Algebraic Structures for Multi-Terminal Communications

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University of Michigan

Oral Defense

Thesis Outline

1. Lattice codes and Gaussian sources (Chap. 2)

2. Group codes and discrete sources (Chap. 3)

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   - Reconstructing linear function of the sources
   - Existence of "good" nested lattice codes
   - New rate region - better for certain parameters
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Presentation Outline

1. Thesis Overview
2. Information Theory: An Introduction
3. Random Codes for Distributed Source Coding
4. Nested Group Codes
5. Distributed Source Coding : An Inner Bound
6. Conclusions

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- Information theory - random unstructured codes ubiquitous
  - Shannon's original proofs based on random codes
  - Good performance. Exponential complexity
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- Unified way to use structured codes in many problems
- Nesting of one linear code inside another
  - No loss in performance vs unstructured codes in point-to-point setting
  - Performance gains in multi-terminal settings
- Existence proofs for "good" nested structured codes
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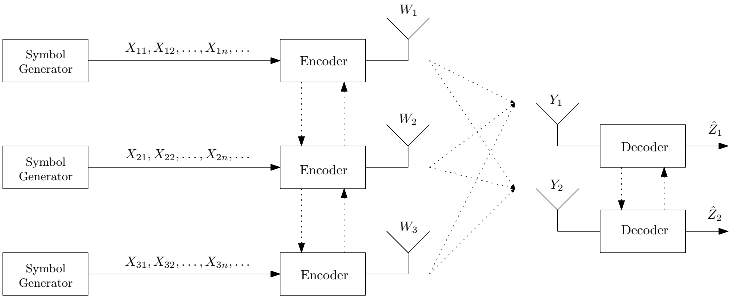
- Big picture: Transmit stochastic sources over noisy channels



Information Theory:An Introduction

- Mathematical theory of information transmission
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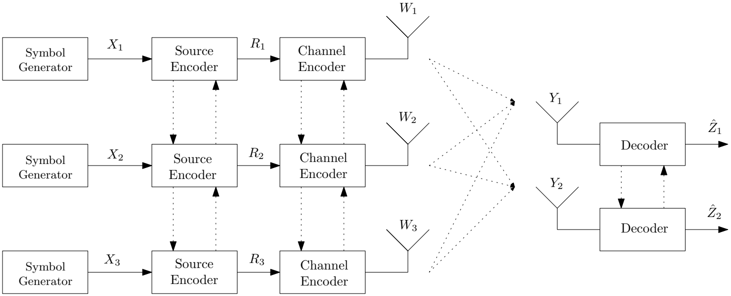
- Split into modules. Shannon's source channel separation



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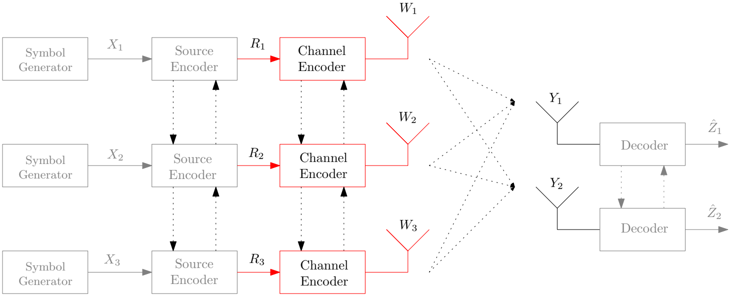

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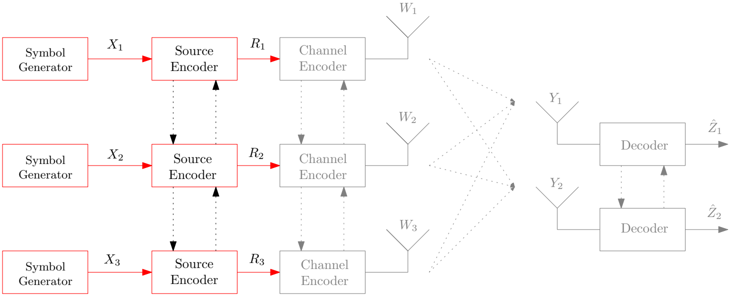

Information Theory:An Introduction

- Mathematical theory of information transmission
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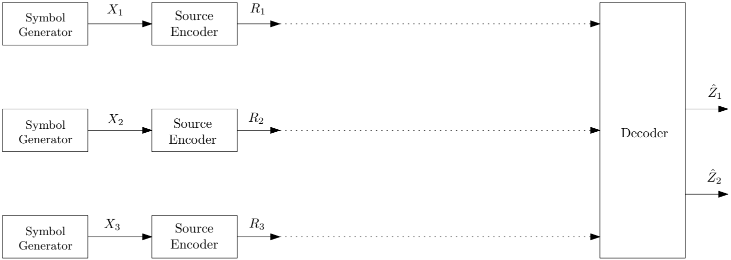
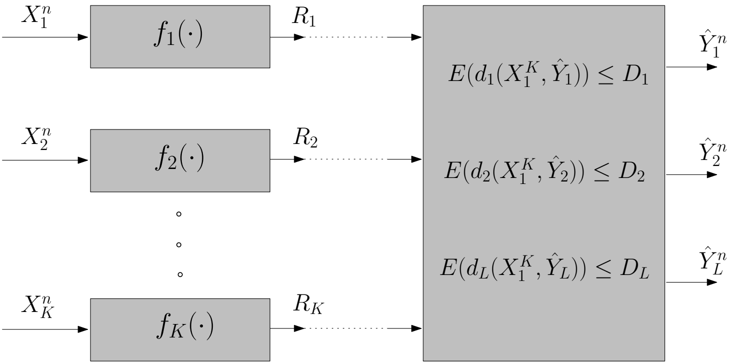
- Distributed source coding. Sensor networks

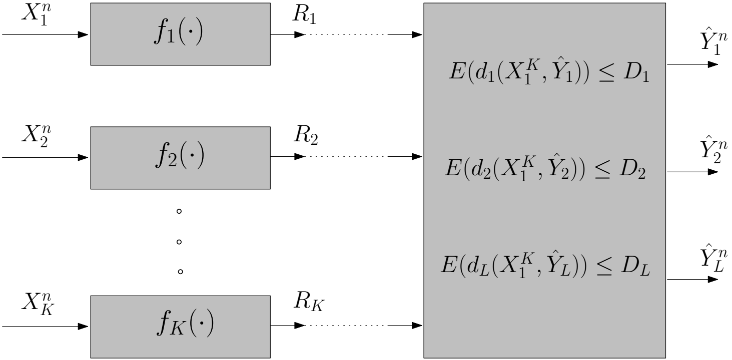


Distributed Source Coding



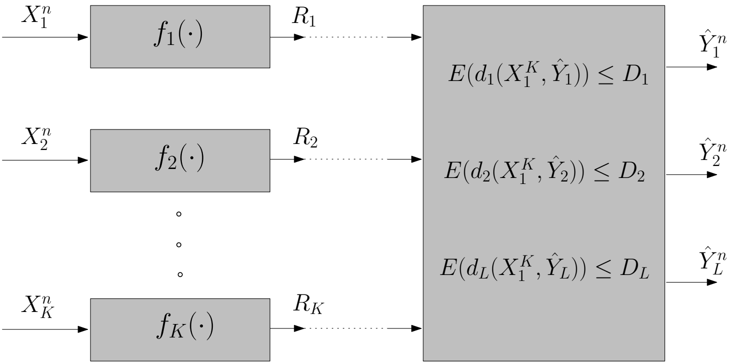
- $X_1, X_2, \ldots, X_K$ - Correlated across space, independent across time

Distributed Source Coding



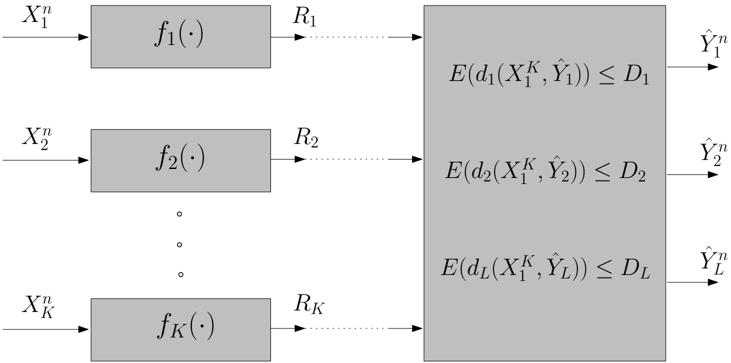
- Encoders $f_i : \mathscr{X}_i^n \to \{1, 2, \ldots, 2^{nR_i}\}, i = 1, \ldots, K$

Distributed Source Coding



- Rate distortion region $\mathscr{R}\mathscr{D}$: set of achievable $(R_1,\ldots,R_K,D_1,\ldots D_L)$

Distributed Source Coding



- Goal: Characterize $\mathscr{RD}$ using single-letter information quantities

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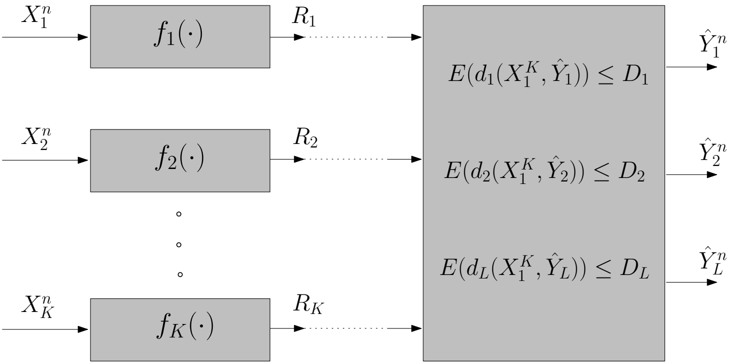

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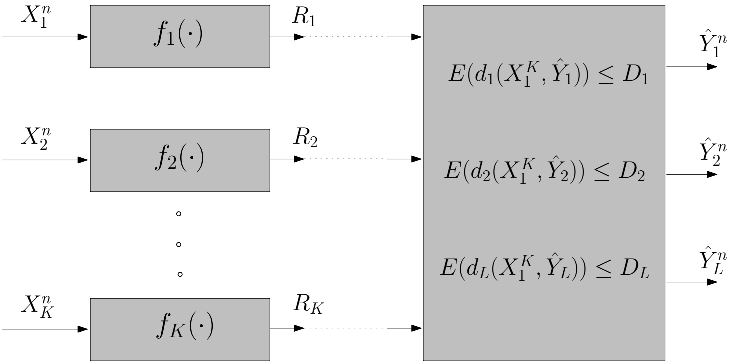
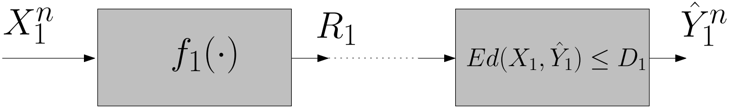


- Goal: Characterize $\mathscr{RD}$ using single-letter information quantities
  - Very hard to solve completely
  - Provide computable inner bounds

Single user source coding



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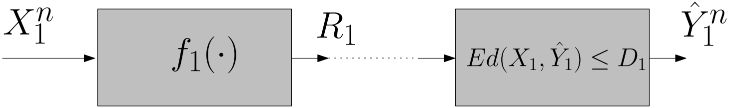
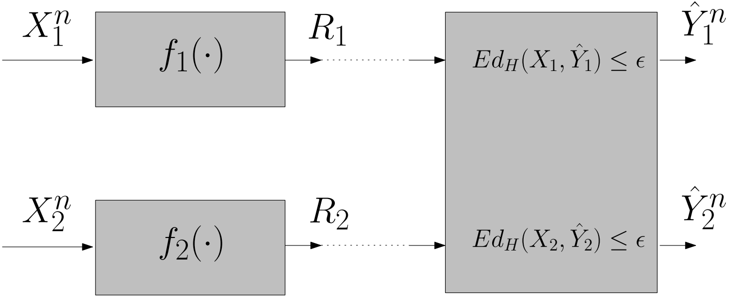


- Solved completely by Shannon

$$R_1 \geq \min_{p_{\hat{Y}_1|X_1} : \, \mathbb{E}d(X_1, \hat{Y}_1) \leq D_1} I(X_1; \hat{Y}_1)$$

Slepian-Wolf problem



- Lossless reconstruction of both sources

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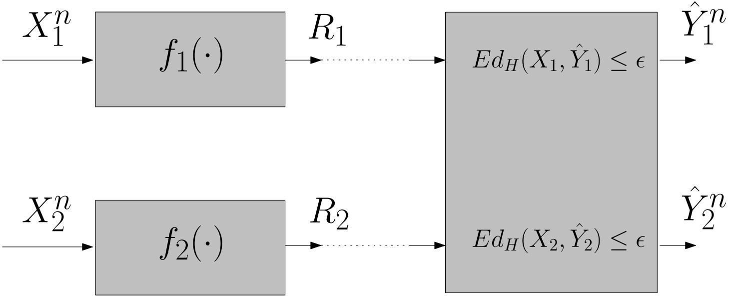
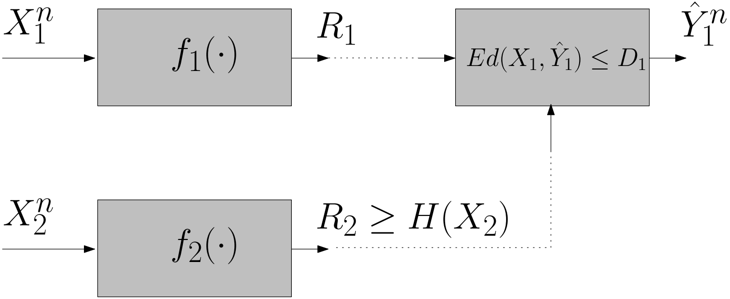


- Lossless reconstruction of both sources

$$R_1 \geq H(X|Y),\ R_2 \geq H(Y|X)$$

$$R_1 + R_2 \geq H(X, Y)$$

Wyner-Ziv problem



- Lossy reconstruction with decoder side information

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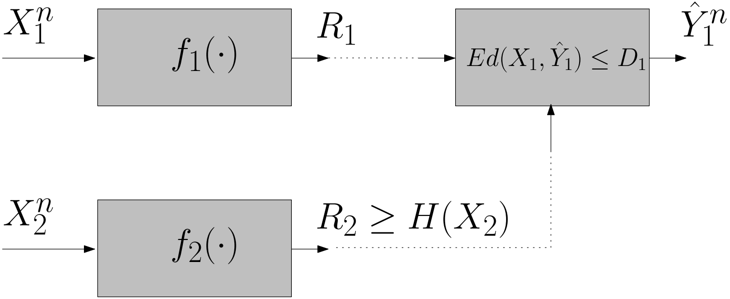

- Lossy reconstruction with decoder side information
- Auxiliary random variable U with Markov chain $U - X_1 - X_2$

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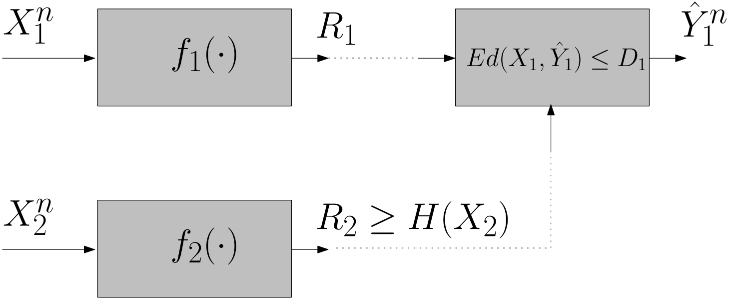

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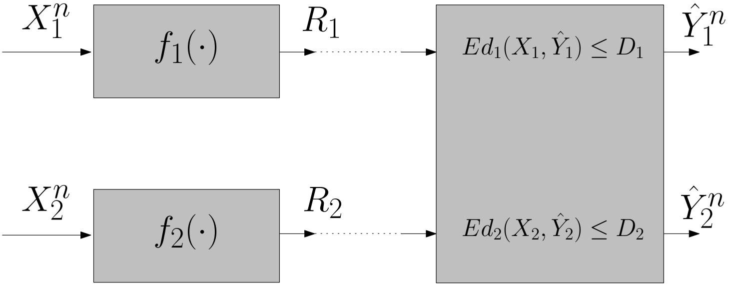
$$R_1 \geq I(X_1; U \mid X_2) = I(X_1; U) - I(X_2; U)$$

Berger-Tung problem



- Independent distortion criteria

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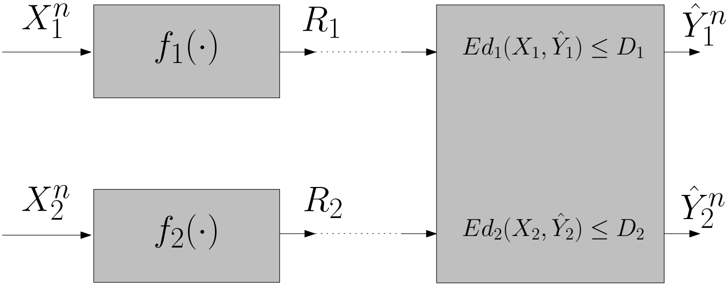

- Auxiliary random variables U, V with $U - X_1 - X_2 - V$

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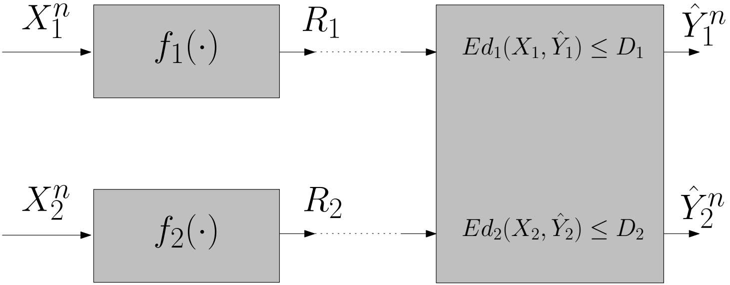

- Inner bound (tightness not known in general):

$$R_1 \geq I(X_1; U \mid X_2),\ R_2 \geq I(X_2; V \mid X_1)$$

$$R_1 + R_2 \geq I(X_1 X_2; UV)$$

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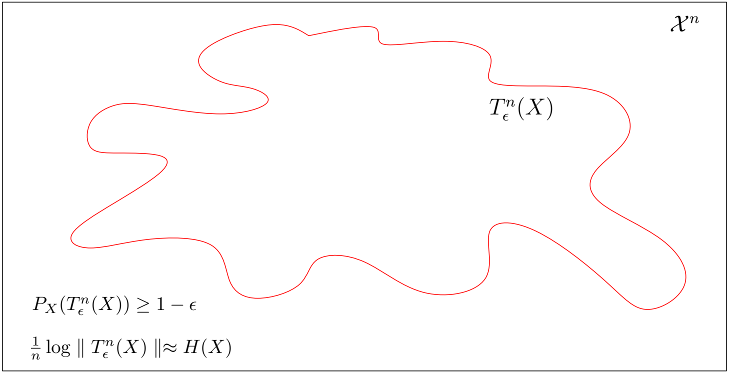
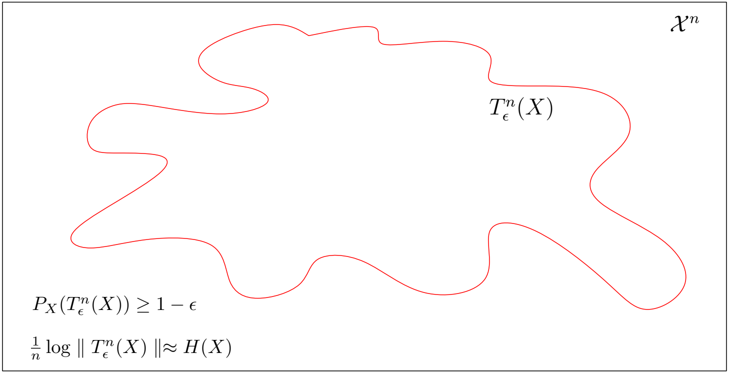
- Typical set: Set of probabilistically significant sequences



Typical proof techniques

- Two parts to all problems - achievability and converse
- Achievability proofs: Operations on the typical set
- Very complex. No low-dimensional characterization



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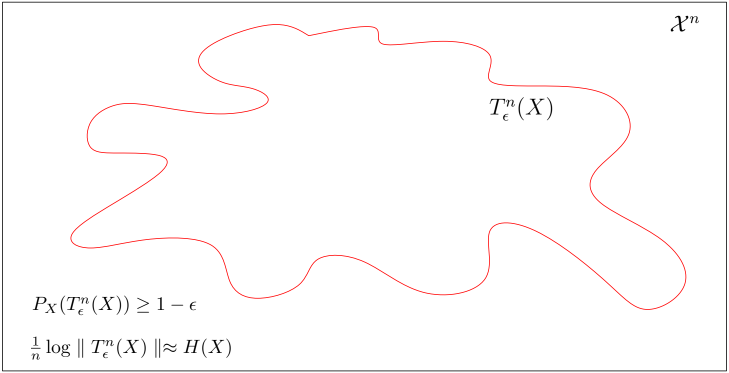
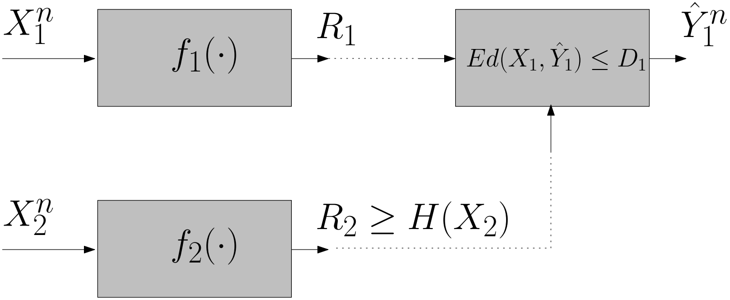
- Quantization (source coding) and binning (channel coding)



Wyner-Ziv problem - Revisited



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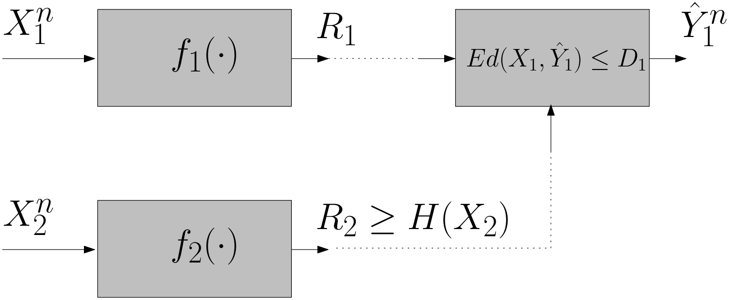

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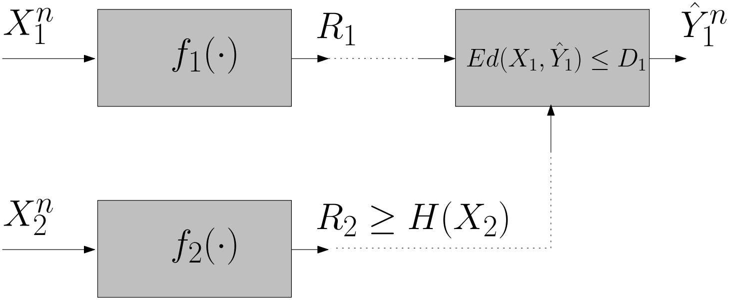

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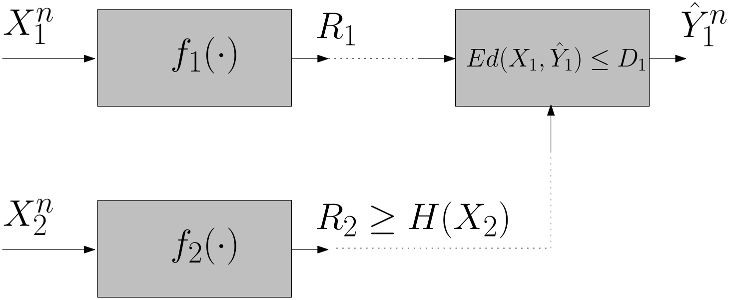

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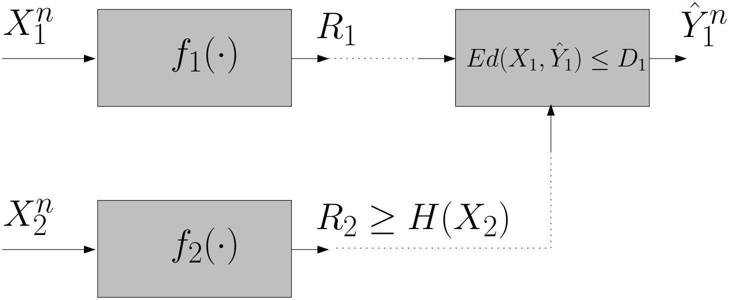

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- Combines aspects of both source and channel coding
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  - Channel coding: Decode U at decoder using X_2

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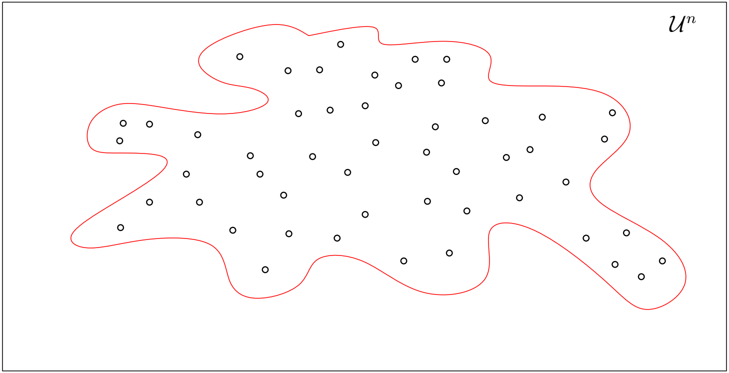
- Codebook $\mathscr{C}$ built from typical set of U



Quantization - Good source codes

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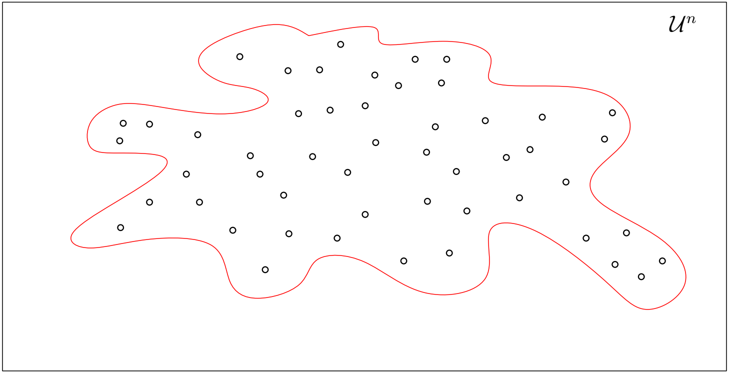

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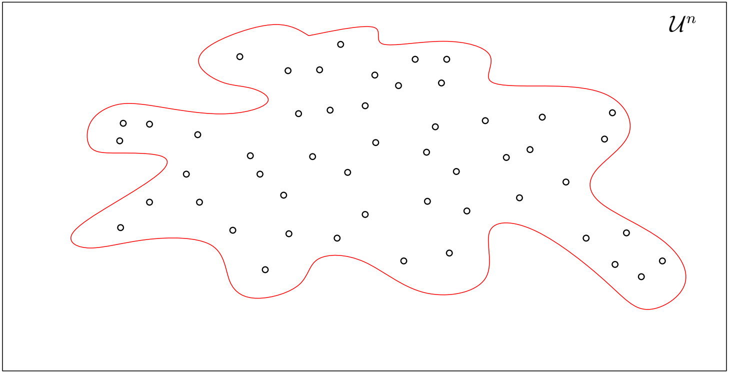
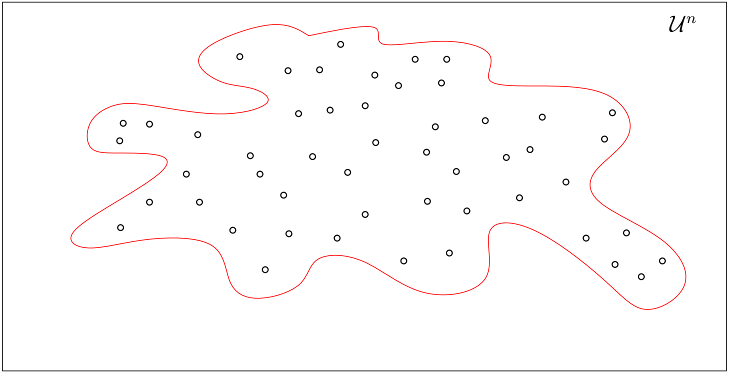
- Size of good code book: $I(X_1;U)$



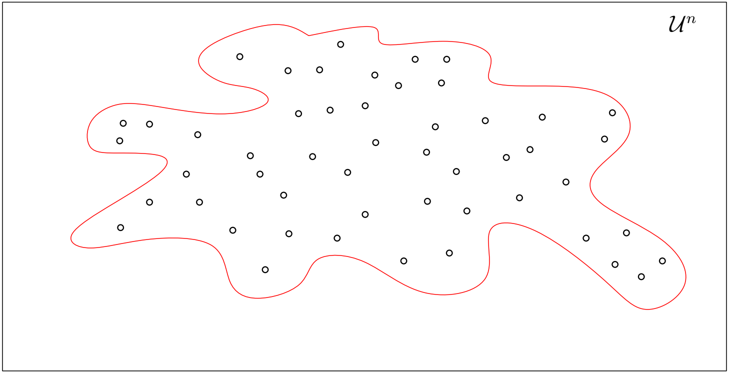
Quantization - Good source codes

- Quantize X_1 to U for a fixed $P_{U|X_1}$
- Codewords chosen at random. No structure.



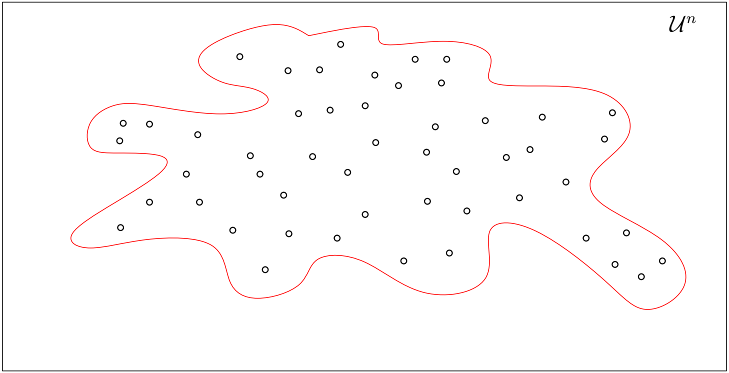
Binning - Good channel codes

- Suppose X_1 already quantized to U



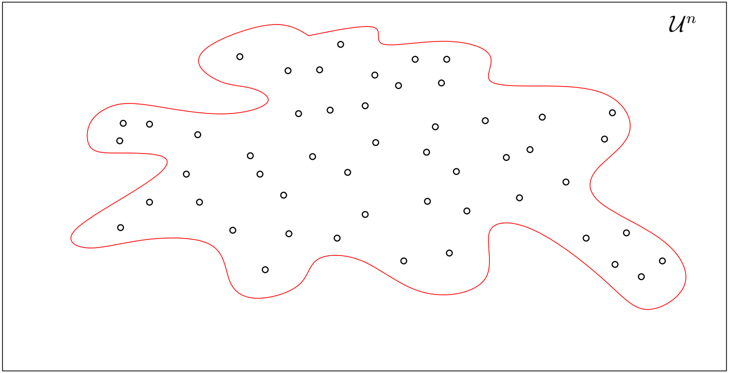
Binning - Good channel codes

- Decoder knows X_2 correlated to U



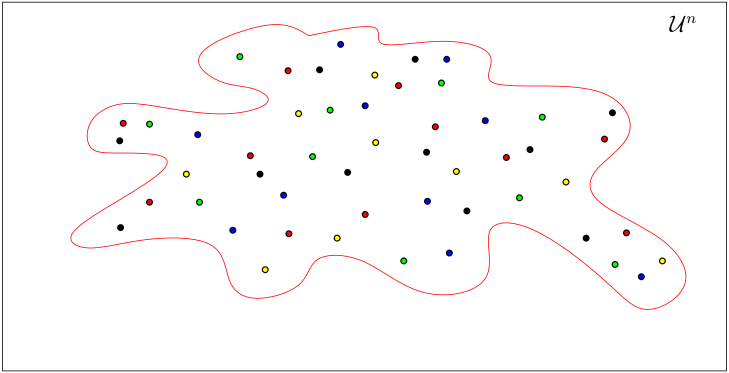
Binning - Good channel codes

- Can this side information be exploited?



Binning - Good channel codes

- Bin the codewords - Transmit only bin index



Binning - Good channel codes

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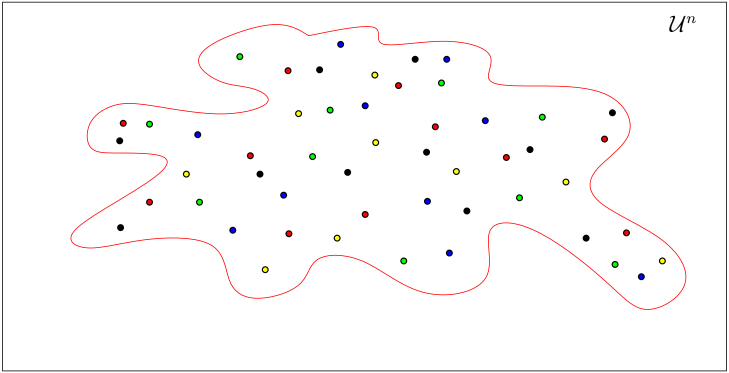
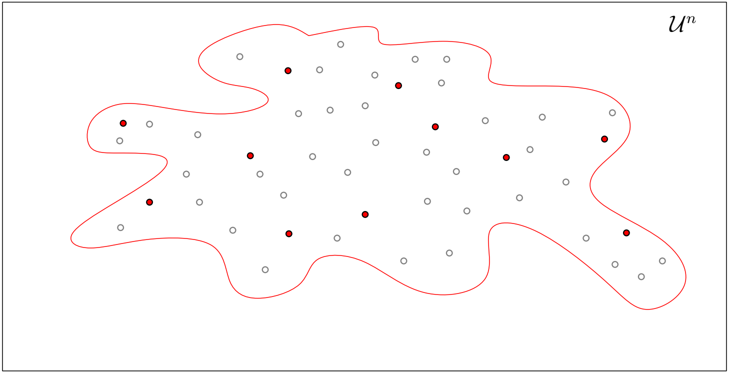
- Each bin:



Binning - Good channel codes

- Bin the codewords - Transmit only bin index
- Each bin: Channel code for channel $P_{X_2|U}$ with input U, output X_2



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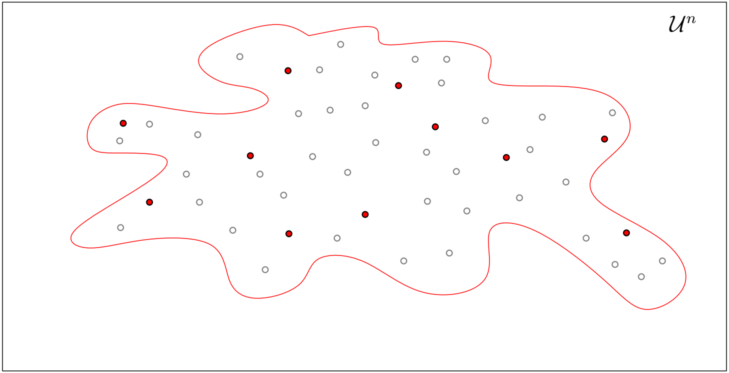
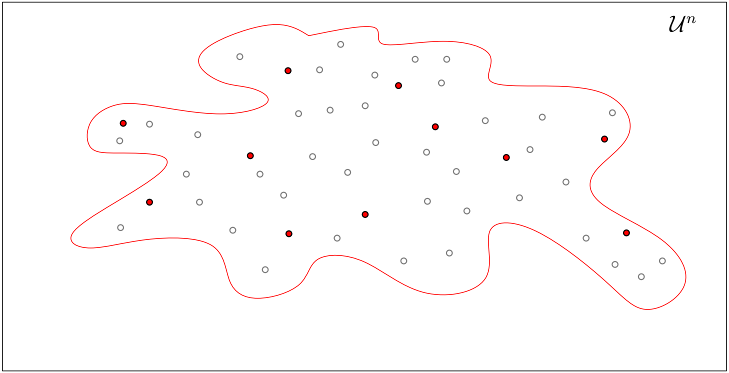


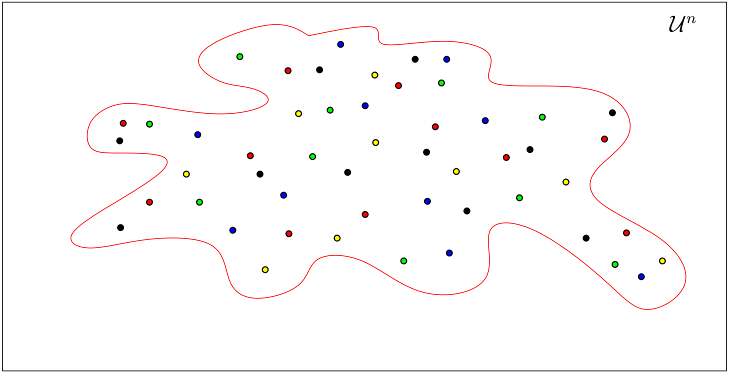
Binning - Good channel codes

- Bin the codewords - Transmit only bin index
- Size of each bin $I(U; X_2)$. Binning done randomly



Binning - Good channel codes

- Bin the codewords - Transmit only bin index
- Overall transmission rate $R = I(X_1;U) - I(U;X_2)$



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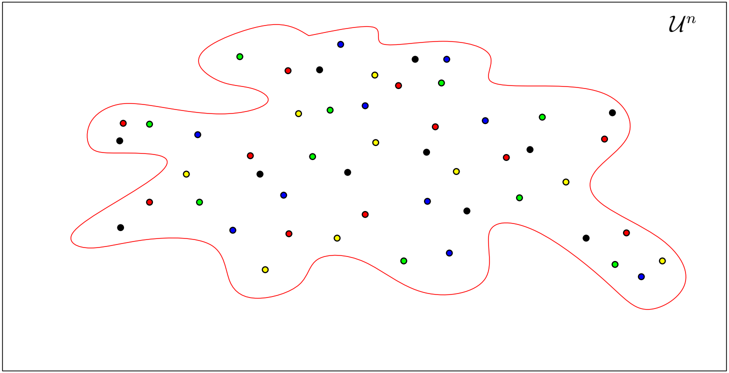
- Nesting of a "good" channel code in a "good" source code



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  - Independent quantization followed by independent binning
- Decoder given excess information
  - First reconstructs auxiliary random variables U, V

Random coding: Some observations

- Random coding for distributed source coding
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Outline

1 Thesis Overview

2 Information Theory: An Introduction

3 Random Codes for Distributed Source Coding

4 Nested Group Codes

5 Distributed Source Coding : An Inner Bound

6 Conclusions

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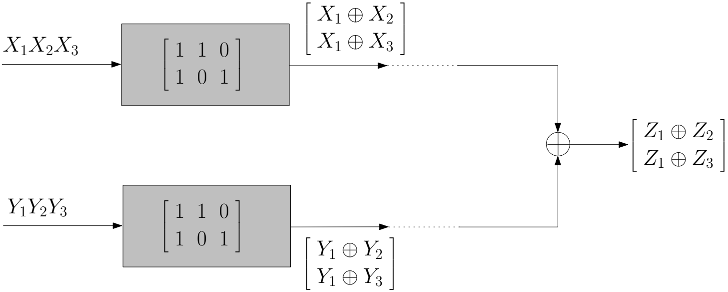

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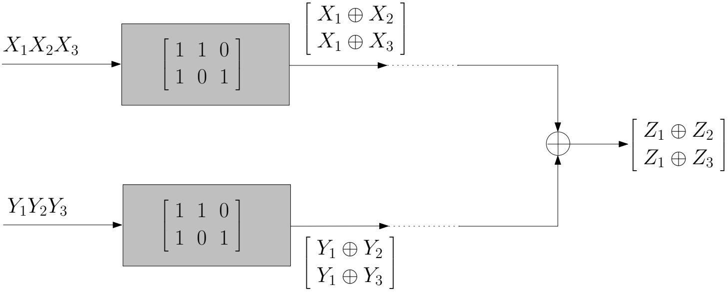

- Significant feature: Identical linear binning

Korner-Marton Coding Scheme

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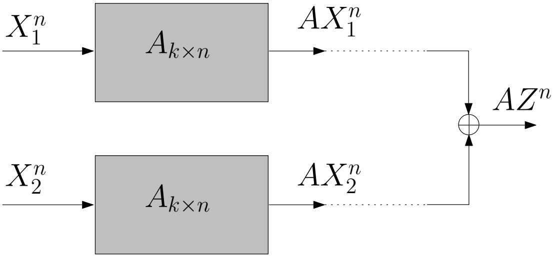

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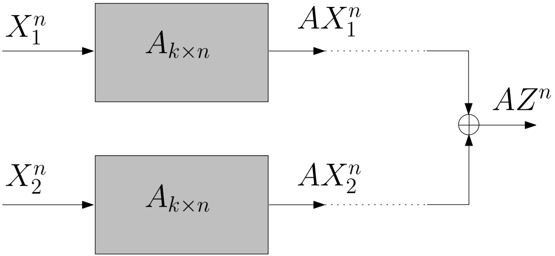

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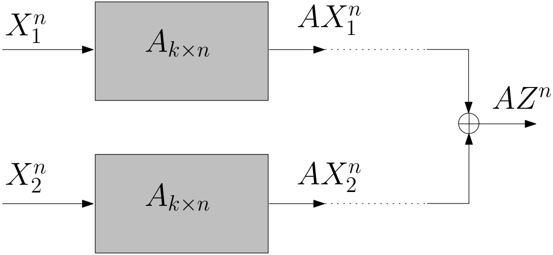

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- Associated code: Good channel code for additive noise Z

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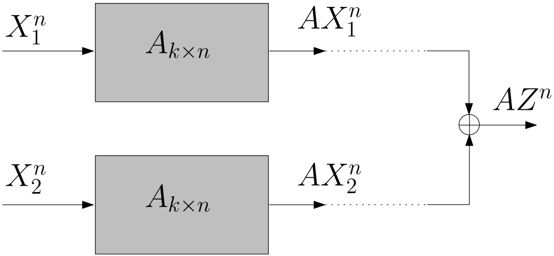

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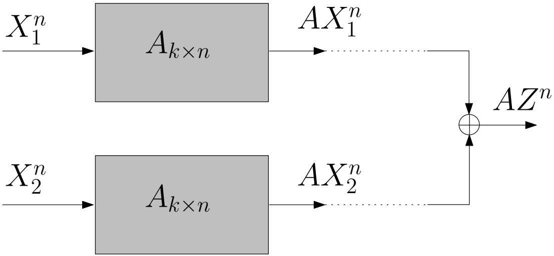

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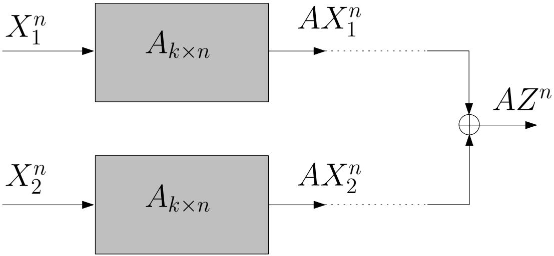

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  - Transmit $f(z^n)$ to decoder. Decoder recovers z^n

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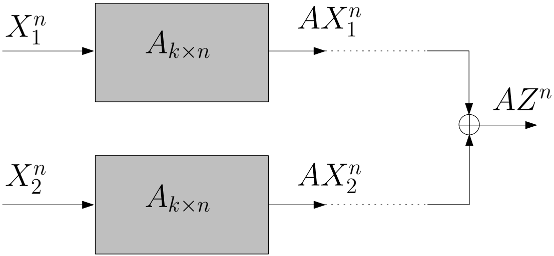

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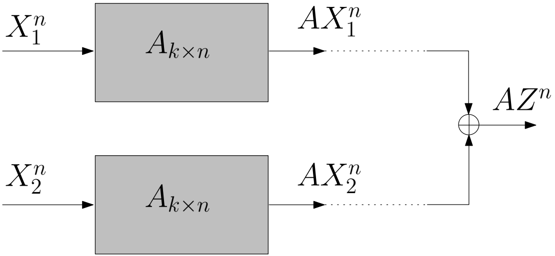

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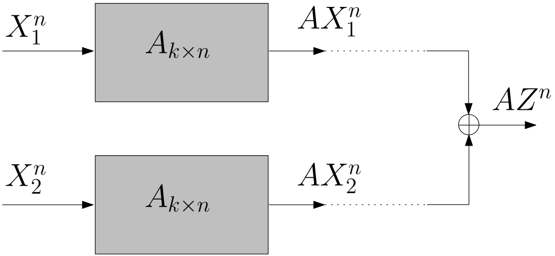

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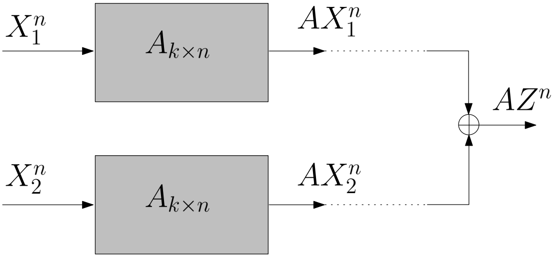

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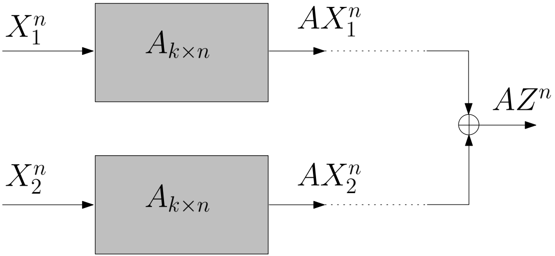

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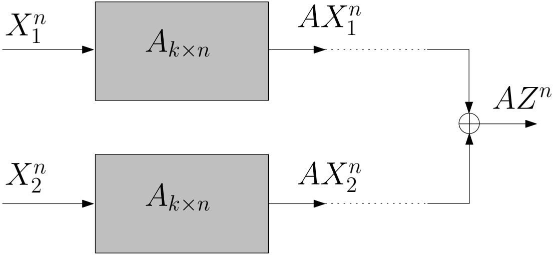

- Identical linear binning:
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  - Lossy coding
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- Injection of some non-linearity seems necessary for optimality

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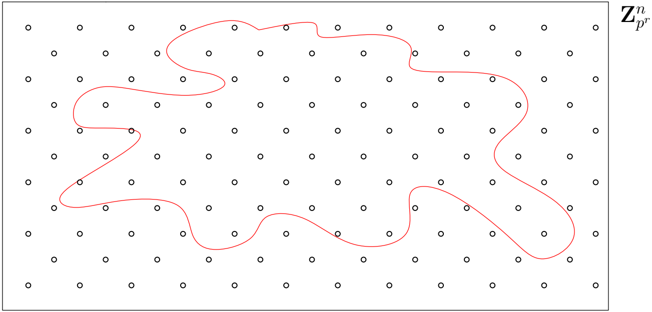
- Group code $\mathscr{C}$ over $\mathbb{Z}_{p^r}$: $\mathscr{C} = \ker(\phi)$ for homomorphism $\phi : \mathbb{Z}_{p^r}^n \to \mathbb{Z}_{p^r}^k$

Good Group Source Codes

- Good group source code $\mathscr{C}_1$ for the triple $(\mathscr{X}, \mathscr{U}, P_{XU})$



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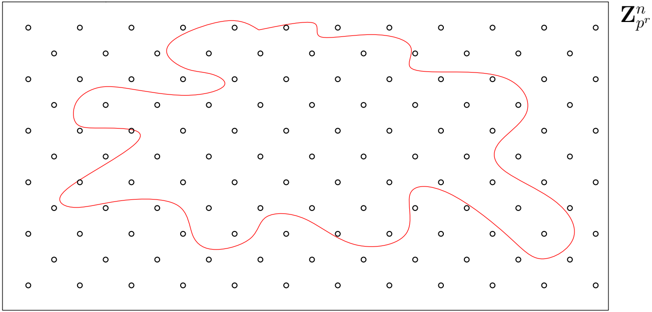
- Assume $\mathscr{U} = \mathbb{Z}_{p^r}$ for some prime p and exponent $r > 0$



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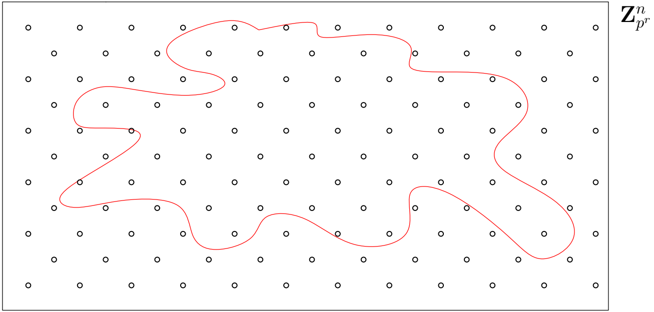

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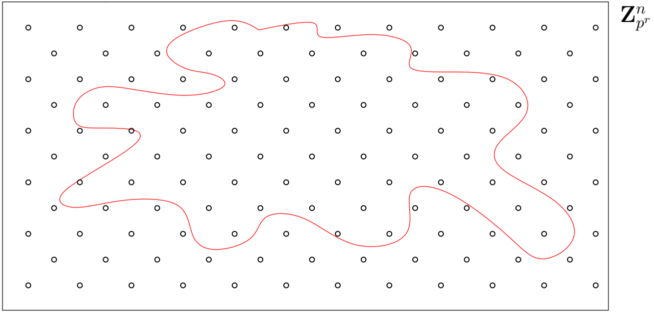
- We showed:

Good group source codes

Exist for large n if $\frac{1}{n}\log|\mathscr{C}_1| \geq \log p^r - r|H(U|X) - \log p^{r-1}|^+$



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- Good: Can find $u^n \in \mathscr{C}_1$ jointly typical with x^n
- No good source code in ensemble if $H(U|X) < \log p^{r-1}$

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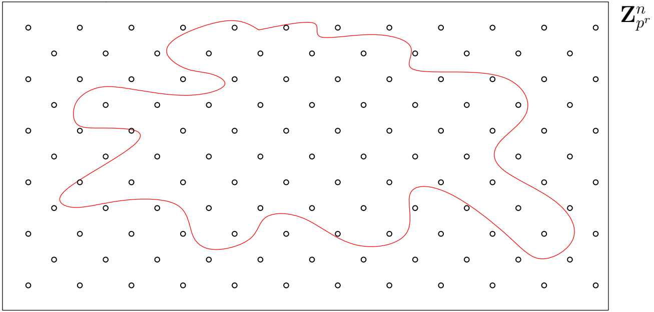

Good Group Source Codes

- Good: Can find $u^n \in \mathscr{C}_1$ jointly typical with x^n
- Else: Bad performance

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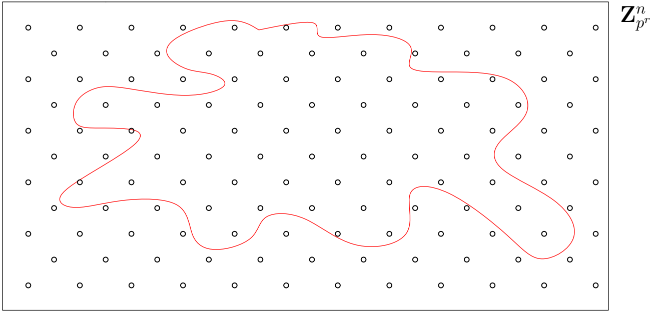
Exist for large n if $\frac{1}{n}\log|\mathscr{C}_1| \geq r(\log p^r - H(U|X))$



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- Good: Can find $u^n \in \mathscr{C}_1$ jointly typical with x^n
- Linear code $(r = 1)$: Still not very good

Good linear source codes

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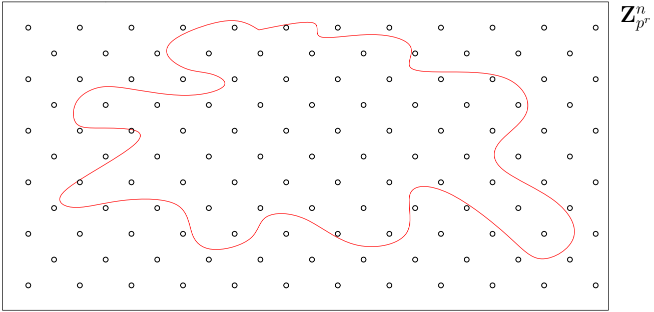

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- Larger than optimal code size: $H(U) - H(U|X)$

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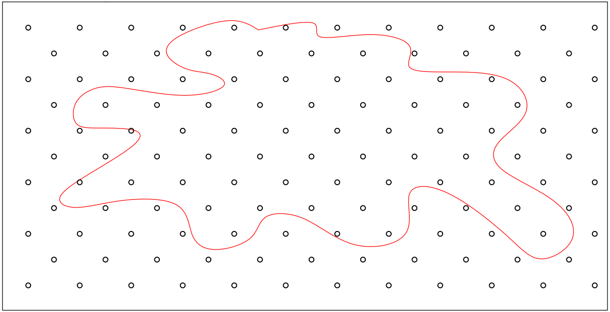
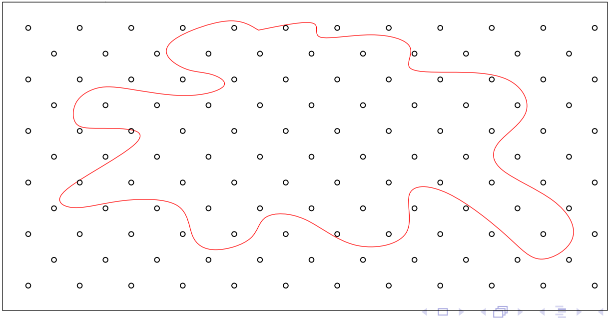


$\mathbf{Z}_{p^r}^n$

Good Group Source Codes contd.

- Linear code not Shannon-good for source coding



$\mathbf{Z}_{p^r}^n$

Good Group Source Codes contd.

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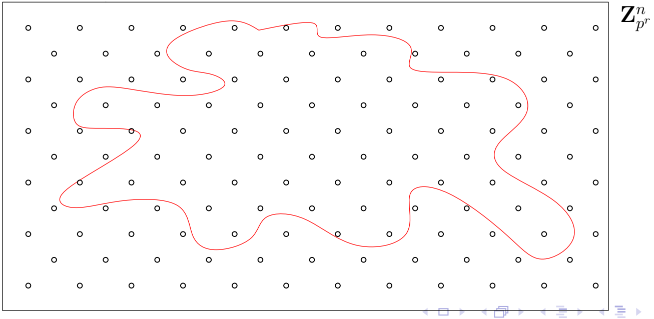

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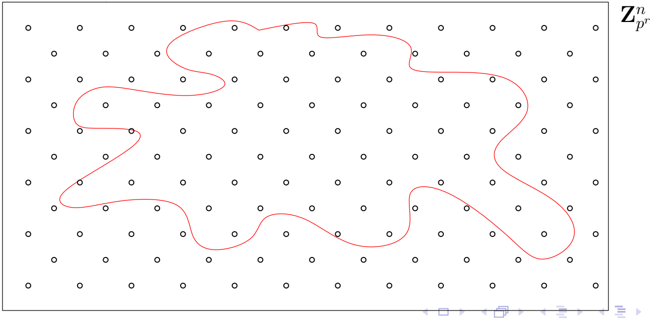

Good Group Source Codes contd.

- Linear code not Shannon-good for source coding
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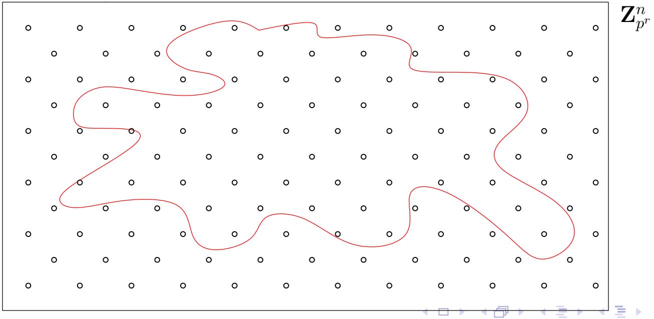
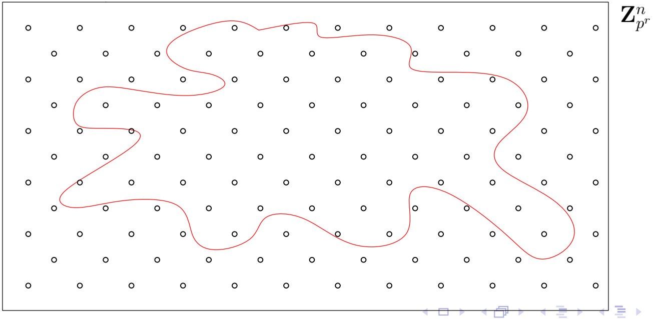


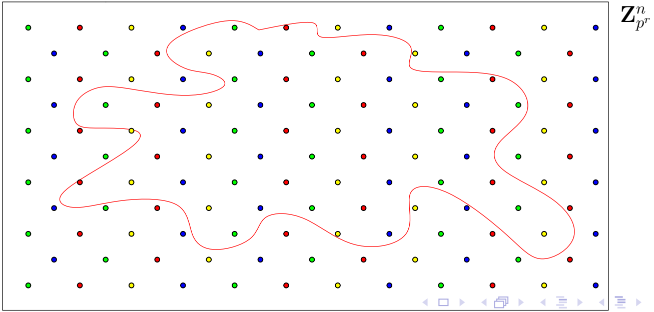
Good Group Source Codes contd.

- Linear code not Shannon-good for source coding
- But contains a Shannon-good source code
- Group codes $(r > 1)$: more penalties for subgroups



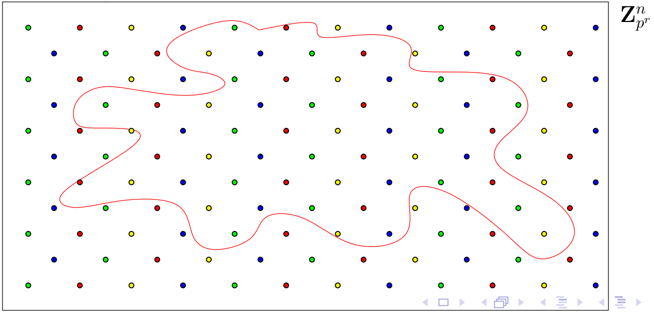
Good Group Channel Codes

- Good group channel code $\mathscr{C}_2$ for the triple $(\mathscr{Z}, \mathscr{S}, P_{ZS})$



Good Group Channel Codes

- Assume $\mathcal{Z} = \mathbb{Z}_{p^r}$ for some prime p and exponent $r > 0$



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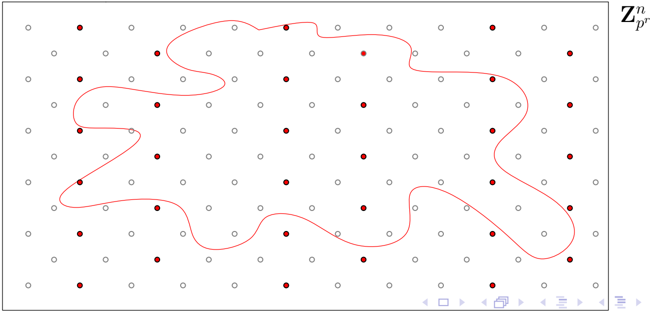
- Good: Can find z^n given its coset(color) and s^n

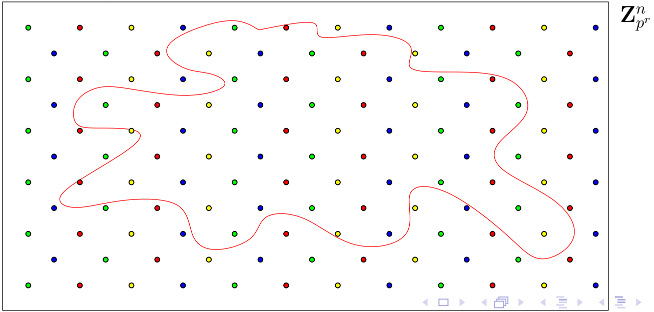


Good Group Channel Codes

- Good: Can find z^n given its coset(color) and s^n
- We showed:

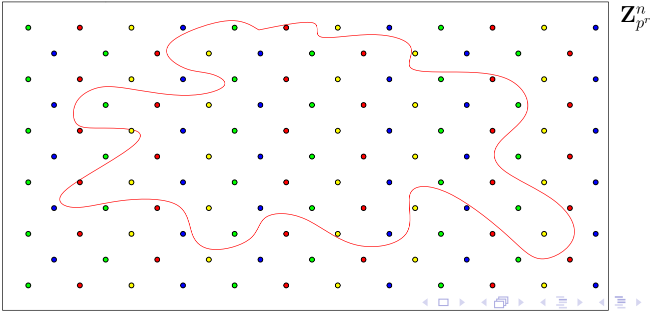
Good group channel codes

Exist for large n if $\frac{1}{n}\log|\mathscr{C}_2| \le \log p^r - \max_{0\le i<r}\left(\frac{r}{r-i}\right)(H(Z|S) - H([Z]_i|S))$



Good Group Channel Codes

- Good: Can find z^n given its coset(color) and s^n
- $[Z]_i$ - random variable taking values over distinct cosets of $p^i\mathbb{Z}_{p^r}$ in $\mathbb{Z}_{p^r}$



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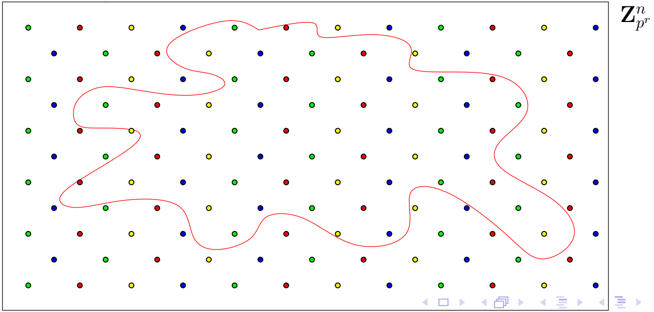
- Good: Can find z^n given its coset(color) and s^n
- Suppose $\mathcal{Z} = \mathbb{Z}_8$. $[Z]_1$ - binary random variable



Good Group Channel Codes

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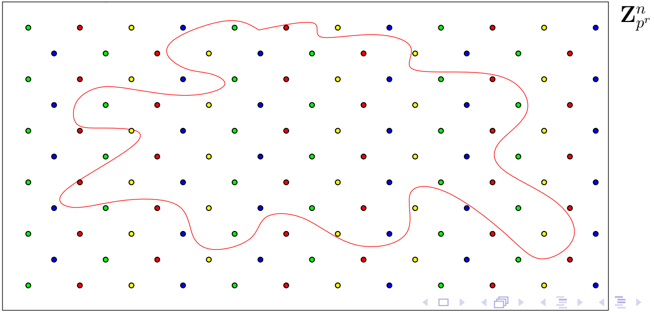
- Symbol probabilities: $(p_0 + p_2 + p_4 + p_6, p_1 + p_3 + p_5 + p_7)$



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- Each subgroup of $\mathbb{Z}_{p^r}$: one term in maximization

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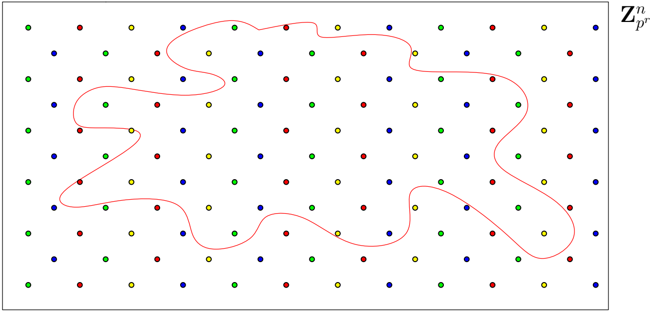

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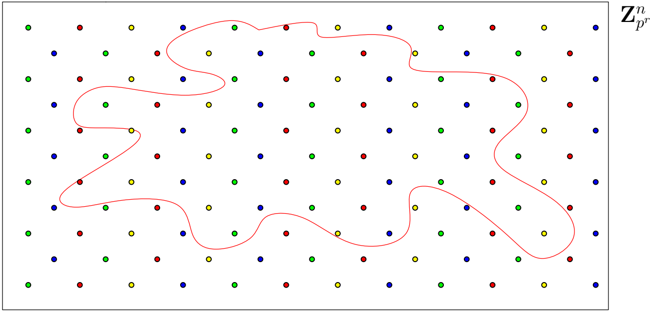

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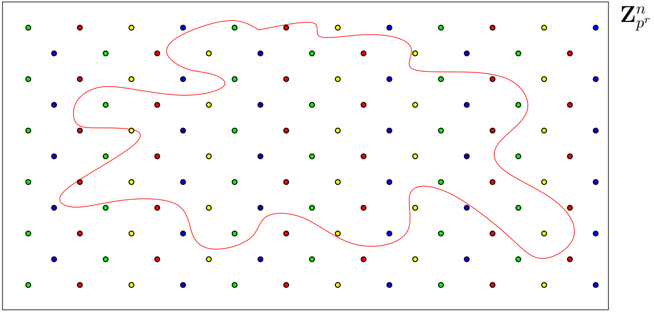

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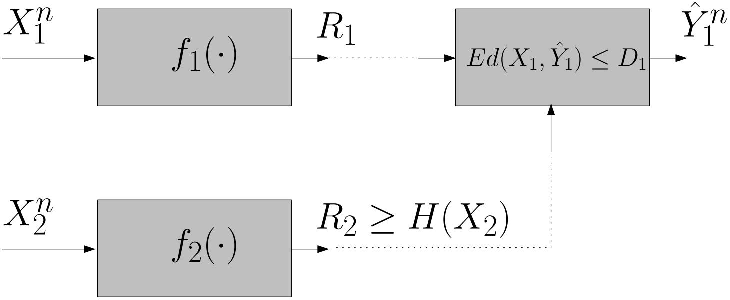
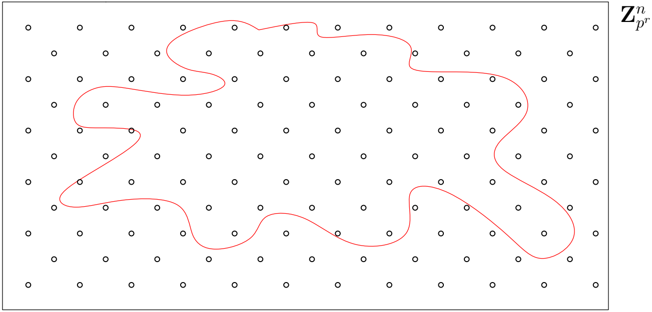


$$R_1 \geq I(X_1; U \mid X_2) = I(X_1; U) - I(X_2; U)$$

Wyner-Ziv via group codes: Quantization

- Group code good for $(\mathscr{X}_1, \mathscr{U}, P_{X_1 U})$



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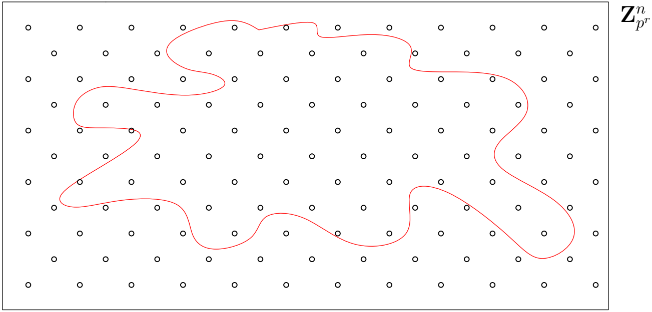

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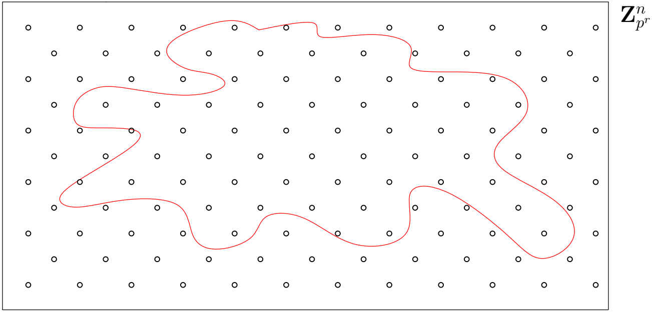

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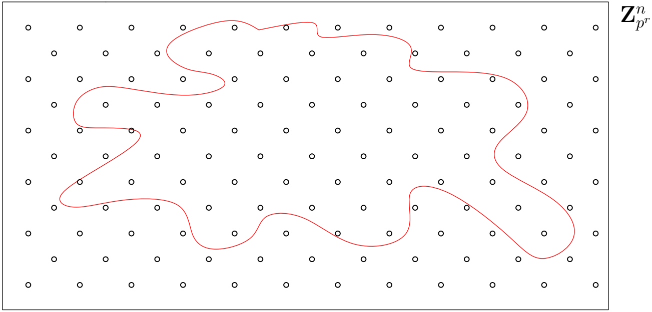

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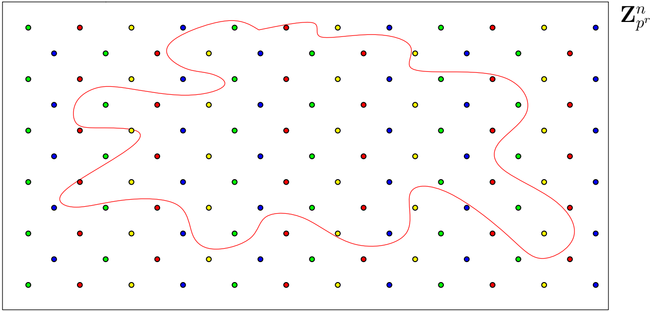

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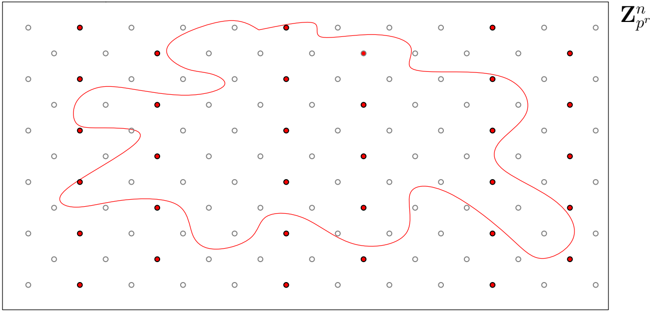

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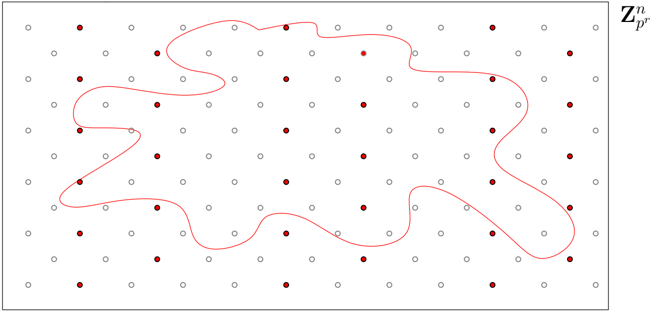

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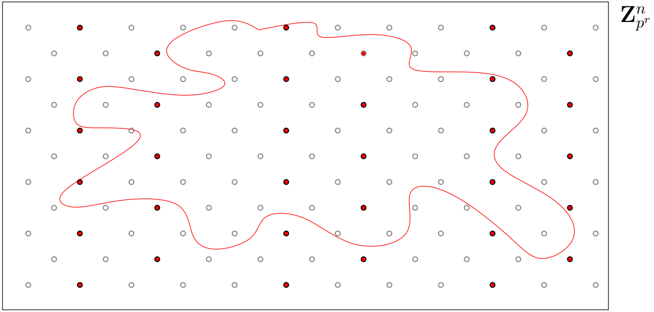
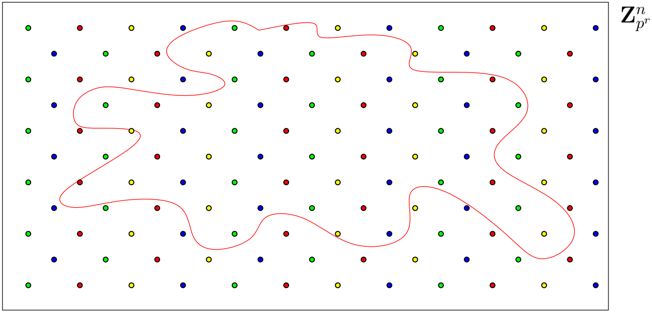


Wyner-Ziv via group codes: Rate Region

- Only coset leaders (colors) get transmitted



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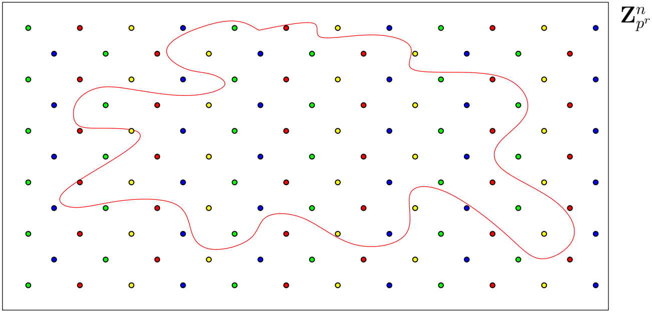

Wyner-Ziv via group codes: Rate Region

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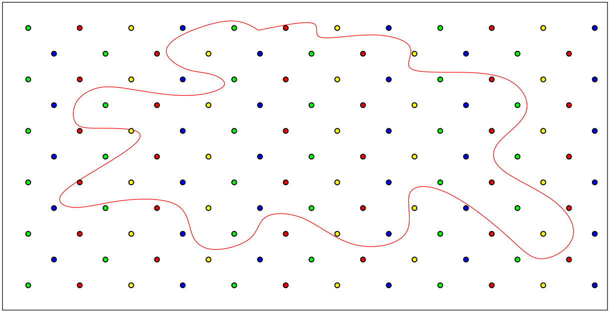

$\mathbf{Z}_{p^r}^n$

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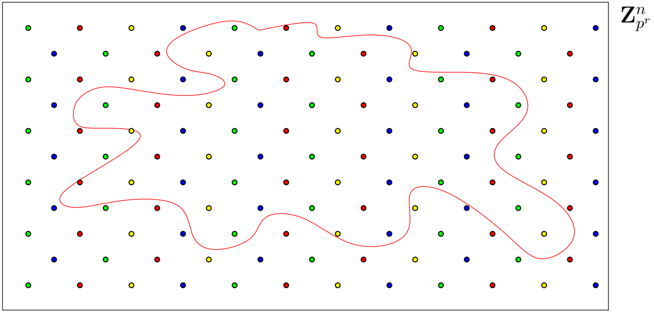

Outline

1. Thesis Overview
2. Information Theory: An Introduction
3. Random Codes for Distributed Source Coding
4. Nested Group Codes
5. Distributed Source Coding : An Inner Bound
6. Conclusions

Overview of the coding scheme

- Fix test channels $P_{X_1 X_2 UV} = P_{X_1 X_2} P_{U|X_1} P_{V|X_2}$
- Decoder interested in some reconstruction function $g(U,V)$
  - $g(U,V)$ group operation in abelian group G: Nested group codes
  - What if it isn't?
  - "Embed" $g(U,V)$ in a suitable abelian group
- Decompose G into primary cyclic groups $G \cong \mathbb{Z}_{p_1^{e_1}} \oplus \mathbb{Z}_{p_2^{e_2}} \cdots \oplus \mathbb{Z}_{p_k^{e_k}}$
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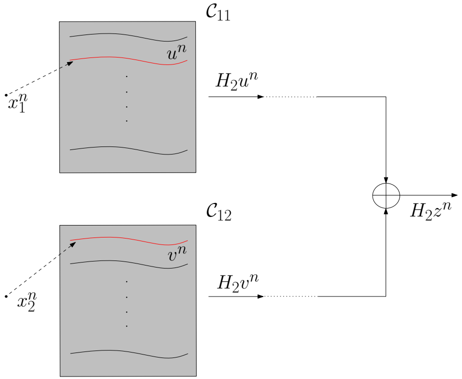

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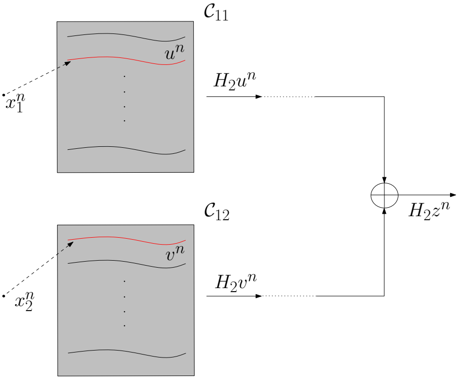

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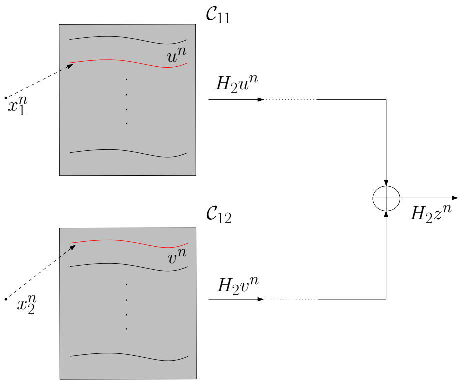

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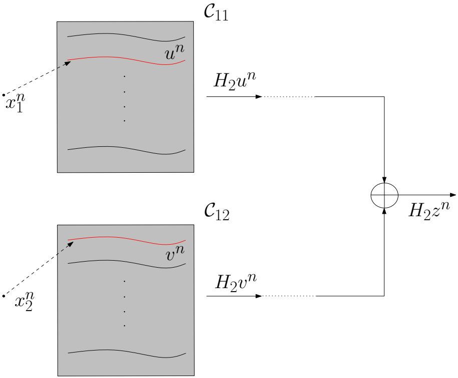

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Achievable Rates

Achievable rates

The set of tuples (R_1, R_2, D) that satisfy

$$R_1 \geq \max_{0 \leq i < r} \left(\frac{r}{r-i}\right) (H(Z) - H([Z]_i)) - r|H(U|X) - \log p^{r-1}|^+$$
$$R_2 \geq \max_{0 \leq i < r} \left(\frac{r}{r-i}\right) (H(Z) - H([Z]_i)) - r|H(V|Y) - \log p^{r-1}|^+$$
$$D \geq \mathbb{E} d(X, Y, g(U, V))$$

are achievable.

- More general rate region possible by
  - Embedding in general groups and using digit decomposition
  - Alternative coding strategy - Encode (U, V) instead of Z

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  - For every group : P_X, P_Z such that that group gives best embedding

P_X	P_Z	$R_{\mathbb{Z}_4}$	$R_{\mathbb{Z}_7}$	$R_{\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2}$	$R_{\mathbb{Z}_4\oplus\mathbb{Z}_4}$
$[\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4}]$	$[\frac{1}{2}\,0\,\frac{1}{4}\,\frac{1}{4}]$	3	3.9056	3.1887	3.5
$[\frac{3}{10}\,\frac{6}{10}\,\frac{1}{10}\,0]$	$[0\,\frac{4}{5}\,\frac{1}{20}\,\frac{3}{20}]$	2.3911	2.0797	2.4529	2.1796
$[\frac{1}{3}\,\frac{1}{10}\,\frac{1}{2}\,\frac{1}{15}]$	$[\frac{3}{7}\,\frac{1}{7}\,\frac{1}{7}\,\frac{2}{7}]$	3.6847	4.5925	3.3495	3.4633
$[\frac{9}{10}\,\frac{1}{30}\,\frac{1}{30}\,\frac{1}{30}]$	$[\frac{3}{20}\,\frac{3}{4}\,\frac{1}{20}\,\frac{1}{20}]$	2.308	2.7065	1.9395	1.7815

Table: Example distributions for which embedding in a given group gives the lowest sum rate.

Lossy Reconstruction of binary XOR

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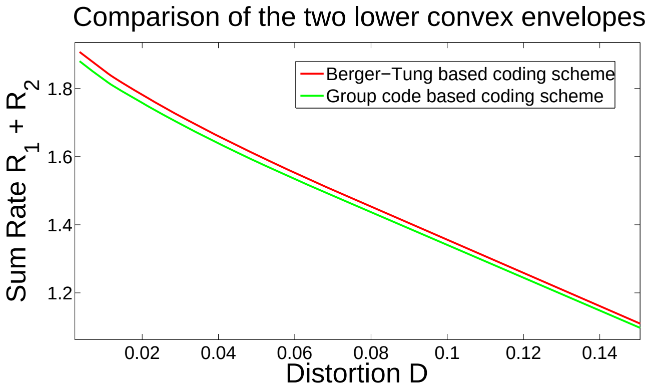
Lossy Example contd.



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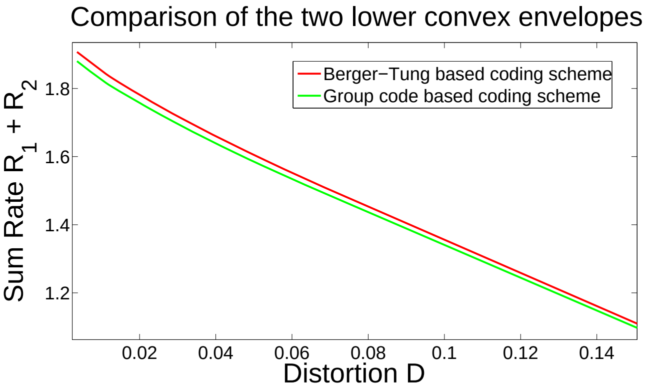

- Rate gains over the Berger-Tung based scheme
- Implies Berger-Tung inner bound not tight for three-user case

Outline

1. Thesis Overview
2. Information Theory: An Introduction
3. Random Codes for Distributed Source Coding
4. Nested Group Codes
5. Distributed Source Coding : An Inner Bound
6. Conclusions

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Thank You

Questions?

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Table: Addition in $\mathbb{F}_4$

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Suen's Inequality

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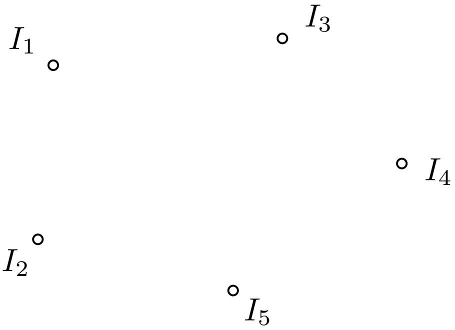

- Each indicator is a node

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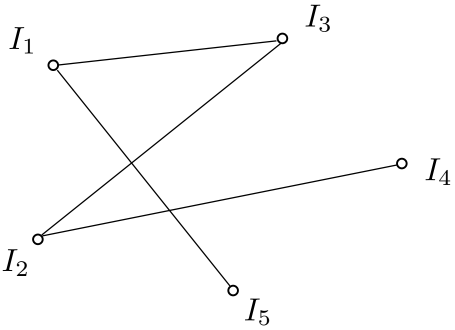

- Edges between dependent indicators

Suen's Inequality

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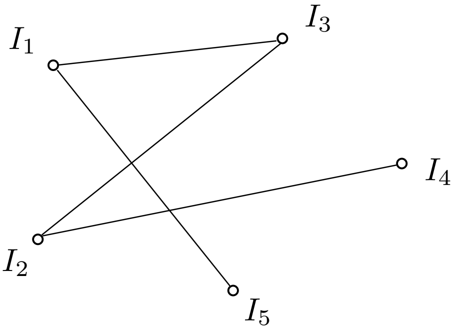

- $\lambda = \sum_i \mathbb{E} I_i$
- $\Delta = \frac{1}{2} \sum_i \sum_{j \sim i} \mathbb{E}(I_i I_j)$
- $\delta = \max_i \sum_{k \sim i} \mathbb{E} I_k$

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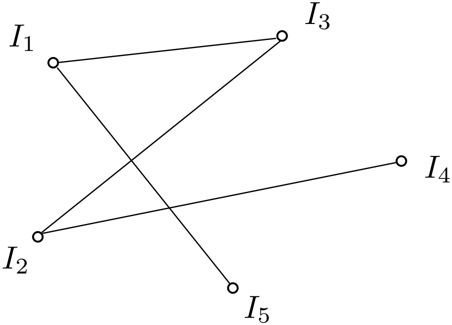

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$$P\left(\sum_i I_i = 0\right) \leq \exp\left\{-\min\left(\frac{\lambda^2}{8\Delta}, \frac{\lambda}{2}, \frac{\lambda}{6\delta}\right)\right\}$$

Proof Techniques - Group Source Codes contd.

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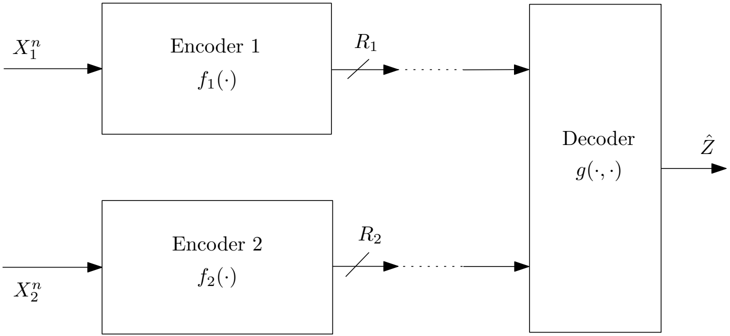
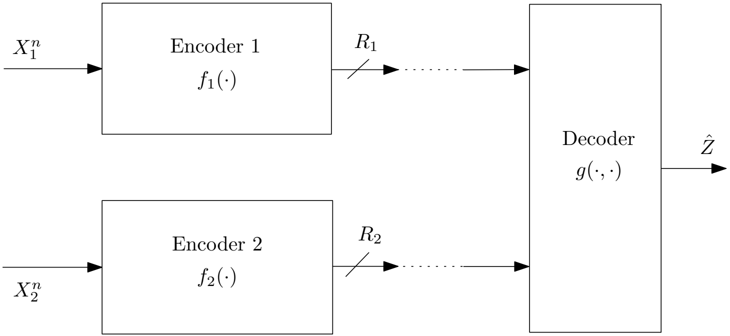
  - Have to estimate the degree of each vertex in the dependency graph

Highlights of the work



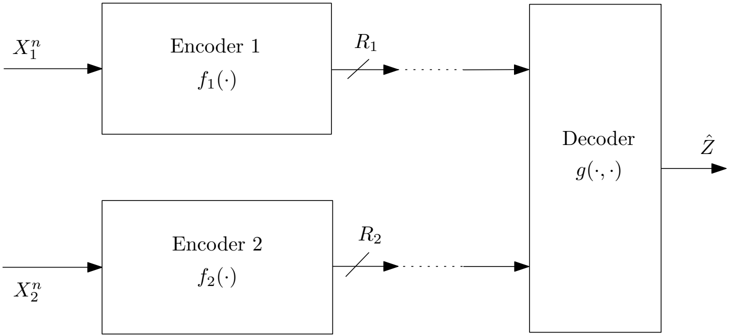
- $X_1, X_2 \sim \mathcal{N}(0,1), \mathbb{E}(X_1 X_2) = \rho > 0$

Highlights of the work



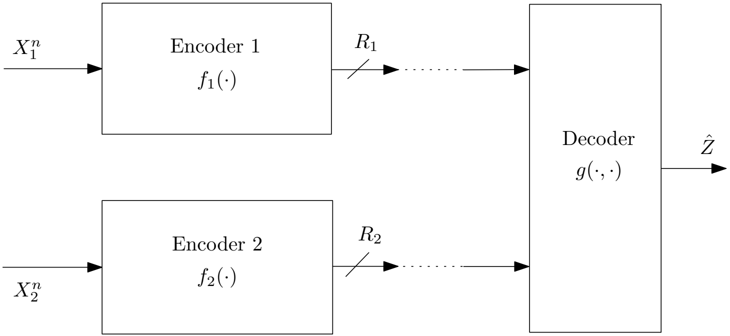
- $\hat{Z} = X_1 - cX_2$, $c > 0$

Highlights of the work



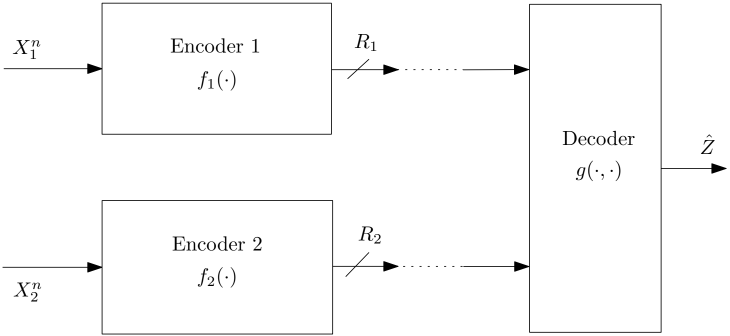
- $\mathbb{E}d(X_1, X_2, \hat{Z}) = \mathbb{E}(X_1 - cX_2 - \hat{Z})^2$

Highlights of the work



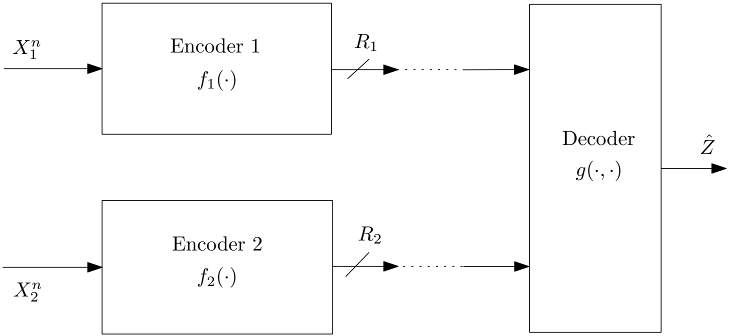
- Objective: Achievable rates (R_1, R_2) at distortion D

Highlights of the work



- Achievable rate region using nested lattice codes

Highlights of the work



- Showed achievability of (R_1, R_2, D) when

$$2^{-2R_1} + 2^{-2R_2} \leq \left(\frac{\sigma_Z^2}{D}\right)^{-1}$$