6. Modal Logic and Metaphysics

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1. In recent years, the research of many logicians has converged toward a solution of the problem of quantification into modal contexts, and in the last decade a new level of understanding has been achieved in this area. Of course, problems still remain; but these can now be treated within the context of a well-articulated logical theory. In this paper I wish to discuss in an informal way some of my own contributions to this development; these consist primarily of semantical completeness theorems for some systems of modal logic with quantifiers.

Perhaps the simplest example of a theorem of this sort is the one obtained by Post in the 1920s for the familiar sentential calculus. Post showed that every formula provable in this system was tautologous, and conversely that every tautology was a theorem: thereby establishing that a notion of validity for such formulas coincides with a notion of provability. Such a theorem was obtained for the predicate calculus of first order with identity in 1930 by Gödel. A stronger result holding for

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2. For a brief history of this research, see Bibliographical Note below, pp. 144–45.
sets of formulas of the predicate calculus was obtained by Henkin in 1947.

My own results are analogues of Henkin's theorem for certain modal systems. There are many systems for which such results have been obtained, but in this presentation I will confine myself to three systems which seem of particular interest—two of them with completeness theorems. (The other results can, for the most part, be obtained from these two by minor changes.) The theorems themselves are mathematical in character, and are proved by algebraic techniques resembling those used in representation theorems. But when discussed in an informal way these results, so far from appearing technical, take on a metaphysical character. The reason for this is that the heuristic notions arising in the semantical interpretation of quantified modal logic are concepts of traditional philosophic concern; and so, in much the same way that a mathematician explaining his work might refer to space and its characteristics, I will be speaking of existence, substances, and accidents.

2. The central task of this paper is to provide an adequate semantical account of a sufficiently rich formal system of modal logic.

By an adequate semantical account, I mean one that is intuitively plausible (and hence, will provide an informally satisfying explanation of what modality is about), as well as technically acceptable. A crucial technical test of adequacy is the establishment of a semantical completeness theorem of the sort described above; but symbolic logic, being a well-articulated discipline, also imposes numerous criteria which any well-made logical theory must meet. So, as well as shaping logical work in the light of heuristic philosophical considerations, I have often found myself modifying metaphysical preconceptions in the light of technical considerations.

By a sufficiently rich formal system of modal logic I mean, for present purposes, something quite specific; the system must contain formal apparatus sufficient to express the following:

- Truth-functional connectives,
- Modal connectives,
- Identity,
- Universal quantification over individuals,
- Definite descriptions.

As we will see, profound difficulties are involved in handling all of these simultaneously, which cannot be settled on a piecemeal basis; for instance, one might possess a satisfactory semantics for all of the above notions except for identity, and yet have no idea of how the theory could be satisfactorily extended to account for identity.

Accordingly, our formal language will include individual variables \(x, y, z, \ldots\); individual constants \(a, b, c, \ldots\); \(t\)-ary \((i \geq 0)\) predicate letters \(P, Q, \ldots\); and sentential connectives \(\to, \sim, \square\). We may also construct identity-formulas \(s = t\), universal quantifications \((x)A\), and descriptions \(\exists xA\). Terms \(t\) and formulas \(A\) are defined inductively as usual. By means of these primitives, we can then define disjunction, conjunction, equivalence, possibility, existential quantification, and unique-existential quantification \((v, \land, \equiv, \Diamond, \exists, \exists!\). We also use the following definitions:

\[
E t = \alpha t (\exists x)x = t
\]

\[
E \square t = \alpha t (\exists x)\square(x = t).
\]

The next three sections discuss some familiar ways of handling the sentential connectives, identity, and universal quantification. I will assume that the reader is familiar with these topics, and will be very brief.

3. For definiteness, let's choose a simple axiomatization of classical sentential logic, consisting of the three axioms-
schemes  
\[ A \supset (B \supset A) \]
\[ (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \]
\[ (\neg A \supset \neg B) \supset (B \supset A), \]
and the rule Modus Ponens.
\[
\begin{array}{c}
A \\
B
\end{array}
\]
\[
\begin{array}{c}
\frac{A \supset B}{B}
\end{array}
\]

All of the systems considered below will possess these axiom-schemes and this rule in unrestricted form.

The connectives \( \neg \) and \( \supset \) have the usual two-valued truth-functional semantics, which is characteristic for the above axiomatization; i.e., there is semantical completeness here.

The connective \( \Box \) will be interpreted according to Kripke's semantics. A model structure consists of a nonempty set \( \mathcal{K} \) (of "possible worlds") and a binary relation \( \mathcal{R} \) (of "alternativeness" or "relative possibility") on \( \mathcal{K} \). A formula \( \Box A \) takes the value \( T \) in a member \( x \) of \( \mathcal{K} \) if and only if \( A \) takes the value \( T \) in all members \( \beta \) of \( \mathcal{K} \) such that \( xR\beta \). By imposing various conditions on \( \mathcal{R} \), we obtain interpretations corresponding to various kinds of modality; e.g., if \( \mathcal{R} \) is assumed reflexive and transitive, the semantics corresponds to Lewis' \( S4 \).

For definiteness, the modality of \( S4 \) will be paradigmatic in this paper; thus, I will posit the axiom-schemes
\[ \Box(A \supset B) \supset (\Box A \supset \Box B) \]
\[ \Box A \supset A \]
\[ \Box A \supset \Box \Box A, \]
and the rule of necessitation,
\[
\begin{array}{c}
A \\
\Box A
\end{array}
\]

3. One exception: the alternative to Russell's theory of descriptions mentioned in n. 8 would involve certain restrictions on the use of sentential axioms containing descriptions. But this alternative is only mentioned in passing.

in unrestricted form in the systems considered below. But the results of this paper will not depend on our choice of \( S4 \); they hold for most familiar systems of sentential modality.

I will refer to the axioms and rules of this section as sentential axioms and rules.

4. For the semantical interpretation of identity we introduce the notion of a domain of individuals; terms are assigned values in this (nonempty) domain, and \( \equiv t \) takes the value \( T \) if and only if \( s \) and \( t \) are assigned the same member of the domain.

Let \( A^s//t \) be any result of replacing any number of free occurrences of \( t \) in \( A \) by occurrences of \( s \) (relettering bound variables, if necessary, to avoid binding a free variable in \( A \)). Then the two schemes
\[
\begin{array}{c}
t = t \\
s = t \supset (A \supset As//t)
\end{array}
\]
yield an axiomatic basis for the classical theory of identity. (The second scheme will be restricted in some of the systems presented below.)

5. According to the classical theory of universality a formula \( (x)A \) takes the value \( T \) if \( A \) is true of all members of the domain. This can be made suitably precise by speaking of an interpretation \( I \) on a domain \( D \); such an interpretation assigns values \( I(x) \) and \( I(A) \) in \( D \) to individual variables and constants \( x \) and \( a \), and relations \( I(P) \) on \( D \) to predicate letters \( P \). Where \( d \) is a member of \( D \), let \( I^d/x \) be the interpretation like \( I \) except that \( I^d/x(x) = d \). Then:
\[
I((x)A) = T \text{ iff for all } d \in D \ I^d/x(A) = T.
\]

Corresponding to this semantics, we have axiom-schemes
\[
\begin{array}{c}
(x)(A \supset B) \supset ((x)A \supset (x)B) \\
(x)A \supset A^x/x
\end{array}
\]
where \( A^x/x \) is the result of replacing all free occurrences of \( x \)
in $A$ by occurrences of $t$ (again, relettering if necessary), and a rule of *conditional universalization*,

$$
A \supset B
\frac{}{A \supset (x)B}
$$

where $x$ does not occur free in $A$.

6. We have not yet considered the theory of descriptions, but let us adopt for the time being Russell's theory. According to this way of handling descriptions, expressions $\lambda \alpha \beta \lambda x \alpha$ are not part of the primitive notation of the language; hence, the only terms are individual variables and constants. Descriptions are introduced by means of a contextual definition, of which the following is an instance:

$$P(t, Q(x)) = \alpha \exists x ((y)P(y) \equiv y = x) \wedge P(x)).$$

Since descriptions are defined, they may be neglected in formulating a semantics for the language, leaving truth-functions, modality, identity, and universality to be considered.

By a *Q1-model structure* (Q1ms), we will understand a system $\langle \mathcal{H}, \mathcal{R}, D \rangle$, where $\langle \mathcal{H}, \mathcal{R} \rangle$ is an S4-model structure and $D$ a nonempty domain. A *Q1-interpretation* $I$ assigns a value $I(a)$ or $I(x)$ in $D$ to each individual constant or variable, respectively, of the language, and to each $n$-ary predicate letter $P$ a subset $I_a(P)$ of $D^n$ for each $\alpha \in \mathcal{H}$ (if $n = 0$, $I_a(P)$ is one of the values $T$ and $F$). Where $I$ is a Q1-interpretation on $\langle \mathcal{H}, \mathcal{R}, D \rangle$ and $d \in D$, let $I^d/x$ be the Q1-interpretation which assigns $x$ the value $d$ and in other respects is just like $I$.

The truth-value $I_a(A)$ of $A$ in $\alpha$ under a Q1-interpretation $I$ on a Q1ms $\langle \mathcal{H}, \mathcal{R}, D \rangle$ (where $\alpha \in \mathcal{H}$), is defined inductively as follows:

1. $I_a(P(t_1 \ldots t_n)) = T$ if $\langle I(t_1), \ldots, I(t_n) \rangle \in I_a(P)$,
   $I_a(P(t_1 \ldots t_n)) = F$ otherwise;
2. $I_a(s = t) = T$ if $I(s) = I(t)$,
   $I_a(s = t) = F$ otherwise;
3. $I_a(A \supset B) = T$ if $I_a(A) = F$ or $I_a(B) = T$,
   $I_a(A \supset B) = F$ otherwise;
4. $I_a(\sim A) = T$ if $I_a(A) = F$,
   $I_a(\sim A) = F$ otherwise;
5. $I_a(\Box A) = T$ if for all $\beta \in \mathcal{H}$ such that $x \mathcal{R} \beta$, $I_a(\lambda x \lambda \beta) = T$,
   $I_a(\Box A) = F$ otherwise;
6. $I_a((x)A) = T$ if for all $d \in D$, $I^d/x_a(A) = T$,
   $I_a((x)A) = F$ otherwise.

(Clauses 3, 4, and 5, of this definition are standard, and will not be repeated in later definitions of satisfaction.)

This definition determines in the usual way notions of *satisfiability, validity, simultaneous satisfiability, and implication*, which we will term Q1-satisfiability, etc.

The question now arises, how to axiomatize the notion of Q1-validity. First, it is readily confirmed that all of the axioms we have presented above for truth-functional and modal connectives, for identity and for quantifiers, are Q1-valid; also, that all the rules presented above preserve Q1-validity. It therefore seems reasonable to choose at least these axioms and rules: namely, the sentential axioms and rules, together with SId, InId, Dist, Inst, and the rule of conditional universalization. These axioms and rules, however, turn out to be Q1-incomplete. First, although

$$\Box(x)A \supset (x)\Box A \quad \text{(CVBP)}^4$$

can be obtained from them, the Q1-valid formula

$$(x)\Box P(x) \supset (x)\Box P(x)$$

cannot be thus obtained. Therefore, we must adopt another...

4. A note on nomenclature: 'BP' is for 'Barcan Principle,' and 'CVBP' for 'Converse Barcan Principle.' NecId asserts that identicals are necessarily identical, and NecDiff that different things are necessarily different.
axiom-scheme,

\((x) \square A \Rightarrow \square(x)A\) \quad (BP)^5

Second, though

\(s=t \Rightarrow \square s=t\) \quad (NecId)

can be obtained from InId and the other axioms, the \(Q1\)-valid formula

\(\Diamond a=b \Rightarrow a=b\)

cannot be thus obtained, and we therefore require still another axiom-scheme,

\(\Diamond s=t \Rightarrow s=t\) \quad (NecDif)^6

We will call the system constituted by these axioms and rules \(Q1\) (in a wider context, the name ‘\(S4Q1\)’ would be more appropriate). In the usual way, notions of \(Q1\)-theorethood, \(Q1\)-deducibility, and \(Q1\)-consistency are determined by the system. As it turns out, these syntactically defined concepts are equivalent in extension to the corresponding semantical concepts; we thus have semantic completeness for \(Q1\), in the strong as well as the weak sense.

7. The ontology corresponding to \(Q1\) is reminiscent of Leibniz. The world consists of a number of substances (the members of \(D\)) which persist through changes (conceiving of the relation \(R\), for the time being, as temporal), in which they may exchange attributes. These substances are never generated or destroyed; their number is fixed and immutable. When, on the other hand, \(R\) is taken to be a relation of metaphysical possibility^7 (and the members of \(K\) are, accordingly, metaphysically possible worlds rather than temporal stages), this corresponds to the two principles that every possible object is existent, and that every existent object is necessarily existent.^8 When possibility is understood in this way, Leibniz holds that there are possible objects that do not exist in all worlds; in fact, he asserts that there are no possible objects which exist in more than one world. \(Q1\) may therefore be regarded as corresponding to Leibniz’ notion of temporal possibility, though not to his notion of metaphysical possibility.

More generally, \(Q1\) is reminiscent of any atomistic ontology (ancient or modern), in which “atom” is construed in the strict sense: no atom can be generated or destroyed. The various possible worlds in \(K\) will correspond in this case to possible configurations of atoms; here, \(S5\) would be a more appropriate sentential modality, unless \(R\) is understood temporally and

5. As we will see, the import of BP is that nothing comes into existence; the import of CxBP is that nothing passes out of existence.

6. Intuitively, the content of NecDif is that things which are distinct in a given situation can never become identical. This suggests a way of interpreting identity so that NecDif would not be valid, by providing for the “merging” of distinct individuals (but without permitting the “separation” of identicals). If carried out, this yields a completeness theorem for the remaining axioms and rules, besides establishing the independence of NecDif.

7. I should mention at this point that I am conceiving of the notion of a model structure (and hence of a possible world), as used in the semantics of modal logic, as an abstract or structural concept having a number of kinds of realizations. (This is true in general of semantical notions—e.g. of the concept of a domain in classical quantification theory; but the abstractness is rather trivial in this case since domains have no structure.) For instance, possible worlds may be interpreted temporally, metaphysically, linguistically (as “state descriptions”) or probabilistically (as sample points in a probability space). In the present paper, I will appeal mostly to the temporal interpretation, since it is the one which seems to be most valuable heuristically where quantification theory is concerned; we have some idea of how to identify individuals across temporal worlds, since we have an idea of what temporal change is like. But “metaphysical change” is more problematic.

8. The reason for this is that CxBP guarantees that since necessarily everything (i.e. every existing thing) exists, everything necessarily exists; while BP guarantees that if everything necessarily has a property, then nothing can come into being which does not have that property. For further information on this point, see the discussion of Q3 below, sections 14 ff.
physical laws are not required to be symmetric with regard
to time.

Further insights into Q1 are gained by considering the
solution which it affords to the following stock modal paradox:

\[ \square(2 < 3) \]
\[ 3 = \text{the number of living Presidents} \]
\[ \therefore \square(2 < \text{the number of living Presidents}) \]

In general, there are a number of ways of handling such a
paradox: denying either of the premises, denying the validity
of the inference but holding that the conclusion (though
apparently false) is innocuous, or holding that any of the
statements in the argument is meaningless. Q1 takes the
third course; though it sanctions the inference, the conclusion,
once formalized according to the Russellian theory of descrip-
tions, has the form

\[ (\exists x)[(y)(NP(y) \equiv x = y) \land \square 2 < x], \]

which is harmlessly true. The paradox is thus explained away,
since despite appearances the conclusion does not turn out
to be a statement of necessity.

8. Two objections to Q1 are suggested by the above
considerations: first, that Q1 supposes that the same individu-
als are found in all possible worlds, and second, that Q1
must employ Russell's theory of descriptions. Perhaps more
should be said at this point concerning the second of these
objections, since all we have claimed so far is that this theory
explains away our Presidential paradox; but surely, explaining
away is not in itself objectionable. Quite apart from this,
however, it seems to me that the Russellian theory is infected
with inelegancies and inadequacies. The inelegancies result
from the unsperspicuous notation and awkward considera-
tions of scope arising from the contextual definition of descrip-
tions. An inadequacy closely related to this is that the formalization

of sentences from natural languages is artificial in many cases;
e.g. 'The largest prime number is even or not even' is formalized
not as an instance of excluded middle, but as an existential
quantification. Another inadequacy is that it is plausible to
regard some sentences about nonexistents as true; e.g. sentences
asserting their nonexistence. All of these objections to the
Russellian theory are interrelated in such a way that a pro-
ponent of Russell's theory can meet them by making distinc-
tions of scope; but a dialogue of this sort soon shows that this
theory results in strained relationships to natural language.
The Russellian must eventually cut his moorings and claim
that natural language is inherently "confused" or "vague."

It is significant that both of the above sources of dissatis-
faction with Q1 point in the same direction: to a revision of
classical quantification theory as regards existential pre-
suppositions. In particular, the principle called into question
is Inst; when contraposed, this yields the principle

\[ A^t/x \vdash (\exists x)A \quad \text{(CvInst).} \]

And the special case

\[ t = t \vdash \text{Et} \]

guarantees that every term will refer. But this is clearly
incompatible with allowing names to designate objects not
in the actual world (e.g. objects which did exist, but don't at
this time), as well as with allowing descriptions to be primitive.
Both modifications force us to give up the assumption that
individual terms always refer.

Now, systems of quantification theory modified in this way
have been proposed and investigated during the past ten years

9. The first two difficulties can be resolved by allowing descrip-
tions to be primitive, and assigning any formula the value F if it contains a description not
meeting the unique-existence condition; if the condition is met, the description
is assigned the obvious designatum. This semantical theory of descriptions is
readily axiomatized and proved complete; but this involves revising classical
sentential logic, and does not meet our third objection to the Russellian theory.
or so by a number of logicians,\textsuperscript{10} and their semantical and syntactical characteristics disclosed. As far as the predicate calculus without modality and descriptions is concerned, at least two semantical interpretations have been proposed: one by van Fraassen, and one by Leblanc and myself. Although van Fraassen’s theory is better adapted to description theory, I will confine my remarks here to the latter of these interpretations, which is easier to formulate. According to this interpretation there are two (disjoint) domains: a nonempty “inner” domain $D$ and an “outer” domain $D'$. Individual terms are assigned values in either domain, and predicate letters assigned relations on the union of the domains; thus $P(a)$, e.g., will be either true or false under any interpretation even if $a$ is assigned a “nonexistent” value, in the outer domain. Bound variables are construed as ranging over the inner domain, so that $I((x)A) = T$ if for all $d \in D$, $I^d(x)A = T$.

Syntactically, the formulas valid under this interpretation may be characterized by axioms and rules sufficient to generate all tautologies, together with the rule of conditional universal generalization, the axiom-schemes $\text{InId}$, $\text{SId}$, $\text{Dist}$, and the following axiom-schemes:

\[
(x)A \supset (Et \supset A^t/x) \quad \text{(EIInst)}
\]

\[
(x)Ex \quad \text{(NEm)}
\]

\[
(x)Ex \quad \text{(EU)}.
\]

Following Lambert’s terminology, we call this logic free quantification theory; the system determined by the above axioms and rules will be called $F$.

9. Free quantification theory is to be used as the basis of a more natural theory of descriptions, and of a modal semantics allowing the generation and destruction of individuals. I will first turn to the theory of descriptions. The account I will describe is similar to one due to van Fraassen; he and Lambert also have a completeness theorem for it.\textsuperscript{11} According to this theory, the value $I(\_xA)$ assigned by $I$ to $\_xA$ is the unique member $d$ of the inner domain, such that $I^d/x(A) = T$, if there exists such a unique member; otherwise, $I(\_xA)$ is an arbitrary member of the outer domain (in description theory, then, we must assume the outer domain to be nonempty).

It turns out that only two simple and plausible axioms are needed to suit this interpretation; these, in conjunction, of course, with $F$, yield a minimal kind of description theory.

\[
E_\_xA \supset (\_!x)A \quad \text{(DE)}
\]

\[
(\_)(y)[A^y/x \equiv x = y] \supset x = \_xA \quad \text{(DI)}
\]

DI is subject to the restriction that $y$ have no free occurrences in $A$.

10. We may now proceed to sketch a semantical interpretation of the whole language: truth-functions, modality, universal quantification, identity, and descriptions. The resulting theory $Q2$ will play a dialectical role in the present paper, serving to introduce problems which (I shall claim) are resolved by the theory $Q3$. Anyone who tries to verify whether particular formulas are valid in $Q2$ will soon find that this notion of validity is very complicated. As it turns out, this complexity is more than just apparent; David Kaplan has informed me that there is no axiomatization of $Q2$—the set of $Q2$-valid formulas is not recursively enumerable. (This result is unpublished, and depends on work which appears in Kaplan’s dissertation.) With this in mind, let’s turn to a semantical characterization (which, of course, will be infinitistic) of $Q2$.

11. Clearly, our semantical interpretation will involve a system of possible worlds, each with a domain. Besides these.

\textsuperscript{10} They differ from the ones given in my “Some Completeness Results for Modal Predicate Calculi,” Proceedings of the 1968 Irvine Philosophy Colloquium, forthcoming.
we will want still another set to serve as an outer domain, since otherwise every singular term (including, e.g., $\gamma \alpha (P \land \neg P)$) would perforce be assigned an individual existing in some possible world.

Individual terms are assigned values in the domains of these possible worlds. When we did this in connection with Q2, we assigned each term a value which was fixed beforehand for all worlds; the value assigned to $a$ in $\alpha$ was the same as the value assigned to $a$ in $\beta$. This procedure seemed natural at the time, because we had supposed that the same domain was associated with all worlds; but this assumption glosses over the problem of identifying individuals across worlds.

In the present more general case, then, we define a Q2-model structure $\langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \mathcal{D}' \rangle$ in such a way that the component $\mathcal{D}$, rather than being a fixed domain, is a function taking worlds $\alpha$ into nonempty domains $\mathcal{D}_{\alpha}$. $\mathcal{D}'$ is a nonempty domain serving as a "limbo" for individuals not existing in any world in $\mathcal{K}$. As before, we suppose that $\langle \mathcal{K}, \mathcal{R} \rangle$ is an S4 model structure.

Before determining how to assign values to terms in general, let us consider a description: say, 'the thing in place $p$.' We fix the designation of this term in a given world according to whether there is a unique individual in the domain of that world having the property of being in place $p$ in that world; if there is, we assign that thing to the term. But individuals can change position, and in general the designatum of $\gamma \alpha P(x)$ in world $\alpha$ and the designatum of $\gamma \beta P(x)$ in world $\beta$ will not be the same.

Applying this idea to terms in general, let us regard a Q2-interpretation, $I$, on a Q2 model $\langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \mathcal{D}' \rangle$ as assigning for each $\alpha \in \mathcal{K}$ a value $I_{\alpha}(t)$ in $\mathcal{D}' \cup \bigcup_{\alpha \in \mathcal{K}} \mathcal{D}_{\alpha}$ to each individual variable or individual constant $t$; and to each description $\gamma \alpha A$ a value $I_{\alpha}(\gamma \alpha A) = d$, where if there is a unique member of $\mathcal{D}_{\alpha}$ such that $I_{\alpha}(x(A)) = T$, then $d = e$, and if there is no such member, then $d \notin \mathcal{D}_{\alpha}$; and finally, to each $n$-ary predicate letter $P$ a relation $I_{\alpha}(P)$ on $\mathcal{D}' \cup \bigcup_{\alpha \in \mathcal{K}} \mathcal{D}_{\alpha}$. Predicate letters, then, are assumed to be defined for all individuals; those in $\mathcal{D}'$, as well as those in the domains of the possible worlds. In this way, excluded middle is preserved.

The definition of Q2-satisfaction is straightforward in the case of atomic formulas:

1. $I_{\alpha}(P(t_1, \ldots, t_n)) = T$ if $\langle I_{\alpha}(t_1), \ldots, I_{\alpha}(t_n) \rangle \in I_{\alpha}(P)$,
   $I_{\alpha}(P(t_1, \ldots, t_n)) = F$ otherwise;
2. $I_{\alpha}(s = t) = T$ if $I_{\alpha}(s) = I_{\alpha}(t)$,
   $I_{\alpha}(s = t) = F$ otherwise.

An atomic formula, e.g., $P(a)$, is true in a possible world if and only if the relation assigned to $P$ in that world holds of the individual assigned to $a$ in that world.

Cases (3), (4), and (5) of the definition are treated as on p. 125, above. Using only these clauses of the definition, it may be seen that

$$s = t \rightarrow (A \supset A^S/t)$$

is valid provided that no occurrence of $t$ in $A$ at which an occurrence of $s$ appears in $A^S/t$ is within the scope of a modal operator (in other words, provided that $s$ is substituted for $t$ only in transparent contexts). Let us call this qualified form of InId "RInId" ("Restricted Indiscernibility of Identicals"). Although RInId is Q2-valid, InId is not. E.g., let $\mathcal{D}_2 = \{0, 1\}$, where $\alpha \neq \beta$, and $\mathcal{R}_\alpha \beta$, and let $I_\alpha(a) = I_\beta(b) = 0$ and $I_\beta(a) = I_\alpha(b) = 1$. Then $I_\alpha(a = b \supset (\Box a = a \supset \Box a = b)) = F$ and $I_\beta(a = b \supset \Box a = b) = F$ as well. The principle

$$\Box s = t \rightarrow (A \supset A^S/t) \quad \text{(In}\Box\text{Id)},$$

12. Strictly speaking, this is sloppy. Actually, Q2-interpretation and truth-value under a Q2-interpretation should be defined by a simultaneous induction.
however, is $Q_2$-valid. It is clear, therefore, that in $Q_2$ the
Presidential paradox (above, p. 128) is solved by denying the
validity of the argument, which depends on the principle InId.
The force of the paradox may then be ascribed to the failure to
distinguish identity from necessary identity, with resultant
confusion of InId with $\square$Id.

All of this may be seen without completing the definition of
satisfaction; at some point, however, we must determine the
truth-conditions of universal formulas. When $A$ contains no
modal operators, it is clear that $I_d((x)A)$ should be $T$ if and
only if for all $d$ in $D_x$, $I_d((x)A) = T$ for all interpretations $I'$
differing from $I$ at most in the value assigned to $x$ in $a$, and
such that $I'_d(x) \in D_a$. But, e.g. in the case of $(x)P(x)$, the
truth-value in $a$ will depend in general not only on the interpreta-
tion of $x$ in $a$, but on the values assigned to $x$ in other worlds
as well. How, then, are we to define the truth-value in such
cases?

There are some radical approaches to this problem which
merit brief discussion. First, one could restrict the language
in such a way that well-formed formulas do not involve any
quantification into modal contexts. When bolstered by
philosophic arguments, this approach may appear plausible,
but at bottom these arguments simply restate the fact that
there is a problem. Thus, this seems to me to be more a suppres-
sion than a solution of the difficulty; certainly, if this technique
were generally applied to areas in which difficult problems
arise, life would be rather dull.

Another solution would be to interpret quantification as
having to do with the substitution of names; we could then say
that $I_d((x)A) = T$ if for all terms $t$ of the language, $I_d(A^t/x) = T$.
This proposal raises a host of technical difficulties. Obviously,
it conflicts with the decisions made above concerning existence,

since Inst is valid under this interpretation. Even if this and
related difficulties were solved, a more vexing problem would
arise, concerning descriptions. The definition of satisfaction
under this interpretation of quantification presupposes that
the truth values of all identities are fixed beforehand, but the
definition of identities involving descriptions presupposes in
turn the definition of satisfaction. This circularity is vicious:
the usual sort of inductive definition breaks down, and there is
no straightforward way of guaranteeing that a satisfaction
function $I$ exists, given an assignment of values to atomic
formulas and atomic terms. Of course, this problem could be
solved by appealing to Russell's theory of descriptions or a
similar theory; but this limitation is surely a defect. Further
technical difficulties arise concerning the definition of implica-
tion, since under the substitution interpretation the rule of
complete induction is valid. It therefore follows from Gödel's
incompleteness theorem (or, equally well, from Tarski's
theorem) that any effective formulation of this quantification
theory will be incomplete as to consequences, since all of the
truths, e.g. of first-order arithmetic, would be implied by the
usual axioms; but the class of arithmetic truths is not effectively
enumerable. To top these difficulties, there is the more philo-
sophical objection that the substitution interpretation is
plausible as an interpretation of quantification only if every-
thing has a name. But this last condition is not always met,
and there does not seem to be any good reason to suppose in
logic that it is. For these reasons, though I feel that the substi-
tution interpretation (i.e. the rule of complete induction) is
worthy of investigation in its own right, it does not seem to me
that substitution is a satisfactory surrogate for quantification
over individuals.

The most attractive remaining alternative—certainly, the
one most in harmony with the semantic approach of this
paper—is the one in which formulas of the sort $(x)A$ are

13. This has been suggested by R. Barcan Marcus.
interpreted universally with respect to all ways of identifying the value of \( x \). We make this more precise by letting a world-line \( d \) on a \( Q2 \)-model structure \( \langle \mathcal{X}, \mathcal{R}, \mathcal{D}, \mathcal{D}' \rangle \) be any function from \( \mathcal{X} \) into \( \mathcal{D}' \cup \bigcup_{\alpha \in \mathcal{X}} \mathcal{D}_\alpha \). Also, let \( I^d/x \) differ from \( I \) in that for all \( \alpha \in \mathcal{X} \), \( I^d/x_\alpha(x) = d_\alpha \). The final clause of \( Q2 \)-satisfaction then reads as follows:

\[
(6) \quad I_\alpha(x)A = T \text{ if for all world-lines } d \text{ on } \langle \mathcal{X}, \mathcal{R}, \mathcal{D}, \mathcal{D}' \rangle \\
\text{such that } \ d_\alpha \in \mathcal{D}_\alpha, \ I^d/x_\alpha(A) = T, \ I_\alpha(x)A = F \text{ otherwise.}
\]

The only technical difficulty with this definition is that the class of world-lines may be nondenumerably infinite, even in cases where \( \mathcal{D}' \) and the \( \mathcal{D}_\alpha \) are denumerable; but this will seem a glaring defect only to those who do not countenance non-denumerable infinities. More problematic is the fact that certain rather counterintuitive theses turn out to be validated in \( Q2 \). An example of this sort is

\[
(\exists x)\square E x;
\]

this ontologically tantalizing conclusion follows from the more general principle

\[
\square(\exists x)A \supset (\exists x)\square A,
\]

where no free occurrence of \( x \) in \( A \) falls within the scope of a modal operator—which also is \( Q2 \)-valid.

12. Though I believe \( Q2 \) and related logics to be worthy of further investigation, I take these results to be symptomatic of a deeper infirmity, having to do with the notion of substance. Recall that above, when we were motivating the sort of reference involved in \( Q2 \), we used descriptions as a paradigm. The effect of this was to suggest that the fundamental way of identifying individuals across worlds was by means of properties (e.g., we identify the thing in place \( p \) by means of a distinguishing property), and this in turn suggested that all world-lines are semantically on a par. When all such identifications are subsumed under this paradigm, one is led to the conclusion that, e.g., Socrates-at-\( t_1 \) is identified with Socrates-at-\( t_2 \) (i.e., shares with Socrates-at-\( t_2 \) a world line which is a value of a variable), because the properties of the one are like the properties of the other. And the slipperiness of this “likeness” soon suggests that it is arbitrary or conventional that these are given the same name: any world-line is, in reality, as worthy to be the value of a variable as any other.

In contrast to the view sketched in the above paragraph, it seems to me that we do—at least, in temporal cases—identify individuals across worlds in a way more absolute than that taken into account by \( Q2 \). Thus, I am inclined to take the sort of reference in which a name ("Socrates") is assigned one thing (Socrates) which is the same in many possible worlds, as primary or paradigmatic. The defect, then, of \( Q2 \), is that it does not allow for this unity: for preferred world-lines which may be regarded as single things remaining fixed through a change. In a word, \( Q2 \) lacks a concept of substance.

13. Before formulating a system \( Q3 \) in accordance with these suggestions, it may be appropriate to support the above opinions by a philosophical consideration or two. It may be that those who held views similar to those motivating \( Q2 \) were led to do so by the impression that identification of properties across worlds is somehow less problematic than identification of individuals across worlds; this, certainly, would be a reason to attempt the elimination in modal logic of singular terms in favor of distinguishing properties. But there is, so far as I can see, absolutely no reason to suppose that this is the case: if anything, the identification of properties is more problematic. How can properties be identified across worlds (without, of course, appealing to substances)?

Certainly, not by means of their extension, for this presupposes identification of individuals. Perhaps, then, by means
of properties of properties; but this obviously leads to an infinite regress. If there is reason to cut this regress anywhere in order to allow a foundation for the whole series, it is at the very beginning: namely, with identification of individuals. Finally, perhaps by intensional criteria; but these would have to be explicated. If the explication were to involve the notion of alternative situations or possible worlds, the problem would remain unsolved; but it is hard to imagine an adequate explication that would not involve this notion in some form.  

These arguments are not without double edges. For instance, if I were pressed as to how individuals are identified and give any sensible answer (say, "by continuity"), I could be led in similar circles. But I do feel that such arguments can legitimately be used to remove prejudices or misconceptions that may stand in the way of Q3.

14. As in the case of Q2, a Q3-model structure is a quadruple \( \langle W, R, D, I \rangle \). Now however, we will construe overlaps of the \( D_\alpha \) as indicative of substances; i.e. we regard an individual as existing in both \( \alpha \) and \( \beta \) if it is a member of \( D_\alpha \cap D_\beta \). More precisely, we will regard all individuals in \( D' \cup \bigcup_{\alpha \in X} D_\alpha \) as already identified across worlds; any individual \( d \) of a Q3ms is the same with respect to all worlds of the model structure. We will construe individual variables as ranging over substances, and thus will take seriously the classical doctrine that

only substances are "beings" in the fullest sense of the word. In practice, this means that if an individual variable is assigned a value in world \( \alpha \), it is automatically assigned the same value in every other world \( \beta \) (whether or not it is in the domains of these worlds). Thus, so far as the interpretation of individual variables goes, we are reverting to the sort of reference we had in Q1.

As regards descriptions, however, we will retain the treatment of Q2; this means that descriptions may refer to non-substances, even though the individual variables range in Q3 over substances; there is thus a major difference in the way Q3 treats individual variables, on the one hand, and descriptions, on the other. But this is precisely the distinction that is wanted, and creates no difficulties since the underlying quantification theory is free; descriptions need not designate values of individual variables.

The question of how to handle individual constants is delicate. Most uses of proper names seem to be of the sort in which substances are intended; e.g. 

\[
(x)(x = \text{Socrates} \Rightarrow \Box x = \text{Socrates})
\]

and

\[
(x)(x = \text{Texas} \Rightarrow \Box x = \text{Texas})
\]

have a ring of truth. This would suggest that in Q3 individual constants should be treated like individual variables. There are, however, some uses of proper names (roughly classifiable as titulary uses) which do not meet this condition. For example, if we call anyone holding a certain political office "Caesar," then various people may be Caesar in different possible worlds, and

\[
(x)(x = \text{Caesar} \Rightarrow \Box x = \text{Caesar}),
\]

or equivalently,

\[
E\Box \text{Caesar}
\]

is false. Similarly with "Miss America," and other cases in which a name is won. "Coriolanus" is perhaps a borderline
case. It seems rather artificial to make a syntactical distinction between these and other sorts of namings, and therefore in Q3 I will treat individual constants more like descriptions than individual variables; any world-line may be assigned to an individual constant. The supposition that an individual constant refers to a substance can, however, always be made explicitly, by means of assertions of the sort

$$E \Box a.$$

15. A **Q3-interpretation** on a Q3ms $\langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \mathcal{D}' \rangle$, then, is an assignment I of members of $\mathcal{D}' \cup \bigcup_{a \in \mathcal{K}} \mathcal{D}_a$ to individual variables, world-lines on $\langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \mathcal{D}' \rangle$ to individual constants, and relations on $\mathcal{D}' \cup \bigcup_{a \in \mathcal{K}} \mathcal{D}_a$ to predicate letters (one for each world in $\mathcal{K}$). This may be summarized by saying that a Q3-interpretation is a Q2-interpretation, except that for all individual variables $x$ and all $\alpha, \beta \in \mathcal{K}$, $I_\alpha(x) = I_\beta(x)$. Descriptions are handled precisely as in Q2.

Let $I^d/x$ differ from $I$ at most in that for all $x \in \mathcal{K}$, $I^d/x_a(x) = d$. Then the most important clauses of the definition of Q3-satisfaction are:

1. $I_\alpha(P(t_1 \ldots t_n)) = T$ if $I_\alpha(t_1), \ldots, I_\alpha(t_n) \in I_\alpha(P)$,
   $I_\alpha(P(t_1 \ldots t_n)) = F$ otherwise;
2. $I_\alpha(s = t) = T$ if $I_\alpha(s) = I_\alpha(t)$,
   $I_\alpha(s = t) = F$ otherwise;
3. $I_\alpha((x)A) = T$ if for all $d \in \mathcal{D}_x, I^d/x_d(A) = T$,
   $I_\alpha((x)A) = F$ otherwise.

16. In seeking an axiomatization of Q3, we must first note that the principle EInst is Q3-invalid. To Q3-satisfy, e.g. the negation of

$$\forall x \exists E x \Rightarrow (E a \Rightarrow \Box E a)$$

(an instance of EInst), let $\mathcal{D}' = \{0\}$, and $\mathcal{D}_a = \mathcal{D}_\beta = \{1\}$, where $\alpha \neq \beta$ and $\alpha \not\in \mathcal{R} \beta$. Then let $I_\alpha(a) = 1$ and $I_\beta(a) = 0$. Now,

$$I_\alpha(x \Box E x) = T \text{ (because } \mathcal{D}_x \subseteq \mathcal{D}_\beta), \text{ and } I_\beta(Ea) = T; \text{ but }$$

$$I_\beta(Ea) = F, \text{ and so } I_\beta(\Box Ea) = F.$$  

The reason for this failure of EInst in Q3 is clear; the variables of Q3 range over substances; i.e. over objects identified across worlds. Thus, terms which fail to refer to the same object in all worlds, as well as terms not referring to existents, may not instantiate such variables. This diagnosis indicates how to remedy the difficulty: in Q3, we posit the axiom-scheme

$$(x)A \Rightarrow (E \Box t \Rightarrow A^t/x) \quad (\text{SInst}).$$

Besides SInst, we will incorporate in Q3 the sentential axioms and rules, Dist, NEm, and EU; also the principles DI and DE, and SId and RnInd. Our treatment of free individual variables renders two further axiom-schemes indispensible:

$$x = y \Rightarrow \Box x = y, \text{ and } \Box y = x \Rightarrow x = y,$$

where $x$ and $y$ are individual variables.\(^\text{16}\)

15. The system Q3Em obtained by dropping the Axiom-scheme EM from Q3 is characterized semantically by relaxing the restriction on Q3 model structures $\langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \mathcal{D}' \rangle$ that $\mathcal{D}_x$ be nonempty for all $x \in \mathcal{K}$.

16. In the proof of semantical completeness given in my "Some Completeness Results for Modal Predicate Calculi," five additional rules must be incorporated in Q3.

$$A \Rightarrow B \quad \text{ where } x \text{ is not free in } A.$$  

$$A \Rightarrow \Box B \quad \text{ where } x \text{ is not free in } A.$$  

$$A \Rightarrow B_1 \ldots \ldots \ldots \searrow B_n \searrow \Box C \quad \text{ where } x \text{ is not free in } A, B_1, \ldots, \text{ or } B_n.$$  

$$A \Rightarrow \sim t = x \quad \text{ where } x \text{ is not free in } A \text{ or in } t.$$  

$$A \Rightarrow B_1 \ldots \ldots \ldots \searrow B_n \searrow \sim t = x \quad \text{ where } x \text{ is not free in } A, B_1, \ldots, B_n, \text{ or } t.$$  

It is not known at present whether or not these rules are redundant.
The system Q3 given by these axioms and rules is semantically complete, in both the weak and strong senses.

17. The above list of axioms and rules, and the definition of Q3-validity, in all likelihood will provide the reader with only a rough idea of what the system Q3 is like. It may be helpful at this point to list some Q3-valid, Q3-invalid, and Q3-satisfiable formulas. Of course, every substitution-instance of a sentential theorem of S4 is Q3-valid; so also is any substitution-instance of a theorem of classical quantification theory without free individual terms. E.g. any formula of the kind \((x)(y)A \supseteq A^x/y\) is Q3-valid. Besides these, all formulas of the following sorts are Q3-valid:

\[ E^a_{x} P(x) \supset P(\bar{x} P(x)), \]
\[ (y)(y = \bar{x} P(x) \equiv (P(y) \land (P(x) \equiv x = y))), \]
\[ \Box (\exists x) \Box \Box x, \]
\[ \Box (x) \Box \Box x, \text{ and} \]
\[ (x)(y)(x = y \supset (A \supset A^x/y)) \text{ InIdS}. \]

The following formulas, however, are Q3-invalid:

\[ (x) \Box P(x) \supset \Box (x) P(x) \]
\[ \Box (x) P(x) \supset (x) \Box P(x) \]
\[ \Box (x) P(x) \supset (x) \Box P(x) \text{ and} \]
\[ a = \bar{x} P(x) \supset (\Box Q(a) \supset \Box Q(\bar{x} P(x))) \]

The first three of these formulas may be falsified in a Q3-model structure with only two worlds \(\alpha\) and \(\beta\), where \(\alpha \not\equiv \beta\). To falsify (1), let \(\mathcal{D}_x = \{0\}\) and \(\mathcal{D}_\beta = \{0,1\}\), and let \(I_\alpha(P) = \{0\}\) and \(I_\beta(P) = \{0\}\). Then \(I_\alpha((x) \Box P(x)) = T\), but \(I_\beta((x) P(x)) = F\), and hence (1) is false. (Note that this is a case of coming to be.) To falsify (2), let \(\mathcal{D}_x = \{0\}\) and \(\mathcal{D}_\beta = \{0\}\), and let \(I_\alpha(P) = \{0,1\}\) and \(I_\beta(P) = \{0\}\). Since \(1^1/x_\alpha(P(x)) = F\), \(1^1/x_\alpha(\Box P(x)) = F\), and hence \(I_\alpha((x) \Box P(x)) = F\), since \(1 \in \mathcal{D}_\alpha\). But, \(I_\beta((x) P(x)) = T\), and hence (2) is false. (Note that this is a case of passing away.) Finally, to falsify (3), let \(\mathcal{D}_x = \{0,1\}\) and \(\mathcal{D}_\beta = \{0\}\), and

\[ I_\alpha(P) = \{0\} \text{ and } I_\beta(P) = \{0\}. \text{ Now, } I^1/x_\alpha(P(x)) = F \text{ and } I^1/x_\beta(P(x)) = F, \text{ so } I^1/x_\alpha(\Diamond P(x)) = F \text{ and hence } I_\alpha((x) \Diamond P(x)) = F, \text{ since } 1 \in \mathcal{D}_\alpha. \text{ But, } I_\beta((x) P(x)) = T, \text{ so } I_\alpha(\Diamond (x) P(x)) = T; \text{ hence (3) is false.} \]

The duals of (1)–(3) are, of course, also invalid.

The formula \((x) \Diamond \sim Ex\) is Q3-satisfiable, as well as the slightly more gloomy formula \((x) \Box \Diamond \sim Ex\). It therefore is consistent in Q3 to suppose that everything is permissible, or even that everything is necessarily permissible. (It would, of course, be Q3-inconsistent though to assert that everything is simultaneously permissible.)

18. Finally, it will be worthwhile to discuss the way in which Q3 solves the presidential paradox; like Q2, the system Q3 renders the argument of the paradox—which has the form

\[ \begin{array}{c}
a = \bar{x} P(x) \\
\Box Q(a) \\
\therefore \Box Q(\bar{x} P(x))
\end{array} \]

and is thus an instance of (4), above—invalid, and so blocks the paradoxical inference. But besides providing a well-articulated semantical theory in which the argument is invalidated, Q3 disarms the paradox in a deeper and more significant sense, by at once explaining why the argument of the paradox is seductive, and at the same time fortifying our resistance to those seductions.

The argument of the paradox is plausible because it is an instantiation of InIdS (i.e. of \((x)(y)(x = y \supset (A \supset Ay/x))\)), which is Q3 valid. But the semantical theory of Q3 makes it clear that the universality of InIdS applies only to substances, not to nonsubstantial world-lines. It is therefore a conflation of substance and accident that makes the paradox plausible; and in retrospect it is not surprising that if this conflation is made, one should be able to generate modal paradoxes. The difficulty of this puzzle was compounded by the uncritical
acceptance during the first half of our century of the principle
Inst; thus, the crucial step in loosening this modal aporia
must be credited to Leonard, Lambert, Hintikka, Leblanc,
Hailperin, and other workers in the logical analysis of existence.

19. Some readers may feel that the present paper has been
long on logic and short on metaphysics. It would have been
possible to render the above material more philosophical and
less technical, but it is my feeling that, at the present stage of
philosophic inquiry into these topics, what is most needed is a
firm technical foundation; and this I have tried to provide.
I hope, though, that the relevance of this discussion to tradi-
tional metaphysical concepts is clear. And—though I have not
made this explicit either—its relevance to modern philosophical
treatments of logical problems concerned with modality,
reference, and existence will be apparent to anyone familiar
with the literature on this topic.

BIBLIOGRAPHY

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of the theory Q1 formulated in the present paper. During the
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several researchers, including Montague, Kanger, and Kripke.
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article [7]; references to other semantical theories of modali-
ty (e.g. the algebraic interpretation of McKinsey and Tarski)
are also given there. During this period, all of the above three
authors were developing or had developed a semantical theory
of quantification as well; quantification is treated explicitly in
Kanger[5] and Kripke [6], and a quite sophisticated theory is
sketched in Montague [11]. In the 1960s these ideas have
crystallized and developed into general, rigorous semantical
theories of modal languages with quantifiers. Much of this
recent research is still unpublished, and to my knowledge, the
mathematical details of a general theory have not yet appeared
in print. But expositions of such theories have been published
during the past six years; among the most important of these
are Hintikka [2] and [3], and Kripke [8].

This account is far from being historically complete; other
logicians who have contributed to the subject are Dana Scott,
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