Information-Theoretic Results on Communication Problems with Feed-forward and Feedback

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1. Source Coding with Feed-forward
   - Point to Point Source Coding (Chap. 2)
     - Problem Statement
     - Rate-Distortion Function
     - Error Exponents
   - Computation (Chap. 4)
   - Multiple Descriptions (Chap. 5)
2. Channel Coding with Feedback (Chap. 3,4)

Thesis Outline

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     - Problem Statement
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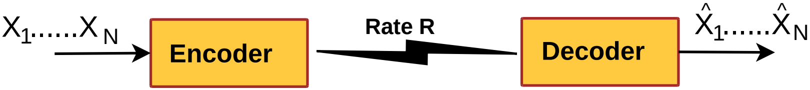
   - Multiple Descriptions (Chap. 5)
2. Channel Coding with Feedback (Chap. 3,4)

Lossy Data Compression



- Source X, reconstruction $\hat{X}$
- Distortion measure $d(X, \hat{X})$

Rate-distortion function

Minimum rate R for distortion level D (Shannon '59)

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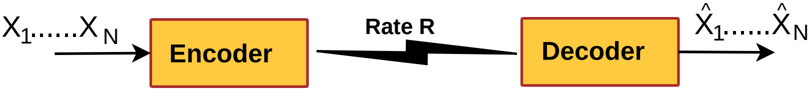

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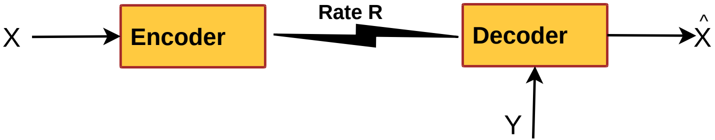
Source Coding with Side-Information



- X, Y correlated random variables
- *Example*: Temperature at nearby cities
- Presence of $Y \Rightarrow$ lower rate for given distortion D

Rate-distortion function $R_{WZ}(D)$ [Wyner,Ziv '76]

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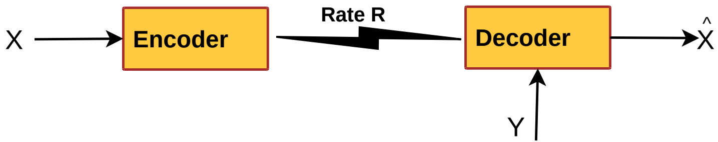

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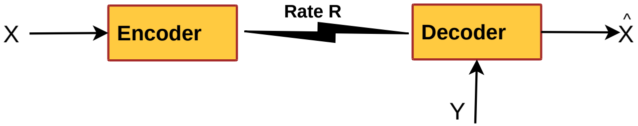

Example: Block length $N = 5$

Time	1	2	3	4	5	6	7	8	9	10
Source	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
Encoder	-	-	-	-	W	-	-	-	-	W
Side Info	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}
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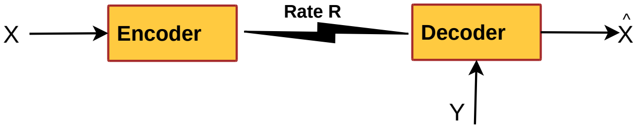

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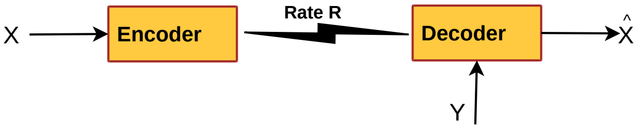

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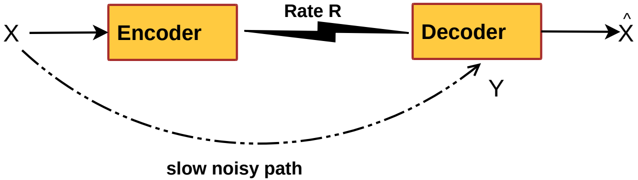

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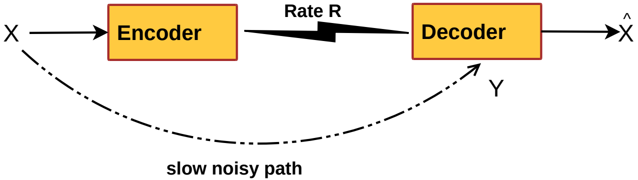

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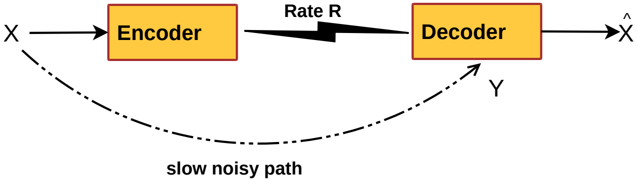

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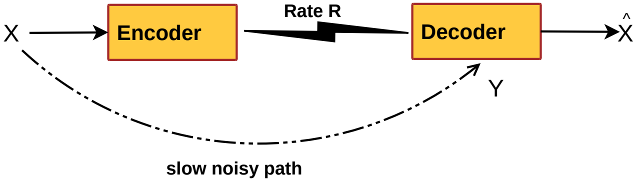

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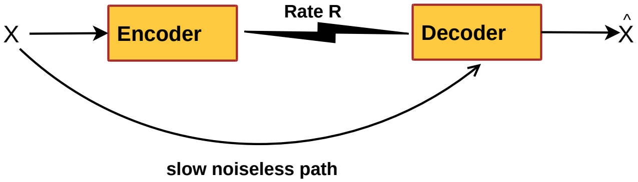
What is feed-forward?

The source field itself available with delay at decoder.



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Here, block length= 5, delay= 6 time units.

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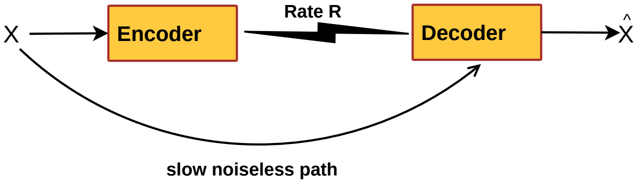

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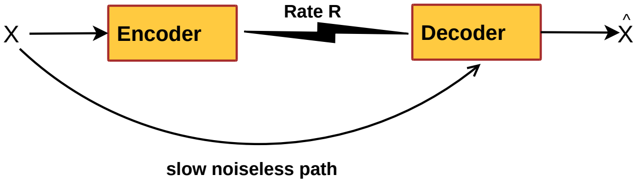

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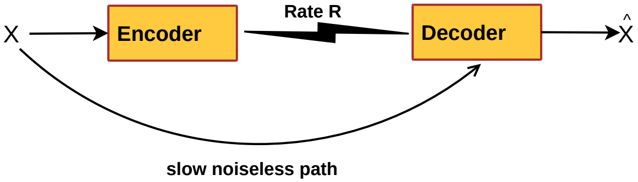

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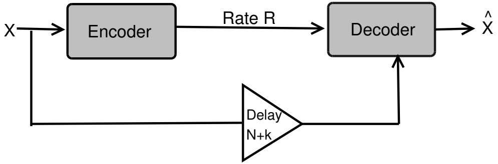
Source Coding with Feed-Forward

- Feed-forward$\Rightarrow$ Decoder knows some of the past source samples.

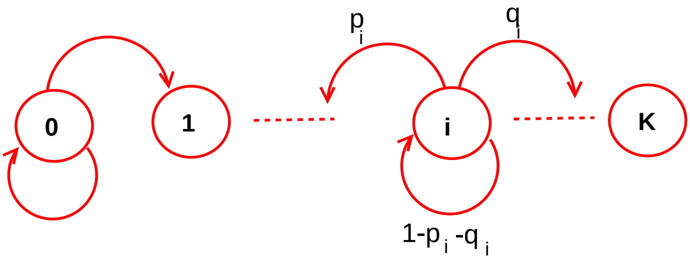


FF with delay k, block length N.

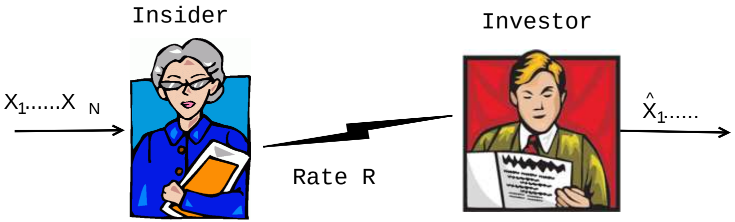
- To produce $\hat{X}_n$, the decoder knows index W and $(X_1, \ldots, X_{n-k})$.

Stock Market Example

- Behavior of a particular stock over an N-day period.
- Stock price- modeled as a $k+1$-state Markov chain.
- Value of stock: Markov source $\{X_n\}$

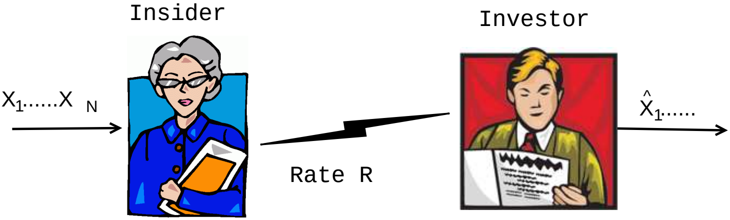


Insider and Investor



- Insider- *a priori* knowledge about behavior of stock
- Investor has stock for N days, needs to know when value drops.
- Insider: gives information to investor at rate R.

Insider and Investor



Reconstruction

- Decision of investor on day n: $\hat{X}_n$
- $\hat{X}_n = 1 \Rightarrow$ price drop from day $n-1$ to n
- Otherwise $\hat{X}_n = 0$
- Distortion $= 1$ if $\hat{X}_n$ is wrong

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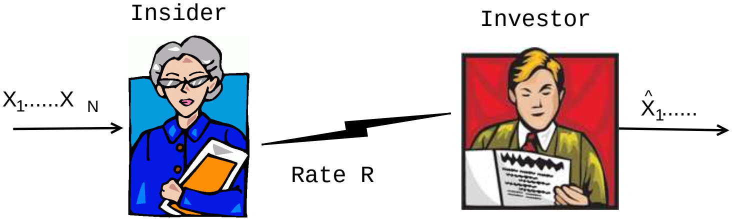

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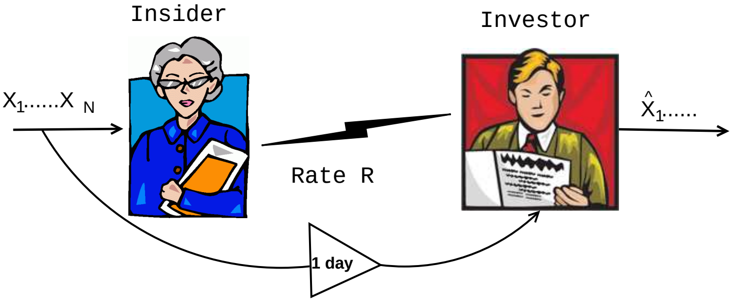
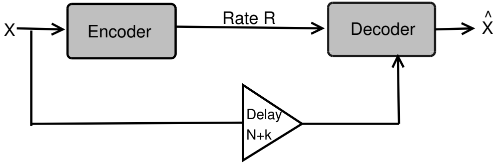


Feed-forward!

- Before day n, investor knows previous stock values X^{n-1}, makes decision $\hat{X}_n$
- Minimum info (in bits/sample) needed to predict drops with distortion D?

Feed-Forward: A Formal Definition

Previous: [Weissman et al 03], [Pradhan 04], [Martinian et al 04]



- **Source** X: Alphabet $\mathcal{X}$, reconstruction alphabet $\widehat{\mathcal{X}}$
- **Encoder**: Rate R , $e : \mathcal{X}^N \rightarrow \{1, \ldots, 2^{NR}\}$
- **Decoder**: knows all the past $(n - k)$ source samples to reconstruct nth sample.

$$g_n : \{1, \ldots, 2^{NR}\} \times \mathcal{X}^{n-k} \rightarrow \widehat{\mathcal{X}}, \quad n = 1, \ldots, N.$$

A Formal Definition (contd.)

- Distortion measure $d_N(X^N, \hat{X}^N)$.

GOAL

Given any source X, find the least R such that

$$E[d_N(x^N, \hat{x}^N)] \leq D.$$

- Rate-Distortion function with Feed-forward!

Directed Information

- Inspired by [Marko '73]: work on *bidirectional communication*
- [Massey '90] The directed information flowing from A^N to B^N

$$I(A^N \to B^N) = \sum_{n=1}^{N} I(A^n; B_n | B^{n-1}).$$

- Interestingly :

$$I(A^N \to B^N) = I(A^N; B^N) - \sum_{n=2}^{N} I(B^{n-1}; A_n | A^{n-1})$$

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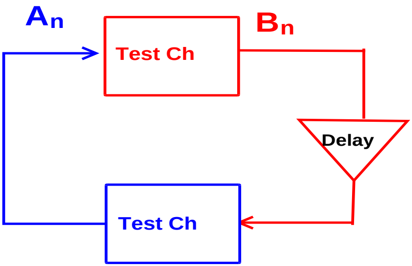
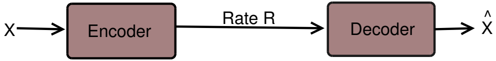
Causality in Information Flow

$$I(A^N; B^N) = I(A^N \to B^N) + I(0B^{N-1} \to A^N)$$

- $I(A^N \to B^N)$: How causal knowledge of A_n's reduces the uncertainty in B_n
- Example: GNP vs Money Supply [Geweke '82]



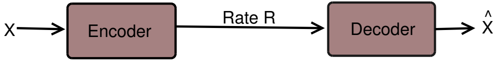
Interpretation of Directed Information



Without FF, need $I(\hat{X}^N; X^N)$ bits to represent X^N with $\hat{X}^N$.

- With feed-forward, to produce $\hat{X}_n$, the decoder knows X^{n-1}.
- Number of bits required is reduced by $I(\hat{X}_n; X^{n-1}|\hat{X}^{n-1})$.

Interpretation of Directed Information



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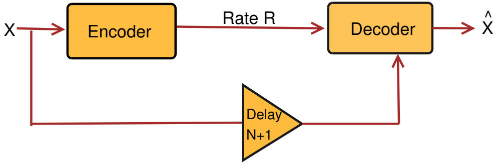

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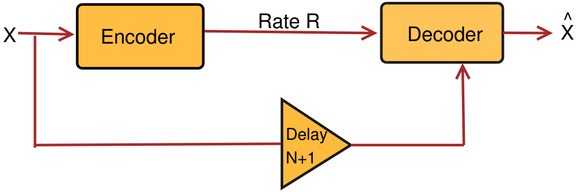
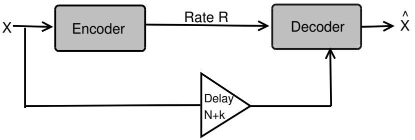


- With delay 1 feed-forward, we need

$$I(\hat{X}^N; X^N) - \sum_{n=2}^{N} I(\hat{X}_n; X^{n-1} | \hat{X}^{n-1}) \quad \text{bits.}$$

- Directed information from $\hat{X}^N$ to X^N !

Delay k feed-forward



- With delay k feed-forward, to produce $\hat{X}_n$, the decoder knows X^{n-k}.
- No. of bits: $I(\hat{X}^N; X^N) - \sum_{n=k+1}^{N} I(\hat{X}_n; X^{n-k}|\hat{X}^{n-1})$
- Not Directed Information- will denote it $I_k(\hat{X}^N \to X^N)$
  - '$k-$directed information'.

Rate-distortion function

- Optimize I_k over $P_{\hat{X}^N|X^N}$ that satisfy distortion
- No savings for discrete memoryless source w/ single-letter distortion measure

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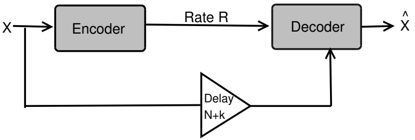

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General source, general distortion measure

- Even when source is stationary,ergodic:
  - with feed-forward, optimal joint distribution may not be.

- Source could be non-stationary, non-ergodic
- Sequence of distortion functions $d_n(.,.)$

- Need information-spectrum methods [Han, Verdu '93, '95]

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Definitions

$a_1, a_2, \ldots$: random sequence

- $\limsup_{in\ prob} a_n = \overline{a}$: Smallest number α such that

$$\lim_{n\to\infty} Pr(a_n > \alpha) = 0.$$

- We will need

$$i_k(\hat{x}^n \to x^n) = \frac{1}{n} \log \frac{P(x^n, \hat{x}^n)}{P(x^n) \cdot \prod_{i=1}^n P(\hat{x}_i | \hat{x}^{i-1}, x^{i-k})}$$

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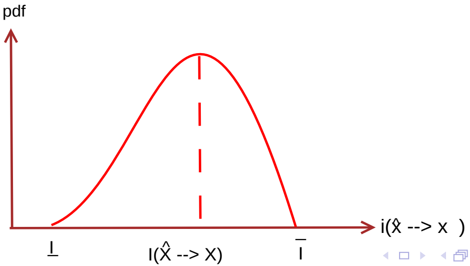

Rate-Distortion Theorem for a general source

- $\mathbf{P_X} = \{P_{X_1}, P_{X^2}, \ldots, P_{X^N}, \ldots\}$
- $\mathbf{P_{\hat{X}|X}} = \{P_{\hat{X}_1|X_1}, P_{\hat{X}^2|X^2}, \ldots, P_{\hat{X}^N|X^N}, \ldots\}$

Theorem

For arbitrary source X with distribution $\mathbf{P_X}$, the rate-distortion function with feed-forward delay k, the infimum of all achievable rates at probability-1 distortion D, is given by

$$R_{ff}(D) = \inf_{\mathbf{P_{\hat{X}|X}} : \rho(\mathbf{P_{\hat{X}|X}}) \leq D} \overline{I}_k(\hat{X} \to X),$$

where

$$\rho(\mathbf{P_{\hat{X}|X}}) \triangleq \limsup_{inprob} d_n(x^n, \hat{x}^n)$$

Rate-Distortion Theorem for a general source

- $\mathbf{P_X} = \{P_{X_1}, P_{X^2}, \ldots, P_{X^N}, \ldots\}$
- $\mathbf{P_{\hat{X}|X}} = \{P_{\hat{X}_1|X_1}, P_{\hat{X}^2|X^2}, \ldots, P_{\hat{X}^N|X^N}, \ldots\}$

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The story so far . . .

- What is feed-forward in source coding?
- Directed information and why it occurs
- Feed-forward rate-distortion result for general sources, distortions

Next . . .

How to compute the rate-distortion function?

The story so far ...

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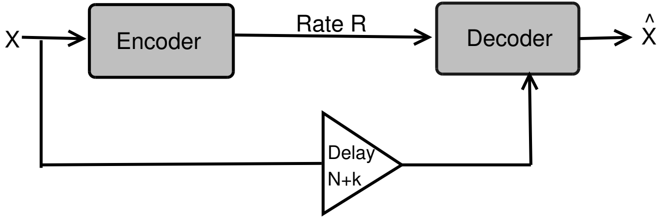
How to compute the rate-distortion function?

Source Coding Optimization



- Source X with distribution $\mathbf{P_X}$.
- Find $\mathbf{P_{\hat{X}|X}}$ to minimize $\bar{I}_k(\hat{X} \rightarrow X)$ s.t

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- Multi-letter optimization- difficult!

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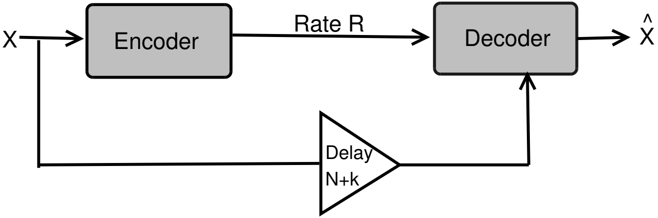

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Approach

- Pick a conditional distribution $\mathbf{P}_{\hat{\mathbf{X}}|\mathbf{X}} = \{P_{\hat{X}^n|X^n}\}$
- For what sequence of distortion measures d_n does $\mathbf{P}_{\hat{\mathbf{X}}|\mathbf{X}}$ achieve the infimum in the rate-distortion formula ?
- Approach- similar in spirit to [Csiszar and Korner]

Structure of Distortion Function

Theorem

A stationary, ergodic source X characterized by $\mathbf{P_X} = \{P_{X^n}\}_{n=1}^{\infty}$ with feed-forward delay k. $\mathbf{P_{\hat{X}|X}} = \{P_{X^n|X^n}\}_{n=1}^{\infty}$ is a conditional distribution such that the joint distribution is stationary and ergodic. Then $\mathbf{P_{\hat{X}|X}}$ achieves the rate-distortion function if for all sufficiently large n, the distortion measure satisfies

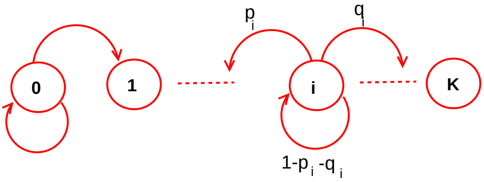
$$d_n(x^n, \hat{x}^n) = -c \cdot \frac{1}{n} \log \frac{P_{X^n,\hat{X}^n}(x^n, \hat{x}^n)}{\vec{P}^k_{\hat{X}^n|X^n}(\hat{x}^n|x^n)} + d_0(x^n),$$

where

$$\vec{P}^k_{\hat{X}^n|X^n}(\hat{x}^n|x^n) = \prod_{i=1}^{n} P_{\hat{X}_i|X^{i-k},\hat{X}^{i-1}}(\hat{x}_i|x^{i-k}, \hat{x}^{i-1})$$

and c is any positive number and $d_0(.)$ is an arbitrary function.

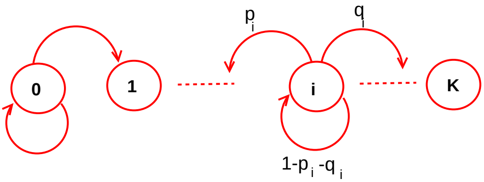
Stock example revisited



- Value of the stock: Markov source $\{X_n\}$

- Decision of investor on day n: $\hat{X}_n$ (0 or 1)

Stock example revisited



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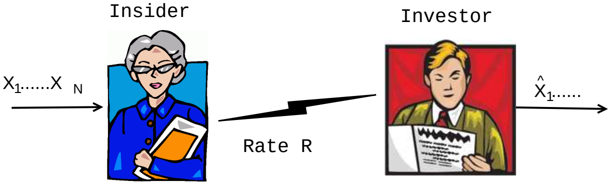

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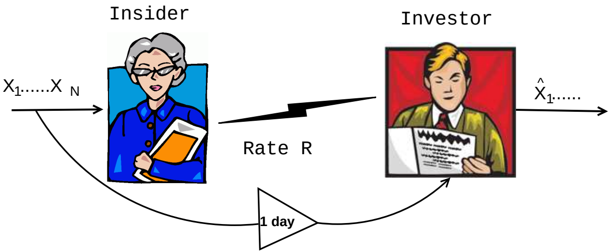

- $R_{ff}(D)$: Minimum rate the investor needs to predict drops with distortion D.

- Try first-order Markov joint distribution
- Distortion can be cast in the required form!

Minimum rate

Proposition

The minimum rate in (bits/sample) is

$$\sum_{i=1}^{k-1} \pi_i \Big[h(p_i, q_i, 1 - p_i - q_i) - h(\epsilon, 1 - \epsilon) \Big] + \pi_k \left(h(q_k, 1 - q_k) - h(\epsilon, 1 - \epsilon) \right)$$

where

- $h()$ *is the entropy function*
- $\left[\pi_0, \pi_1, \cdots, \pi_k \right]$ *is the stationary distribution of the stock*
- $\epsilon = \frac{D}{1-\pi_0}$

Computing Rate-distortion function with FF

1. 'Predict' a conditional distribution
2. Check if distortion function can be put into required form.

Next . . . A multi-terminal problem

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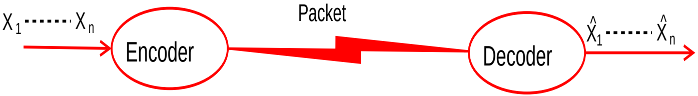
Multiple Descriptions



- Source X: compressed into packets
- Packets may be dropped
- Compress X into two *different* packets

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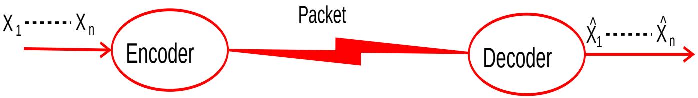

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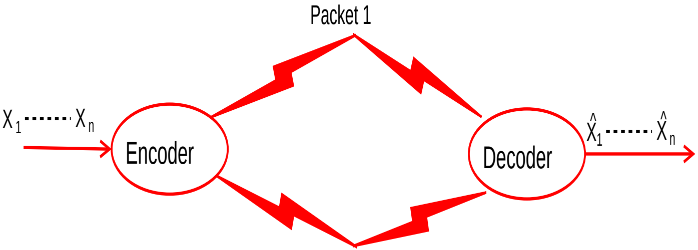

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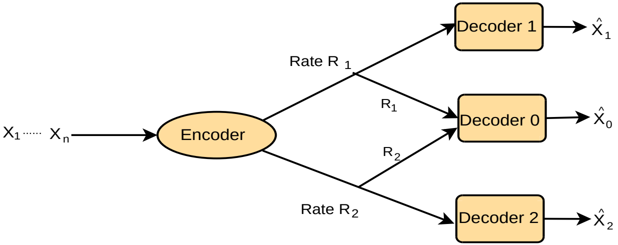

- Rate R_1 yields reconstruction $\hat{X}_1$ with distortion D_1
- Rate R_2 yields reconstruction $\hat{X}_2$ with distortion D_2

- Want better quality D_0 if both packets received
- $\hat{X}_1$ and $\hat{X}_2$ need to *refine* each other!

Rate	Distortion
R_1 bits/sample	D_1
R_2 bits/sample	D_2
$R_1 + R_2$ bits/sample	D_0

GOAL

Given i.i.d source P_X:

Find all achievable $(R_1, R_2, D_1, D_2, D_0)$

- Still an open problem
- Studied by [Cover, El Gamal], [Ahlswede], [Zhang, Berger], . . .
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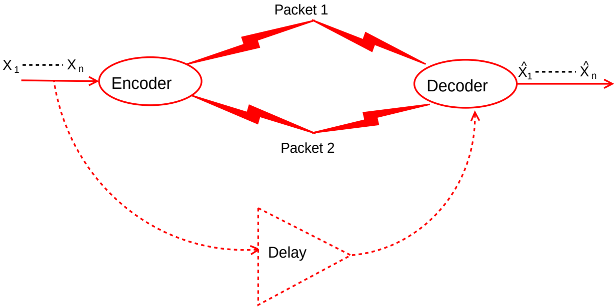
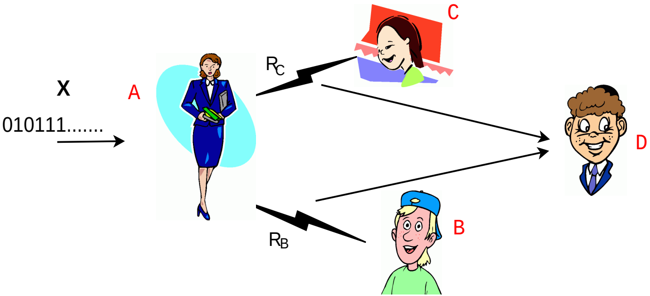
Network example with feed-forward



- Source propagates to destination with delay
- To reconstruct $\hat{X}_n$, decoder has packet and X^{n-k}

Example

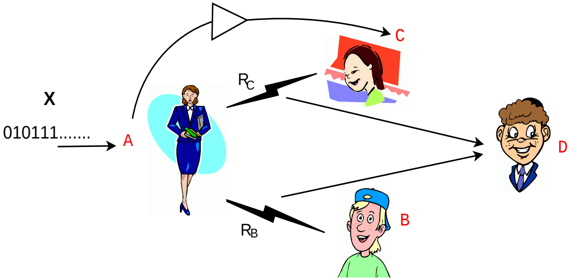


- Alice has an i.i.d binary source $\sim$ Bernoulli(1/2)
- Bob and Carol: distortion d using R_B, R_C bits/sample
- Dave gets Bob's bits and Carol's bits- needs to reconstruct perfectly!

Characterize

$r_{sum}(d) \triangleq$ Smallest sum-rate $R_B + R_C$ for distortion $(d, d, 0)$

Feed-forward



Same model as before, one extra feature. . .

- After Carol reconstructs each sample, Alice reveals the value to her. *Feed-forward*
- Before reconstructing each sample, Carol knows *past* source samples

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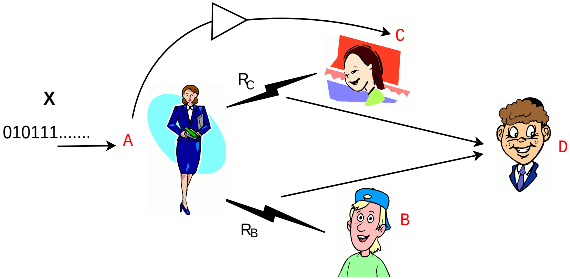

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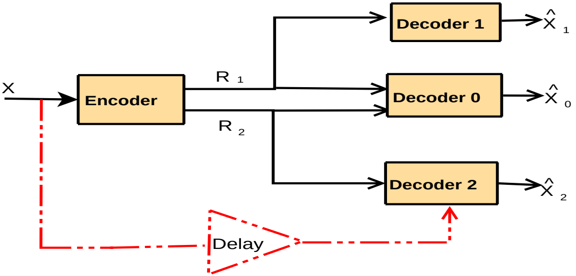
General feed-forward model



- Encoder mappings: $e_m : \mathcal{X}^N \to \{1, \ldots, 2^{NR_m}\}, \quad m = 1, 2.$
- Mappings for decoders 1 and 0:

$$g_1 : \{1, \ldots, 2^{NR_1}\} \to \widehat{\mathcal{X}}_1^N$$

$$g_0 : \{1, \ldots, 2^{NR_1}\} \times \{1, \ldots, 2^{NR_2}\} \to \widehat{\mathcal{X}}_0^N$$

- A sequence of mappings for decoder 2:

$$g_{2n} : \{1, \ldots, 2^{NR_2}\} \times \mathcal{X}^{n-k} \to \widehat{\mathcal{X}}_2, \quad n = 1, \ldots, N.$$

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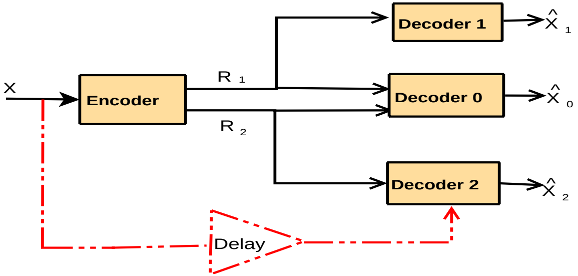

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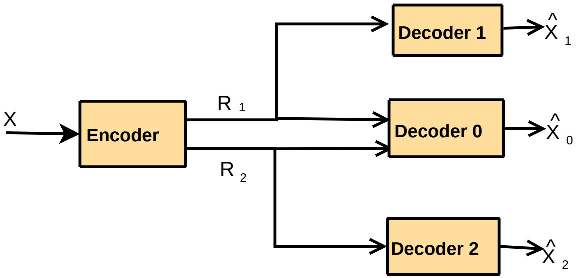
Rate region w/o FF



Zhang-Berger '87

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Correlation in MD

Stringent Decoder 0 distortion $\Rightarrow$ Need correlation in $\hat{X}_1, \hat{X}_2$

'Cloud' Center

- U: Cloud center of X sent to all decoders
- Rate $I(U; X)$ each for decoder 1 and 2

Penalty Term

- Can't have $\hat{X}_1 \sim P(\hat{X}_1|XU)$ and $\hat{X}_2 \sim P(\hat{X}_2|XU)$ indep'ly
- Need to be *jointly* distributed: $\sim P(\hat{X}_1, \hat{X}_2|XU)$
- $I(\hat{X}_1; \hat{X}_2|XU)$: Penalty in sum-rate

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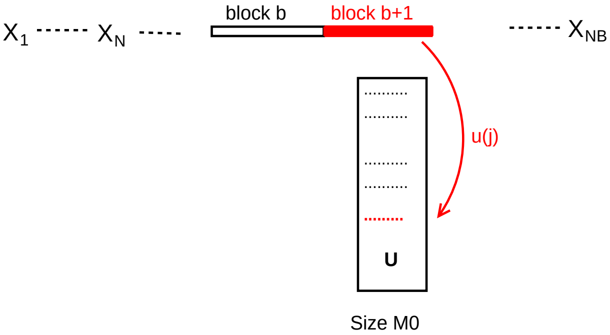
Coding Strategy

- Consider B long blocks of source samples

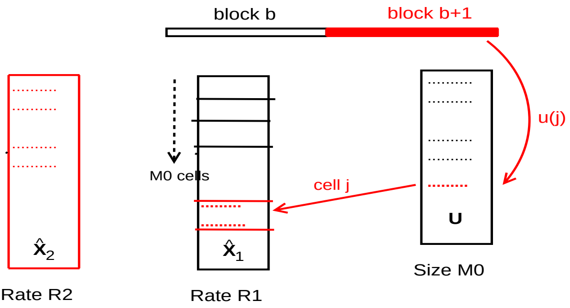
$$X_1 \ldots X_N \quad X_{N+1} \ldots X_{2N} \qquad \ldots\ldots X_{NB}$$

- While encoding one block, give 'preview' of next block



Coding Strategy

Restricted encoding for user 1- within cell j



After reconstructing block b:

- User 1 gets 'preview' of block $b+1$
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Block-Markov, superposition; [Cover,Leung], [Willems] for MAC

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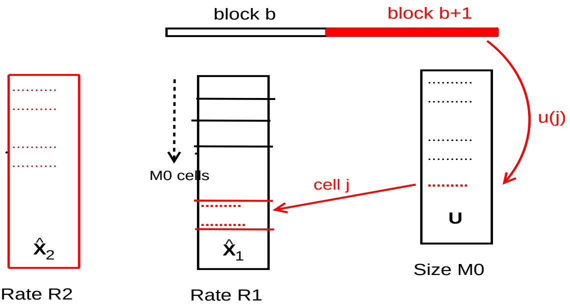

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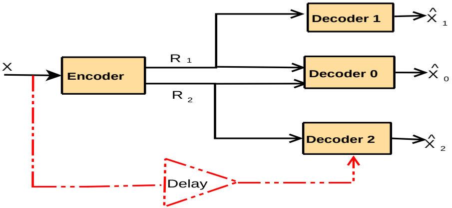

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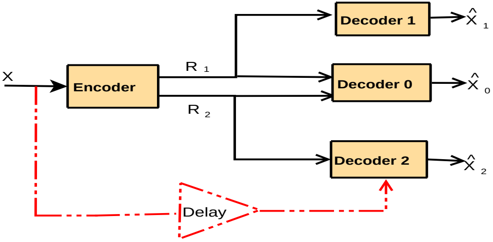
Improvement over [Zhang-Berger]



- Fix $R_1 = I(X; \hat{X}_1 U) + \epsilon$
- If $\max\{0, R_1 - I(X\hat{X}_2; \hat{X}_1 | U)\} = 0$:
  - Savings in $R_2 = I(U; X)$ bits/sample
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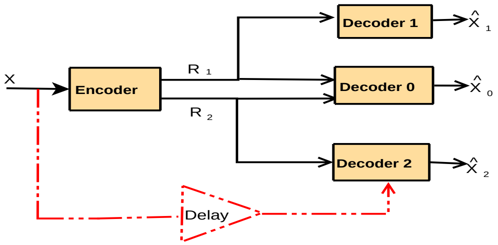

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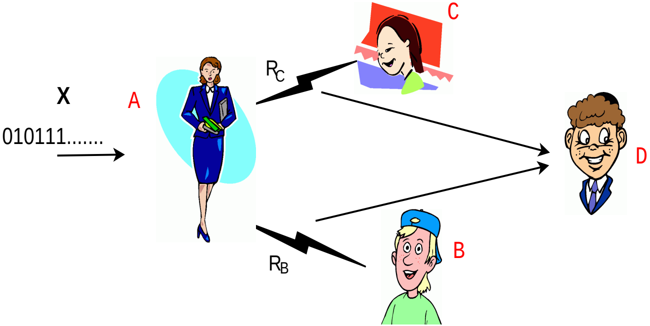
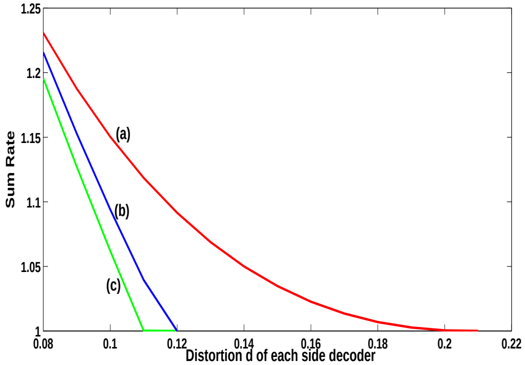


Without FF [Zhang Berger '87]

$$r_{sum}(d) \geq 2 - h\left(\frac{4d+1-\sqrt{12d^2-4d+1}}{2}\right)$$



- (a): Lower bound without feed-forward [Zhang, Berger '87]
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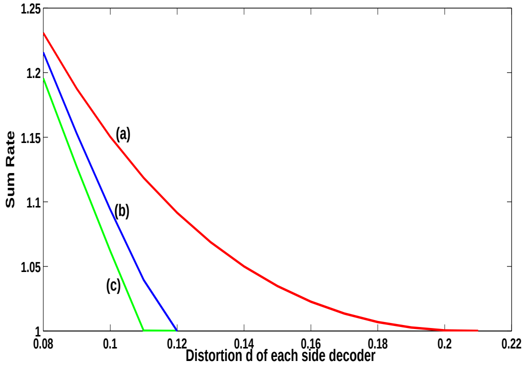
Comparison



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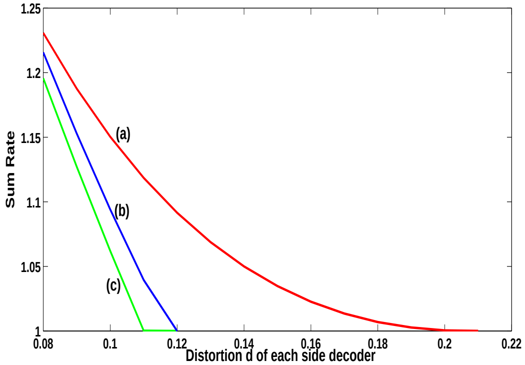

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Summary

- Feed-forward: helps build correlation
- Single-letter achievable region for MD with FF
- FF helps even for an i.i.d source

- How to use feed-forward to all decoders?
- FF for Gaussian multiple descriptions [Pradhan IT '07]

Summary

- Source Coding with feed-forward

  -Directed Information

  -Rate-Distortion Function

  -How to evaluate optimization

  -Multiple Descriptions with FF

- Channel Coding with feedback

Some questions...

- Feedback/FF in multi-terminal setting
- Noisy feedback/FF
- Applications of directed-info like quantities

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