gamma_{E'GO}$ can be computed from one another:

$$D_{1+\gamma}(\Gamma_{E'GO}\|\Phi) = -\frac{1+\gamma}{\gamma} \log K + D_{1+\gamma}(\Gamma_{EGO}\|\Sigma)$$

(H.34)

$$D_{1+\gamma}(\Gamma_{E'}\|I) = -\frac{1+\gamma}{\gamma} \log K + D_{1+\gamma}(\Gamma_E\|\Gamma).$$

(H.35)

Applying Proposition H.3 to $(D, E')$, we find that

$$D_{1+\gamma}(\Gamma_{E'GO}\|\Phi) - D_{1+\gamma}(\Gamma_{E'}\|I) \leq -N\Delta(v, h, q, \kappa, r).$$

(H.36)

By (H.34)–(H.35), the same bound holds when $E', \Phi, I$ are replaced $E, \Sigma, \Gamma$. Since $D_{1+\gamma}(\Gamma_E\|\Gamma) = 0$, the desired inequality is obtained.

The following corollary of Proposition H.4 provides final preparation for the proof of the main result.

**Corollary H.5.** Let $\epsilon > 0$. Then, there exists a positive semidefinite operator $\Gamma_{EGO}$ which is classical with respect to the systems $E$ and $G$ such that

$$\|\Gamma_{EGO} - \Gamma_{EGO}\|_1 \leq \epsilon$$

(H.37)

and

$$D_{\max}(\Gamma_{EGO}\|\Sigma) \leq -N \cdot \Delta(v, h, q, \kappa, r) + \frac{\log(2/\epsilon^2)}{qkr}$$

(H.38)

(\text{where } \Delta \text{ and } \Sigma \text{ are as in Proposition H.2 and Proposition H.4 respectively}).

**Proof.** This follows from Corollary D.3.

**H.4 The Security of Protocol A’**

Let $s$ denote the event that Protocol A’ succeeds, and let $\Gamma_{EGO}^s$ denote the corresponding (subnormalized) operator on $E \otimes G \otimes O$.

**Proposition H.6.** There exists a continuous real-valued function $R(v, h, q, \eta, q, \kappa, r)$ such that the following holds.

1. Let $\epsilon > 0$. If Protocol A’ is executed with parameters $(v, h, N, q, \eta, D)$, then

$$H_{\min}^c(\Gamma_{EGO}^s | EG) \geq N \cdot R(v, h, q, \eta, q, \kappa, r) - \frac{\log(2/\epsilon^2)}{qkr}.$$  

(H.39)
2. The following equality holds:

\[
\lim_{(q,\kappa) \to (0,0)} R(v, h, \eta, q, \kappa, r) = \min_{s \in [0,1]} \left[ \pi(s) + \frac{vs - \eta}{r} \right]
\]  

(H.40)

Proof. The “success” event for Protocol A’ is defined by the inequality

\[
\sum_i \xi_i \leq (h/2 + \eta)qN.
\]  

(H.41)

Let \( S \subseteq G \otimes O \) be the span of the vectors \( |go\rangle \) where \((g, o)\) varies over all pairs of sequences satisfying (H.41). For any operator \( X \) on \( E \otimes G \otimes O \) which is classical-quantum with respect to \((GO|E)\), let \( X^s \) denote the restriction of \( X \) to \( E \otimes S \). Applying this construction to the operators \( \Gamma_{\text{EGO}}, \Gamma_{\text{EGO}}' \) and \( \Sigma \) from Corollary H.5, and using the fact that \( D_{\text{max}} \) and \( \| \cdot \|_1 \) are monotonically decreasing under restriction to \( S \), we find that

\[
D_{\text{max}}^e(\Gamma_{\text{EGO}}^s \| \Sigma^s) \leq -N \cdot \Delta(v, h, q, \kappa, r) + \frac{\log(2/e^2)}{qkr}.
\]  

(H.42)

In order to give a lower bound on the smooth min-entropy of \( \Gamma_{\text{EGO}}' \), we need to compute its divergence with respect to an operator on \( E \otimes G \otimes O \) that is of the form \( X \otimes I_O \), where \( X \) is a density matrix. Define a new operator \( \Sigma' \) on \( E \otimes G \otimes O \) by

\[
\Sigma' = \Gamma_E \otimes \left( \sum_{(g, o) \in S} (1-q)^{\Sigma(1-g)}q^\Sigma g \cdot 2^{(h/2 + \eta)N/r} |go\rangle \langle go| \right)
\]  

(H.43)

(recalling that \( \gamma = qkr \)). Comparing this definition with (H.30) and using the success criterion (H.41), we find that \( \Sigma' \geq \Sigma^s \). Therefore, the bound in (H.42) holds also when \( \Sigma^s \) is replaced by \( \Sigma' \).

When we let \( \Psi \) be the operator on \( E \otimes G \otimes O \) defined by

\[
\Psi = \Gamma_E \otimes \sum_{g \in \{0,1\}^N} (1-q)^{\Sigma(1-g)}q^\Sigma g |g\rangle \langle g|
\]  

(H.44)

and rewrite \( \Sigma' \) as

\[
\Sigma' = 2^{(h/2 + \eta)N/r} (\Psi \otimes I_O),
\]  

(H.45)

we find (using the rule \( D_{\text{max}}^e(X \| Y) = \log c + D_{\text{max}}^e(X \| cY) \)) that

\[
D_{\text{max}}^e(\Gamma_{\text{EGO}}^s \| \Psi \otimes I_O) \leq (h/2 + \eta)N/r - N \cdot \Delta(v, h, q, \kappa, r) + \frac{\log(2/e^2)}{qkr}.
\]

Since \( \Psi \) is a density matrix, we have

\[
H_{\text{min}}^e(\Gamma_{\text{EGO}}^s | EG) \geq -D_{\text{max}}^e(\Gamma_{\text{EGO}}^s \| \Psi \otimes I_O).
\]  

(H.46)

Therefore if we let

\[
R(v, h, \eta, q, \kappa, r) = -\frac{h/2 + \eta}{r} + \Delta(v, h, q, \kappa, r),
\]  

(H.47)

condition 1 of the theorem is fulfilled. Condition 2 follows easily from the formula for the limit of \( \Delta \) (H.26). \( \square \)
A final improvement can be made on the previous result by optimizing the coefficient $r$.

**Theorem H.7.** There exist continuous real-valued functions $T(v, h, q, \eta, \kappa)$ and $E(v, h, q, \eta, \kappa)$ such that the following holds.

1. If Protocol $A'$ is executed with parameters $(v, h, N, q, \eta, D)$, then for any $\epsilon \in (0, \sqrt{2}]$ and $\kappa \in (0, \infty)$,

   $$H_{\min}^e(\Gamma^s_{EGO} \mid EG) \geq N \cdot T(v, h, \eta, q, \kappa) - \frac{\log(\sqrt{2}/\epsilon)}{q\kappa} E(v, h, \eta, q, \kappa). \quad (H.48)$$

2. The following equalities hold, where $\pi$ denotes the function from Theorem E.2.

   $$\lim_{(q, \kappa) \to (0, 0)} T(v, h, \eta, q, \kappa) = \pi(\eta/v), \quad (H.49)$$
   $$\lim_{(q, \kappa) \to (0, 0)} E(v, h, \eta, q, \kappa) = -\frac{2\pi'(\eta/v)}{v}. \quad (H.50)$$

**Proof.** Let

$$r_0 = \min \left\{ \frac{v}{-\pi'(\eta/v)}, \frac{1}{q\kappa} \right\}. \quad (H.51)$$

Define the function $T$ by

$$T(v, h, \eta, q, \kappa) = R(v, h, \eta, q, \kappa, r_0). \quad (H.52)$$

By substitution into Proposition H.6 the bound (H.48) will hold when we set $E$ to be equal to $2/(r_0)$.

To prove (H.49), note that

$$\lim_{(q, \kappa) \to (0, 0)} r_0 = \frac{v}{-\pi'(\eta/v)} \quad (H.53)$$

and therefore

$$\lim_{(q, \kappa) \to (0, 0)} T(v, h, \eta, q, \kappa) = \min_{s \in [0, 1]} \left[ \pi(s) + r_0^{-1}(vs - \eta) \right] \quad (H.54)$$

$$= \min_{s \in [0, 1]} \left[ \pi(s) - \pi'(\eta/v)(s - \frac{\eta}{v}) \right] \quad (H.55)$$

The function enclosed by square brackets in (H.55) is a convex function of $s$ (by Theorem E.2) and its derivative at $s = \eta/v$ is zero. Therefore, a minimum is achieved at $s = \eta/v$, and the expression in (H.55) thus evaluates simply to $\pi(\eta/v)$.

Equality (H.50) is immediate. This completes the proof. \qed

## I Randomness Expansion from an Untrusted Device

In this section, we will combine the results of previous sections to prove that randomness expansion from an untrusted device is possible.
I.1 The Trust Coefficient of a Strong Self-Test

Corollary G.3 proves that if $G$ is a strong self-test, then for some $\delta_G > 0$, the behavior of an untrusted device under $G$ can be simulated by a partially trusted device with parameters $(\delta_G, 2f_G)$. Let us say that the trust coefficient of $G$ is the largest value of $\delta_G$ which makes such a simulation possible.

As a consequence of the theory in section F, we have the following formal definition for the trust coefficient of $G$.

**Definition 15.** Suppose that $G$ is an $n$-player binary XOR game. Then the trust coefficient of $G$, denoted $v_G$, is the maximum value of $c \geq 0$ such that there exists a Hermitian operator $N$ on $(\mathbb{C}^2)^ \otimes n$ satisfying the following conditions.

1. The square of $N$ is the identity operator on $(\mathbb{C}^2)^ \otimes n$.

2. The operator $N$ anticommutes with the operator $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I \otimes \ldots \otimes I$.

3. For any complex numbers $\zeta_1, \ldots, \zeta_n \in \{\zeta \mid |\zeta| = 1, \text{Im}(\zeta) \geq 0\}$, the operator given by

$$M = \begin{bmatrix} \ldots & a_{00\ldots1} & \ldots \\ & \ddots & \vdots \\ & a_{11\ldots0} & \ldots \\ & \vdots & \ddots \\ & a_{11\ldots1} & \ldots \end{bmatrix},$$

where

$$a_{b_1, \ldots, b_n} = P_G(\zeta_1^{(-1)^{b_1}}, \zeta_2^{(-1)^{b_2}}, \ldots, \zeta_n^{(-1)^{b_n}}),$$

satisfies

$$\|M - cN\| \leq q_G - c.$$ (I.3)

I.2 The Security of Protocol R

Combining Theorem H.7, Corollary G.3, and the definition from the previous subsection, we have the following. As with Protocol $A'$, let us record the outputs of Protocol $R$ as bit sequences $G = (g_1, \ldots, g_N)$ and $O = (o_1, \ldots, o_N)$, where $o_i = 0$ if the outcome of the $i$th round is $H$ or $P$, and $o_i = 1$ otherwise. If $E$ is a purifying system for the device $D$ used in Protocol $R$, then we denote by $\Gamma_{EGO}$ the state of $E$, $G$, and $O$, and by $\Gamma_{EGO}^s$ the subnormalized state corresponding to the “success” event.

**Theorem I.1.** There exists continuous real-valued functions $T(v, h, \eta, q, \kappa)$ and $E(v, h, \eta, q, \kappa)$ (with the domains specified in Figure 5) such that the following statements hold.

1. Let $G$ be an $n$-player strong self-test. Let $D$ be an untrusted device with $n$ components, and let $E$ be a purifying system for $D$. Suppose that Protocol $R$ is executed with parameters $N, \eta, q, G, D$. Then, for any $\kappa \in (0, \infty)$ and $\epsilon \in (0, \sqrt{2}]$, the following bound holds.

$$H_{\min}^e(\Gamma_{EGO}^s \mid EG) \geq N \cdot T(v_G, 2f_G, \eta, q, \kappa) - \left(\frac{\log(\sqrt{2}/\epsilon)}{q\kappa}\right) E(v_G, 2f_G, \eta, q, \kappa),$$ (I.4)
2. The following limit conditions are satisfied, where \( \pi \) denotes the function from Theorem [E.2]

\[
\lim_{(q, \kappa) \to (0, 0)} T(v, h, \eta, q, \kappa) = \pi(\eta/v), \quad (I.5)
\]

\[
\lim_{(q, \kappa) \to (0, 0)} E(v, h, \eta, q, \kappa) = -2\pi'(\eta/v). \quad (I.6)
\]

The following corollary shows that the linear rate of Protocol R can be lower bounded by the function \( \pi \) from Theorem [E.2].

**Corollary I.2.** Let \( \eta > 0 \) and \( \delta > 0 \) be real numbers. Then, there exist positive reals \( b \) and \( q_0 \) such that the following holds. If Protocol R is executed with parameters \( N, \eta, q, G, D \), where \( q \leq q_0 \), then

\[
H_{\text{min}}^e(\Gamma_{EGO}^s \mid EG) \geq N \cdot (\pi(\eta/v) - \delta), \quad (I.7)
\]

where \( e = \sqrt{2} \cdot 2^{-bqN} \).

**Proof.** By the limit conditions for \( T \) and \( E \), we can find \( q_0, \kappa_0 > 0 \) sufficiently small and \( M > 0 \) sufficiently large so that for any \( q \in (0, q_0] \) and \( \kappa \in (0, \kappa_0) \),

\[
T(v_G, 2f_G, \eta, q, \kappa) \geq \pi(\eta/v) - \delta/2 \quad (I.8)
\]

\[
E(v_G, 2f_G, \eta, q, \kappa) \leq M. \quad (I.9)
\]

Let \( b = \delta\kappa_0/(2M) \), and let \( e = \sqrt{2} \cdot 2^{-bqN} \). Then, provided that \( q \leq q_0 \), the output of Protocol R satisfies

\[
H_{\text{min}}^e(\Gamma_{EGO}^s \mid EG) \geq N \cdot T(v_G, 2f_G, \eta, q, \kappa_0) - \left(\frac{\log(\sqrt{2}/e)}{q\kappa_0}\right) E(v_G, 2f_G, \eta, q, \kappa_0) \quad (I.10)
\]

\[
\geq N(\pi(\eta/v) - \delta/2) - \left(\frac{bqN}{q\kappa_0}\right) M \quad (I.11)
\]

\[
= N(\pi(\eta/v) - \delta/2) - (\delta/2)N, \quad (I.12)
\]

which simplifies to the desired bound.

The corollary stated above is not quite at full strength. We wish to show that the output register \( O \) has high min-entropy conditioned on any external information — including the original inputs to the device \( D \). The above corollary takes into account the biased coin flips \( g_1, \ldots, g_N \) used in the protocol, but it does not take into account the inputs that are given to \( D \) during game rounds.

For each \( k \in \{1, \ldots, N\} \), let \( I_k \) denote a classical register consisting of \( n \) bits which records the input used at the \( k \)th round. Let \( I \) be the collection of the all the registers \( I_1, \ldots, I_N \).

**Corollary I.3.** Let \( G \) be a strong self-test, and let \( \eta > 0 \) and \( \delta > 0 \) be real numbers. Then, there exist positive reals \( b, K, \) and \( q_0 \) such that the following holds. If Protocol R is executed with parameters \( N, \eta, q, G, D \), where \( q \leq q_0 \), then

\[
H_{\text{min}}^e(\Gamma_{EGI}^s \mid EG I) \geq N \cdot (\pi(\eta/v) - \delta), \quad (I.13)
\]

where \( e = K \cdot 2^{-bqN} \).
Proof. Let \( \delta' = \delta/2 \). By Corollary I.3, we can find \( b' \) and \( q_0 \) such that whenever Protocol R is executed with \( q \leq q_0 \),

\[
H_{\min}^\varepsilon(\Gamma_{sEGO}^s | EG) \geq N \cdot (\pi(\eta/v_G) - \delta/2),
\]

where \( \varepsilon' = \sqrt{2} \cdot 2^{-b'qN} \). By decreasing \( q_0 \) if necessary, we will assume that \( q_0 < \delta/(2n) \).

For each \( k \in \{1, 2, \ldots, \lfloor N\delta/(2n) \rfloor \} \), let \( I_k \) denote the input string that was given to the device \( D \) on the \( k \)th game round. If there were fewer than \( k \) game rounds, then simply let \( I_k \) be the sequence \( 00 \ldots 0 \). Let \( \overline{I} \) denote the collection of the registers \( I_1, \ldots, I_{\lfloor N\delta/(2n) \rfloor} \).

Let \( d \) denote the event that

\[
\sum G_i \leq N\delta/(2n).
\]

(That is, \( d \) denotes the event that the number of game rounds is not more than \( N\delta/2 \).) By the Azuma-Hoeffding inequality,

\[
P(d) \leq e^{-N[\delta/(2n) - q_0]^2/2}.
\]

Let \( \varepsilon \) be the sum of \( \varepsilon' \) and the quantity on the right of (I.16), and let \( sd \) denote the intersection of the event \( d \) and the success event \( s \). Observe the following sequence of inequalities, where we first use the fact that the operator \( \Gamma_{sdEGIO}^s \) can be reconstructed from the operator \( \Gamma_{sdEGIO}^s \), and then use the fact that the register \( \overline{I} \) consists of \( \leq (N\delta/2) \) bits.

\[
H_{\min}^\varepsilon(\Gamma_{sEGIO}^s | EGI) \geq H_{\min}^\varepsilon(\Gamma_{sdEGIO}^s | EGI) \geq H_{\min}^\varepsilon(\Gamma_{sdEGIO}^s | EGT),
\]

\[
\geq H_{\min}^\varepsilon(\Gamma_{EGO} | EG) - N\delta/2, \quad \text{(I.19)}
\]

\[
\geq H_{\min}^\varepsilon(\Gamma_{EGO} | EG) - N\delta/2, \quad \text{(I.20)}
\]

\[
\geq N \cdot (\pi(\eta/v_G) - \delta), \quad \text{(I.21)}
\]

as desired.

We wish to show that \( \varepsilon \) is upper bounded by a decaying exponential function of \( qN \) (i.e., a function of the form \( J \cdot 2^{-cqN} \), where \( J \) and \( c \) are positive constants depending only on \( \delta, \eta, \) and \( G \)). We already know that \( \varepsilon' \) has such an upper bound. The expression on the right side of (I.16) also has such a bound — indeed, it has a bound of the form \( J \cdot 2^{-cN} \), which is stronger. Therefore \( \varepsilon \) (which is the sum of the aforementioned quantities) is also bounded by a decaying exponential function. This completes the proof. \( \square \)

Remark I.4. Corollary I.3 implies that if \( \pi(\eta/v_G) > 0 \), then (provided \( q \) is sufficiently small) a positive linear rate of output entropy is achieved by Protocol R. Using the formula for \( \pi \) from Remark E.3, this means that a positive linear rate is achieved if \( \eta < 0.11 \cdot v_G \).

Recall that \( \lim_{y \to 0} \pi(y) = 1 \). The next corollary follows easily from Corollary I.3.

Corollary I.5. Let \( \delta > 0 \) be a real number. Then, there exists positive reals \( K, b, q_0, \) and \( \eta \) such that the following holds. If Protocol R is executed with parameters \( N, \eta, q, G, D \), where \( q \leq q_0 \), then

\[
H_{\min}^\varepsilon(\Gamma_{EGIO} | EGI) \geq N \cdot (1 - \delta),
\]

where \( \varepsilon = K \cdot 2^{-bqN} \). \( \square \)
I.3 Example: The GHZ game

Let $H$ denote the 3-player binary XOR game whose polynomial $P_H$ is given by

$$P_H(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{4} (1 - \zeta_1 \zeta_2 - \zeta_2 \zeta_3 - \zeta_1 \zeta_3).$$  \hspace{1cm} (I.23)

This is the Greenberger-Horne-Zeilinger (GHZ) game.

**Proposition I.6.** The trust coefficient for the GHZ game $H$ is at least $0.14$.

For the proof of this result we will need the following lemma (which the current authors also used in [24]):

**Lemma I.7.** Let $a, b, c$ be unit-length complex numbers such that $\text{Im}(a) \geq 0$ and $\text{Im}(b), \text{Im}(c) \leq 0$. Then,

$$|1 - ab - bc - ca| \leq \frac{\sqrt{2}}{2}.$$  \hspace{1cm} (I.24)

**Proof.** We have

$$-1 + ab + bc + ca = (-1 + bc) + a(b + c).$$  \hspace{1cm} (I.25)

The complex number $(b + c)$ lies at an angle of $\pi/2$ (in the counterclockwise direction) from $(-1 + bc)$. Since $a$ has nonnegative imaginary part, the angle formed by $a(b + c)$ and $(-1 + bc)$ must be an obtuse or a right angle. Therefore,

$$|(-1 + bc) + a(b + c)|^2 \leq |-1 + bc|^2 + |a(b + c)|^2$$  \hspace{1cm} (I.26)

$$\leq 4 + 4$$  \hspace{1cm} (I.27)

$$= 8.$$  \hspace{1cm} (I.28)

The desired result follows. □

**Proof of Proposition I.6** We proceed from Definition I.15. Let $N$ be the reverse-diagonal matrix

$$N = \begin{bmatrix}
1 & & & & 1 \\
& 1 & & & \\
& & -1 & & \\
& & & -1 & \\
1 & & & & 1
\end{bmatrix}.$$  \hspace{1cm} (I.29)

Clearly, $N$ anticommutes with $\sigma_x \otimes I \otimes \ldots \otimes I$.

Let $\zeta_1, \zeta_2, \zeta_3$ be unit-length complex numbers with nonnegative imaginary part, and let $M$ be the operator given by (I.1)–(I.2). We wish to show that the operator norm of $M - (0.14)N$ is bounded by $q_H - 0.14 = 0.86$.

Note that

$$\left| \frac{1}{4} (1 - \zeta_1 \zeta_2 - \zeta_2 \zeta_3 - \zeta_1 \zeta_3) - 0.14 \right| \leq \left| \frac{1}{4} (\zeta_1 \zeta_2 + \zeta_2 \zeta_3 + \zeta_1 \zeta_3) \right|$$  \hspace{1cm} (I.30)

$$\leq 0.11 + 0.75$$  \hspace{1cm} (I.31)

$$= 0.86.$$  \hspace{1cm} (I.32)
Also, by applying Lemma [I.7],

\[
\left| \frac{1}{4} \left( 1 - \zeta_{16} - \zeta_{26} - \zeta_{15} - \zeta_{15} \right) + 0.14 \right| \leq \left| \frac{1}{4} \left( 1 - \zeta_{16} - \zeta_{26} - \zeta_{15} - \zeta_{15} \right) \right| + 0.14 \tag{I.33}
\]

\[
\leq \frac{\sqrt{2}}{2} + 0.14 \tag{I.34}
\]

\[
\leq 0.86. \tag{I.35}
\]

Applying similar arguments shows that every reverse-diagonal entry of \((M - 0.14 \cdot N)\) has absolute value bounded by 0.86. This completes the proof.

\[\square\]

**Remark I.8.** By the above result and Remark [I.4], we have the following. If \(\eta\) is a positive real smaller than 0.0154 (= 0.11 \cdot 0.14) and if \(q > 0\) is sufficiently small, then executing Protocol R with the GHZ game yields a positive linear rate of entropy.

### J Unbounded expansion

In this section, we prove a general result, the Concatenation Lemma (Lemma [J.2] below), that implies Corollary [I.5] straightforwardly. This general result says that untrusted-device protocols for generating randomness can be composed sequentially with additive soundness errors, even if only two untrusted devices are used. We shall first describe the general framework for rigorously reasoning about those protocols before formally describing such a protocol and giving the proof for the Lemma.

We use \(\hat{I}\) to denote the density operator for the totally mixed state in a Hilbert space. The space of linear operators on a Hilbert space \(H\) is denoted by \(L(H)\).

Since we will use the accepted output of a device as the input for another device, we will use the same syntax for the state space. A protocol state space \(\mathcal{H}\) is a three-part Hilbert space

\[
\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_D \otimes \mathcal{H}_E, \tag{J.1}
\]

where \(C, D, E\) are referred to as the classical input, device, and adversary sub-systems, respectively. We also represent a protocol space by the triple \((C, D, E)\).

We will consider subnormalized Classical-Quantum-Quantum states \(\rho \in L(H)\) only, i.e., subsystem \(C\) is classical, \(\rho \geq 0\) and \(\text{Tr}(\rho) \leq 1\). Those states correspond to the accepting (or non-aborting) portion of the states in our protocols. Denote by \(\hat{\rho} = \rho / \text{Tr}(\rho)\) the corresponding normalized state. We call a subnormalized quantum state \(\rho \in L(H)\) device-uniform, adversary-uniform, or global-uniform if in \(\hat{\rho}\), \(X\) is uniform with respect to \(D\), or \(E\), or \(DE\), respectively. For any \(\epsilon \in [0, 2]\), \(\rho\) is said to be \(\epsilon\)-device-uniform if there exists a subnormalized state \(\hat{\rho}_D\) such that

\[
\|\rho_{XD} - \hat{I}_X \otimes \hat{\rho}_D\|_\text{tr} \leq \epsilon. \tag{J.2}
\]

Similarly define \(\epsilon\)-adversary-uniform and \(\epsilon\)-global-uniform.

A (untrusted-device) protocol \(\Pi\) is a completely positive and trace non-increasing super-operator from states in protocol space \((X, D, E)\) to those in \((Y, D, XE)\), where \(X\) is used as the control in controlled-operations, and the new adversary sub-system is \(XE\). The soundness error of \(\Pi\) on a set \(S\) of states is the infimum of \(\epsilon\) such that for any \(\rho \in S\), \(\Pi(\rho)\) is \(\epsilon\)-adversary uniform. Note that we allow \(\Pi\) to reduce the trace of the input state with the intention that the residual trace is the probably of aborting.
A protocol \( \Pi \) is a single-device protocol if \( \Pi \) acts as the identity on \( E \) and is controlled-operation on \( XD \), using \( X \) as the control.

The Equivalence Lemma of Chung, Shi and Wu [5] is the following.

**Theorem J.1** (The Equivalence Lemma [5]). For any single-device protocol \( \Pi \), its soundness error on device-uniform states is equal to its soundness error on global-uniform states.

Thus we shall refer to the two equal soundness errors simply as the soundness error of \( \Pi \).

We now formally define the concatenation of protocols using two devices.

**Definition 16.** Let \( D_0 \) and \( D_1 \) be two untrusted quantum devices and \( T \geq 1 \) be an integer. We call \( \Sigma \) a \((T+1)\)-round cross-feeding protocol using \( D_0 \) and \( D_1 \), if it is the composition of a sequence of single-device protocols \( \Sigma_i \), \( i = 0, 1, \ldots, T \), such that \( \Sigma_i \) uses \( D_{i \mod 2} \) and the output of \( \Sigma_i \) is used as the input to \( \Sigma_{i+1} \). The input to \( \Sigma \) is the input to \( \Sigma_0 \), and the output is that of \( \Sigma_T \).

**Lemma J.2** (Concatenation Lemma). Let \( T \geq 1 \) be an integer and \( \Sigma \) be an \((T+1)\)-round cross-feeding protocol using two untrusted quantum devices \( D_0 \) and \( D_1 \). Let \( \Sigma_i, 0 \leq i \leq T \), be the \( i \)th single-device protocol with input \( X_i \), output \( X_{i+1} \), and soundness error \( \epsilon_i \). Let \( A_i, 0 \leq i \leq T + 1 \), be an arbitrary \( X_i \)-controlled trace-preserving, completely-positive super-operator from

\[
L(\mathcal{H}_{X_i} \otimes \mathcal{H}_{X_{i+1}}) \rightarrow L(\mathcal{H}_{X_i} \otimes \mathcal{H}_{X_{i+1}}).
\]

Let

\[
\Sigma := A_{T+1} \Sigma_T A_T \Sigma_{T-1} A_{T-1} \cdots \Sigma_0 A_0.
\]

Then the soundness error of \( \Sigma \) on device-uniform states is \( \leq \sum_{i=0}^{T} \epsilon_i \).

**Proof.** The proof relies on two simple observations. The first is that if \( \Pi \) is a single-device protocol with \( \leq \epsilon \) soundness error, and \( \rho \) is \( \delta \)-device-uniform, \( \Pi(\rho) \) is \( \epsilon + \delta \)-adversary-uniform. The second is that any state \( \omega_{YD_0(D_1E)} \) that is adversary-uniform for \( D_0 \) is device-uniform for \( D_1 \).

We prove by induction that for each \( k, 0 \leq k \leq T \), and for any device-uniform \( \rho_{X_0D_0(D_1E)} \), with \( \Sigma_k := A_k \Sigma_{k-1} A_{k-1} \cdots \Sigma_0 A_0 \), \( \rho^{(k)} := \Sigma_k(\rho) \) is \( \epsilon^{(k)} := \sum_{i=0}^{k} \epsilon_i \)-adversary-uniform (with respect to device \( D_{k \mod 2} \)).

For \( k = 0 \), \( A_0 \) applies to \( XD_1E \) thus does not change the reduced state on \( XD_0 \). Therefore \( A_0(\rho) \) remains device-uniform (for \( D_0 \)).

Suppose that we have \( \rho^{(k)} \) being \( \epsilon^{(k)} \)-adversary-uniform with respect to device \( D_{k \mod 2} \). Since \( D_{k+1 \mod 2} \) is part of the adversary system, \( \rho^{(k)} \) is in particular \( \epsilon^{(k)} \)-device-uniform for \( D_{k+1 \mod 2} \). Thus \( \Pi_{k+1} \rho^{(k)} = \epsilon^{(k+1)} = \epsilon^{(k)} + \epsilon_{k+1} \)-adversary uniform with respect to \( D_{k+1 \mod 2} \). The \( X_{k+1} \)-controlled super-operator \( A_{k+1} \) does not change the reduced operator on \( X_{k+1}D_{k+1 \mod 2} \), thus \( \Sigma_{k+1}(\rho) \) remains \( \epsilon^{(k+1)} \)-adversary uniform. The Lemma holds.

Applying the Concatenation Lemma to our one-shot, exponentially expanding randomness expansion protocol yields Corollary [1.5] One can also apply the Lemma to any interleaving of our protocol and those of Vazirani and Vidick [41,40], though the resulting protocol would not be robust if the non-robust Vazirani-Vidick protocol for expansion is used.

**K Untrusted-device quantum key distribution**

In this section, we shall first formally define what we mean by a key distribution protocol using untrusted quantum devices. We then present Protocol \( R_{kd} \), a natural adaption of Protocol \( R \) for untrusted-device quantum key distribution, then we prove its correctness (Corollary [1.7]).
K.1 Definitions

A min-entropy untrusted-device key distribution (ME-UD-KD) protocol $\Pi_{kd}$ is a communication protocol in the following form between two parties Alice and Bob who have access to distinct components of an untrusted quantum device. Before the protocol starts, they share a string that is uniformly random to the device. They communicate through a public, but authenticated, channel. At each step, both the message they send and the new input to their device components are a deterministic function of the initial randomness, the messages received, and the previous output of their device component. The protocol terminates with a public bit $S$, indicating if the protocol succeeds or aborts, and Alice and Bob each has a private string $S_A$ and $S_B$, respectively.

The protocol is said to have an yield $N$ with a soundness error $\epsilon_s$ if both the following conditions hold.

(a) the joint state $(S, S_A, E)$ is $\epsilon_s$-close to a state that is a mixture of Abort and one that has $N$ extractable bits, and

(b) the joint state $(S, S_A, S_B)$ is $\epsilon_s$-close to a state such that $S_A = S_B$.

The protocol is said to have a completeness error $\epsilon_c$ with respect to a non-empty class of untrusted devices $U_{\text{honest}}$, if for any device in this class, the protocol aborts with probability $\leq \epsilon_c$.

If in the above definition, Condition (a) has “$N$ extractable bits” replaced by “$N$ uniformly random bits”, then we call the protocol simply an untrusted-device key distribution protocol with those parameters.

K.2 The protocol

Protocol $R_{kd}$ is described in Fig. 6. There are two main steps in the proof for Corollary 1.7. The first is to show that for an appropriate range of the parameters, Protocol $R$ has a soundness and completeness error of $\exp(-\Omega(qN))$ with the ideal state being that $O$ and $B$ differ in at most a $(1/2 - \lambda)$ fraction, for a constant $\lambda$. The second step is to construct the Efficient Information Reconciliation Protocol that works on the ideal state and for Bob to correct the differences with some small failure probability. We present those two steps in two separate subsections, which are followed by the proof for the Corollary.

K.3 Error rate

The completeness error is straightforward, so our focus will be on the soundness error.

Our result applies to a broader class of games than the strong self-tests.

Definition 17. Let $f : (0, 1) \to (0, 1)$ be such that $\lim_{\theta \to 0} f(\theta) = 0$. A game $G$ is said to be $f$-self-testing in probability if there exists an input $x_0$ such that for any $\theta \in (0, 1)$ and any quantum strategy that wins with probability $(1 - \theta)w_G$, the game wins on input $x_0$ with probability $\geq (1 - f(\theta))w_G$.

The following theorem uses the notion of second-order robust self-test (which in this paper we call strong self-test) from the paper [24] by Miller and Shi.

Theorem K.1. Let $G$ be a strong self-test. Then there exists a constant $C > 0$ such that $G$ is $C\sqrt{\theta}$-self-testing in probability.
Arguments:

$G$: An $n$-player nonlocal game that is a strong self-test [24].

$D$: An untrusted device (with $n$ components) that can play $G$ repeatedly and cannot receive any additional information. Alice interacts with the first component while Bob interacts the rest of the device. No communication is allowed among the components during Step 1-4 of the protocol. All random bits chosen by Alice and Bob together are assumed to be perfectly random to $D$.

$N$: A positive integer (the output length).

$\lambda$: A real $\in (0, w_G - 1/2)$. ($1/2 - \lambda$ is the key error fraction.)

$\eta$: A real $\in (0, \frac{1}{2})$. (The error tolerance.)

$q$: A real $\in (0, 1)$. (The test probability.)

Protocol:

1. Alice and Bob choose a bit $g \in \{0, 1\}$ according to a biased $(1-q, q)$ distribution.
2. If $g = 1$ (“game round”), then Alice and Bob choose an input string at random from $\{0, 1\}^n$ according the probability distribution specified by $G$. They give their part(s) of $D$ the corresponding input bit, exchange their output bits and record a “P” (pass) or an “F” (fail) according to the rules of the game $G$.
3. If $g = 0$ (“generation round”), then the input string 00...0 is given to the device. Alice records the output of her device component in the $O$ register for this round. Bob stores in the $B$ register for this round the unique bit that XOR’ing with the output bit(s) of his device component(s) would constitute a win for the game. That is, their bits are the same if and only if they win the game.
4. Steps 1–3 are repeated $N - 1$ (more) times.
5. If the total number of failures is more than $(1 - w_G + \eta)qN$, the protocol aborts.
6. If not yet aborted, they run the Efficient Information Reconciliation (EIR) Protocol (Fig. 7) on the registers $O$ and $B$ with the parameters $\lambda$ and $\epsilon = \exp(-qN)$. If the EIR Protocol does not abort, the register $B$ is replaced with Bob’s guess $\tilde{B}$ of $O$. The protocol terminates successfully with Alice and Bob’s version of the key being their $N$-length sequence from the alphabet $\{P, F, H, T\}$ representing the outcomes of each round ($H$ and $T$ for 0 and 1, respectively, stored in $O$ and $\tilde{B}$).

Figure 6: Protocol $R_{kd}$

Proof. Since $G$ is strongly self-testing, there is a unique quantum strategy which achieves the optimal winning probability $w_G$. Let $x_0$ be an input string (which occurs with nonzero probability in $G$) such that, if the optimal strategy is applied on input $x_0$, the winning probability is at least $w_G$. 

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If a given quantum strategy for $G$ achieves a score of $(1 - \theta)w_G$, then by the strong self-testing property its probability distribution is $C_1 \sqrt{\theta}$-close to that of the optimal strategy, for some constant $C_1$. The result follows.

Consequently all strong self-tests are $O(\sqrt{\theta})$-self-testing in probability.

We now fix a game $G$ that is $f$-self-testing in probability for some function $f$. Let $w_i$ and $w_i^0$, $1 \leq i \leq N$, denote the chances of winning the $i$th round game, if the game is played using full input distribution or the randomness generating input, respectively. Define the random variables

$$W_i := \begin{cases} 1 & \text{the output at round } i \text{ is “P” or “H”,} \\ 0 & \text{otherwise.} \end{cases} \quad (K.1)$$

**Theorem K.2.** Let $G$ be a game that is $f$-self-testing in probability for some $f$. Consider Protocol $R$ (Fig. 2) using $G$ and an arbitrary $q \in (0, 1)$. For any $\lambda', 0 < \lambda' < w_G - 1/2$, there exists $\bar{\eta} = \bar{\eta}(\lambda') > 0$, such that

$$\mathbb{P} \left[ \left( \sum_i g_i(1 - W_i) \leq (1 - w_G + \eta)N \right) \land \left( \sum_i W_i \leq (1/2 + \lambda)N \right) \right] \leq \exp \left( -\frac{(\bar{\eta} - \eta)^2}{3} qN \right) + \exp \left( -\frac{(\lambda' - \lambda)^2}{2} N \right). \quad (K.2)$$

We will make use of a refined Azuma-Hoeffding inequality [12].

**Lemma K.3.** Suppose that $S_1, S_2, ..., S_N$ is a Martingale with

$$|S_{i+1} - S_i| \leq 1,$$

and

$$\text{Var} [S_{i+1} - S_i \mid S_1, ..., S_i] \leq w,$$

for all $i, 1 \leq i \leq N - 1$. Then for any $\epsilon > 0$,

$$\mathbb{P} [S_N \geq \epsilon wN] \leq \exp \left( -\frac{\epsilon^2 w}{2} N \left( 1 - \frac{1}{3} \frac{1}{\epsilon} \right) \right). \quad (K.4)$$

In particular if $\epsilon \leq 1$, we have

$$\mathbb{P} [S_N \geq \epsilon wN] \leq \exp \left( -\frac{\epsilon^2 w}{3} N \right). \quad (K.5)$$

Consider

$$T_i := \sum_{j=1}^{i} (g_i(1 - W_i) - q(1 - w_i)). \quad (K.6)$$

Since $E[g_i(1 - W_i) - q(1 - w_i) \mid T_1, ..., T_{i-1}] = 0$, and

$$\text{Var} [T_i - T_{i-1} \mid T_1, ..., T_{i-1}] = q(1 - w_i)[1 - q(1 - w_i)] \leq q, \quad (K.7)$$

applying the above Lemma, we have
Corollary K.4. For any $\epsilon \in (0, 1)$,
\[
\mathbb{P} \left[ \sum_i g_i (1 - W_i) - q \sum_i (1 - w_i) \geq \epsilon q N \right] \leq \exp \left( -\frac{\epsilon^2 q N}{3} \right). \tag{K.8}
\]

Consider now
\[
S_i := \sum_{j=1}^i (W_i - (1 - q)w^0_i - qw_i).
\tag{K.9}
\]
Then $S_i$ is a Martingale with $|S_i - S_{i-1}| < 1$. Thus the following Corollary follows from the standard Azuma-Hoeffding bound.

Corollary K.5. For any $\epsilon > 0$,
\[
\mathbb{P} \left[ \sum_i (W_i - (1 - q)w^0_i - qw_i) \leq -\epsilon N \right] \leq \exp \left( -\frac{\epsilon^2}{2} N \right). \tag{K.10}
\]

We will make use of the following consequence of the $f$-self-testing property.

Proposition K.6. For any quantum strategy and any $\theta > 0$,
\[
1 - \frac{w}{w_G} \leq f(\theta) + \frac{1}{\theta} (1 - w/\bar{w}_G). \tag{K.11}
\]

Proof. If $w/\bar{w}_G \geq 1 - f(\theta)$, LHS $\leq f(\theta) \leq$ RHS. Otherwise, by definition, $w/\bar{w}_G < 1 - \theta$. Thus LHS $\leq 1 \leq$ RHS. \hfill $\square$

Proof of Theorem K.2 Fix arbitrary $\lambda, \lambda', \delta, \delta', \theta$ with $0 < \lambda < \lambda' < w_G - 1/2$, $0 < \delta < \delta'$, and $\theta \in (0, 1)$. For $\epsilon_1, \epsilon_2 \in (0, 1)$ to be determined later, define the following two events
\[
E_1 := \sum_i g_i (1 - W_i) > q \sum_i (1 - w_i) - \epsilon_1 q N, \tag{K.12}
\]
\[
E_2 := \sum_i (1 - g_i)W^0_i > (1 - q) \sum_i w^0_i - \epsilon_2 (1 - q)N. \tag{K.13}
\]

Apply Corollaries (K.4) and (K.5) with $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$, respectively, we have
\[
\mathbb{P}[\bar{E}_1] \leq \exp \left( -\frac{\epsilon_1^2 q N}{3} \right), \tag{K.14}
\]
\[
\mathbb{P}[\bar{E}_2] \leq \exp \left( -\frac{\epsilon_2^2}{2} N \right). \tag{K.15}
\]

Denote the event in Eqn. (K.2) by $E$. Then
\[
\mathbb{P}[E] \leq \mathbb{P}[\bar{E}_1] + \mathbb{P}[\bar{E}_2] + \mathbb{P}[E \wedge E_1 \wedge E_2]. \tag{K.16}
\]

Denote by
\[
\bar{w} := \frac{1}{N} \sum_i w_i/\bar{w}_G, \quad \text{and,} \quad w^0 := \frac{1}{N} \sum_i w^0_i/\bar{w}_G. \tag{K.17}
\]

The event $E \wedge E_1 \wedge E_2$ implies
\[
(1 - \bar{w} < (\eta + \epsilon_1)/\bar{w}_G) \wedge \left( (1 - q)(1 - \bar{w}^0) + q(1 - \bar{w}) > \frac{w_G - (1/2 + \lambda + \epsilon_2)}{w_G} \right). \tag{K.18}
\]
Together with Proposition (K.6), this implies,

\[(\eta + \epsilon_1) > w_G \cdot \left[ \frac{w_G - (1/2 + \lambda + \epsilon_2)}{w_G} - (1 - q) f(\theta) \right] \bigg/ \left( \frac{1 - q}{\theta} + q \right) \quad (K.19)\]

\[\geq w_G \cdot \theta \left[ \frac{w_G - (1/2 + \lambda + \epsilon_2)}{w_G} - f(\theta) \right], \quad (K.20)\]

for any \( \theta \in (0, 1) \). For any \( \lambda, \lambda', 0 < \lambda < \lambda' < w_G - 1/2 \), we set \( \epsilon_2 = \lambda' - \lambda \), denote by \( \bar{\eta} = \bar{\eta}(\lambda') \) the supremum of the R.H.S. of (K.20) for \( \theta \in (0, 1) \). Since \( f(\theta) \to 0 \) when \( \theta \to 0 \), we have \( \bar{\eta} > 0 \).

Thus for any \( \eta \), if \( 0 < \eta < \bar{\eta} \), setting \( \epsilon_1 = \bar{\eta} - \eta \), Event (K.19) does not occur. In such a case, by Eqn. (K.16) and Eqn. (K.14),

\[\mathbb{P}[E] \leq \exp \left( -\frac{(\bar{\eta} - \eta)^2}{3} qN \right) + \exp \left( -\frac{(\lambda' - \lambda)^2}{2} N \right) \quad (K.21)\]

Thus the theorem holds.

**K.4 Efficient Information Reconciliation**

We now arrive at the problem of resolving differences between Alice and Bob’s keys under the promise that the differences (referred to as errors) are strictly less than 1/2-fraction. We hope that the solution is efficient, not just in term of local computational complexity, but also, most critically, the bits communicated. This is because any bit communicated in this stage will be subtracted from the min-entropy guarantee. To that end, we define a quantity to describe the limit of surviving fraction of min-entropy.

**Definition 18** (Efficient Information Reconciliation). Let \( \lambda \in (0, 1/2), \epsilon \in (0, 1), N, R \) and \( M \) be integers, and \( T \) be a function on \( N, \lambda \) and \( \epsilon \). An information reconciliation protocol with those parameters is a communication protocol between two parties Alice and Bob with the following property. On any \( N \)-bit strings \( A \) and \( B \), known to Alice and Bob, respectively, and of Hamming distance \( |A \oplus B| \leq (1/2 - \lambda)N \), for some \( \lambda \in (0, 1/2) \), they start the protocol with a shared \( R \)-bit string, communicate \( M \) bits, output strings \( A', B' \) of \( N \) bits from Alice and Bob, respectively, such that \( A' \) and \( B' \) differ with \( \leq \epsilon \) probability. The computation complexity of the protocol is \( \leq T \).

The protocol is said to be efficient for a constant \( \lambda \), if \( R = O(\log N / \epsilon) \), \( M = (1 - c)N \) for some constant \( c = c(\lambda) \), and \( T = \text{poly}(\log N / \epsilon) \).

For a fixed constant \( \lambda \), the residual fraction is the supremum of \( c(\lambda) \) such that an efficient protocol exists.

Time efficient Information Reconciliation has been studied before for smaller error rate and without constraints on the amount of randomness used. Our problem requires that the amount of randomness used is small when the error rate is \( \geq 1/4 \). We show this is possible with the following parameters using several known results.

The first is the idea of Bennett et al. [3] for information reconciliation through error correcting codes. Their method, however, can only deal with \( < 1/4 \) relative errors (i.e. \( \lambda \leq 1/4 \)). To correct \( 1/2 - \lambda \) relative errors, we will have to resort to list-decodable binary linear codes. To pin down the actual error, we use an approximate universal hashing [26, 2] to exploit the fact that there is only a polynomial in \( N \) number of candidates to examine after list-decoding. We state some of those results below.
Theorem K.7 (Guruswami [17]). There exists a constant \( c > 0 \) such that for any \( \lambda \in (0, 1/2) \), and for an infinite number of integers \( N > 0 \), there exists a binary linear code of block length \( N \), relative error \( 1/2 - \lambda \), rate \( R := \Omega(\lambda^3 / \log(1/\lambda)) \), that can be list-decoded into a list of size \( \exp(O(\log 1/\lambda)) \) in \( O(N^c) \) encoding and decoding time.

There are other constructions (such as [19]) that have different parameters but are still good for our purpose.

Theorem K.8 ([26, 2]). For any natural number \( \ell \), any real \( \epsilon \in (0, 1/2) \) and all sufficiently large integer \( M \), there exists a distribution on \( \{0, 1\}^M \) such that any \( \ell \) bits are \( \epsilon \)-close to the uniform \( \ell \) bits in variation distance, and the distribution can be generated using

\[ O(\log \log M + \ell + \log 1/\epsilon) \]

number of random bits and constructed in polynomial in \( M \) and \( \log 1/\epsilon \) time.

Note that an \( \epsilon \)-almost 2\( k \)-wise independent string of \( 2^N \) bits form an \( \epsilon \)-almost pairwise independent hash function family from \( \{0, 1\}^N \) to \( \{0, 1\}^k \). Thus Theorem K.8 implies the following corollary with \( M = 2^N \) and \( \ell = 2k \).

Corollary K.9 (of Theorem K.8). For any real \( \epsilon \in (0, 1/2) \), any integer \( k \geq 1 \) and all sufficiently large \( N \), there exists a hash function family from \( \{0, 1\}^N \) to \( \{0, 1\}^k \) that is \( \epsilon \)-close to a pairwise independent hash function family and uses

\[ O(\log N + k + \log 1/\epsilon) \]

number of random bits.

We are ready to present our protocol for Efficient Information Reconciliation and prove its correctness.

Theorem K.10. The Protocol in Fig. 7 is an efficient Information Reconciliation protocol [18]. In particular, if \( \lambda > 1/4 \), it succeeds with certainty without consuming any randomness. If \( \lambda \in (0, 1/4] \), the number of shared random bits is \( O(\log N + \log 1/\epsilon) \).

Proof. The length of Alice’s message and the correctness of Bob’s output follow from the properties of \( C \). For the case of \( \lambda \leq 1/4 \), the length of the shared randomness follows from the property of \( H \). Furthermore, the chance of aborting is no more than the existence of \( i' \neq i \) such that \( h(Y + D_i) = h(Y + D_{i'}) \). This probability is no more than

\[ \frac{L}{M} + \epsilon/2 \leq \epsilon. \]

\[ \Box \]

K.5 Proof for Corollary 1.7

Proof of Corollary 1.7 We set \( s = \sup_{\lambda < w_G - 1/2} c(\lambda') \), where \( c(\cdot) \) is the residual fraction (Definition 18), shown to be well-defined by Theorem K.10. For an arbitrarily small \( \delta > 0 \), we choose arbitrary \( \kappa \in (0, \delta) \). The choice of \( \kappa \) will affect our final error parameter, so for practical applications may be chosen to minimize the final error.

Applying Theorem 1.1 with the \( \delta \) parameter there set to be \( \delta_1 := \delta - \kappa \), we get parameter upper bounds, \( q_0 \) and \( \eta_0 \), to ensure the soundness and completeness properties for Alice’s output \( A \).
Arguments:

- \( \lambda \): A real \( \in (0, 1/2] \).
- \( X, Y \): Binary strings of length \( N \) such that \(|X \oplus Y| \leq (1/2 - \lambda)N\).
- \( \epsilon \): A failure probability. Equals 0 for \( \lambda \in (1/4, 1/2] \).

- \( A \): The check matrix of an explicitly constructible (i.e., encoding and decoding in polynomial time) binary linear error-correcting code \( C \) of length \( N \), relative error \( 1/2 - \lambda \), and a linear rate \( R = R(\lambda) \). If \( \lambda \in (1/4, 1/2) \), \( C \) is uniquely decodable; otherwise \( C \) is efficiently list-decodable with a list size \( L = poly(N, \log 1/\epsilon) \). Such codes exist ([11][18]).

- \( \mathcal{H} \): If \( \lambda \leq 1/4 \), \( \mathcal{H} \) is an explicitly constructible \( \epsilon/2 \)-almost pairwise independent hash function family from \( \{0, 1\}^N \) to \( \{0, 1\}^k \), where \( k = \lceil \log(2L/\epsilon) \rceil \). The number of random bits used to draw a random \( h \in \mathcal{H} \) is \( O(\log N + k + \log 1/\epsilon) \). Such \( \mathcal{H} \) exists according to Corollary K.9.

Protocol:

1. Alice sends Bob \( AX \in \{0, 1\}^{(1-R)N} \).
2. If \( C \) is uniquely decodable, Bob computes the error syndrome \( AY + AX = A(X + Y) \), runs the decoding algorithm to obtain the unique \( D \) with \(|D| \leq (1/2 - \lambda)N \) and \( AD = A(X + Y) \). The protocol terminates with Bob outputting \( Y + D \).
3. Otherwise (\( C \) is list-decodable with list size \( L \)), Bob list-decodes from \( A(X + Y) \) to obtain a list \( \{\Delta_1, \Delta_2, \ldots, \Delta_L\} \), where by the property of \( C \), \( X + Y = \Delta_i \) for some \( i, 1 \leq i \leq L \).
4. Alice and Bob draw a random \( h \in \mathcal{H} \), and Alice sends Bob \( h(X) \). Bob checks if there exists a unique \( \Delta_i \) such that \( h(Y + \Delta_i) = h(X) \). If no, aborts. Otherwise, the protocol terminates with Bob outputting \( Y + \Delta_i \).

**Figure 7:** An Efficient Information Reconciliation protocol

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Let \( \lambda \) be such that \( c(\lambda) = s - \kappa \). Choose an arbitrary \( \lambda' \in (\lambda, w_G - 1/2) \). Like \( \kappa \), the choice of \( \lambda' \) affects final error parameters. Replace \( \eta_0 \) by \( \bar{\eta}(\lambda') \) as determined in Theorem K.2 if the latter is smaller.

Fix arbitrary \( q, \eta, N \) with \( q \leq q_0 \) and \( \eta \leq \eta_0 \). Then the conclusions of both Theorem 1.1 and K.2 hold. Therefore, Alice’s output \( O \) in Protocol \( R_{kd} \) before information reconciliation has \( (1 - \delta + \kappa)N \) extractable bits with \( \epsilon_{s1} = \exp(-bqN) \) soundness error, for \( b \) determined in Theorem 1.1. The information reconciliation protocol [7] consumes \( (1 - c(\lambda))N \) bits, thus the remaining conditional min-entropy in \( O \) is

\[
[(1 - \delta + \kappa) - (1 - c(\lambda))] \cdot N = [c(\lambda) - \delta + \kappa] \cdot N = (s-\delta)N.
\]

Thus after information reconciliation, Alice’s key has \( s - \delta \) extractable bits with a soundness error \( \epsilon_{s1} \).

We now bound the probability of the event \( E_\neq \) that the protocol does not abort and Alice and Bob’s keys disagree. Let \( D \) be the event that \(|O + B| \leq (1/2 - \lambda)N \). Denote by \( E \) the event in Enq. K.2, that is, the protocol before information reconciliation not aborting yet \( D \) is true. Denote by \( E_{ic} \) the event that the Efficient Information Reconciliation Protocol [7] does not abort yet \( \bar{B} \neq O \). Then

\[
P[E_\neq] \leq P[E] + P[E_{ic} | D].
\]  

(K.22)

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By Theorem [K.2], \( P[E] \leq \text{the R.H.S. of (K.3)} \), which can be made \( \leq \exp(-b'qN) \) for some appropriate constant \( b' = b'(\bar{\eta}, \lambda, \lambda') \), possibly by lowering \( \eta_0 \) so that it is bounded away from the above by \( \bar{\eta} \). By Theorem [K.10], \( P[E_{IC} | D] \leq \exp(-qN) \). Thus replacing \( b \) by \( \min\{b, 1\} \), we have the soundness error \( \exp(-bqN + O(1)) \).

The completeness error follows from that for randomness expansion protocol, with an additional probability of aborting in information reconciliation. The proof is a standard application of Azuma-Hoeffding inequality thus we leave the proof for the interested reader.

The number of random bits used in the expansion protocol is \( O(Nh(q)) \). That used in the Efficient Information Reconciliation Protocol is either 0 or \( O(\log N/\epsilon) \), where \( \epsilon = \exp(-qN) \), thus \( O(Nh(q) + \log N + qN) \), which is \( O(Nh(q) + \log N) \) (and is \( O(Nh(q)) \) when \( qN = \Omega(1) \)).

References


