Toward a New Approach to Distributed Information Processing: Harnessing Group Structure

S. Sandeep Pradhan
(Joint work with Dinesh Krithivasan)

University of Michigan, Ann Arbor

Distributed Information Processing

- Proliferation of wireless sensor network applications
- Supported by distributed information processing
- Look at distributed source coding problems
- Information-theoretic perspective

Information and Coding theory: Traditional Approach

Information Theory:

- Develop efficient information processing strategies for communication
- Obtain computable performance limits
- Random coding: probability distribution on a collection of communication systems
- Show good average performance
- Encoding and decoding have exponential complexity

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Coding Theory:

- Approach these limits using structured codes (Ex: linear codes)
- Fast encoding and decoding algorithms
- Objective: use structured codes for practical implementability

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- Prob. distribution on a collection of codebooks (ensemble)
- Lot of bad codebooks in the ensemble
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  - Binary Symmetric case: almost all linear codes achieve expurgated bound without expurgation

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Linear Codes: Built on Galois Fields

Caution: Even in point-to-point set-up

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  - Shannon rate-distortion function
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- Linear codes do not achieve in general
  - Shannon rate-distortion function
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- Injection of some non-linearity appears to be necessary for optimality

Prior Work: Linear codes for multi-terminal communication

- Linear codes for symmetric source/channel coding problems
- Lattice codes for Gaussian source/channel coding problems

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- Linear codes for symmetric source/channel coding problems
- Lattice codes for Gaussian source/channel coding problems
- Examples: (incomplete list)
  - Korner-Marton
  - Han-Kobayashi
  - Ahlswede-Han
  - Forney-Barg
  - Philosof-Zamir-Erez
  - Nazer-Gastpar
  - Krithivasan-Pradhan
  - Viswanath
  - ...

Contributions of the present work

- A unified approach to distributed source coding problem
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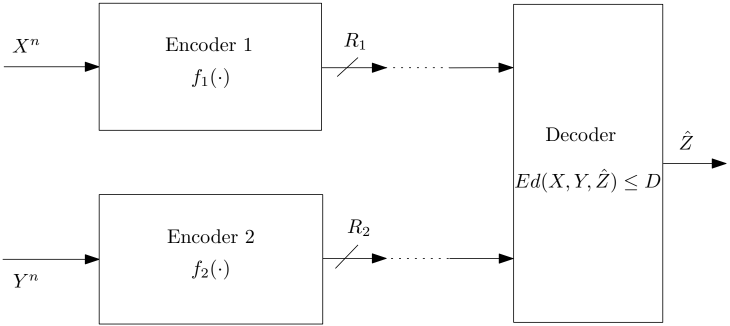
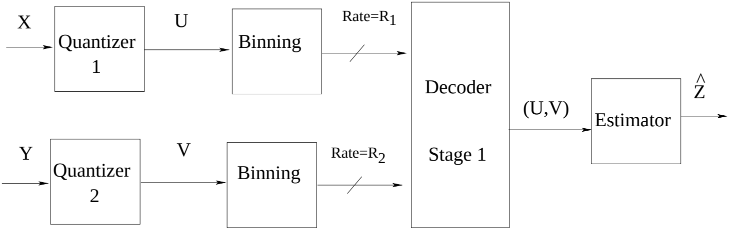
- A unified approach to distributed source coding problem
- Discrete memoryless setting
- Applicable to general source statistics and distortion functions
- Based on abstract abelian groups: groups capture structure
- Nested codes over groups
- New rate-distortion region
- Previously known rate distortion regions can be achieved using nested linear codes

A Distributed Source Coding Problem

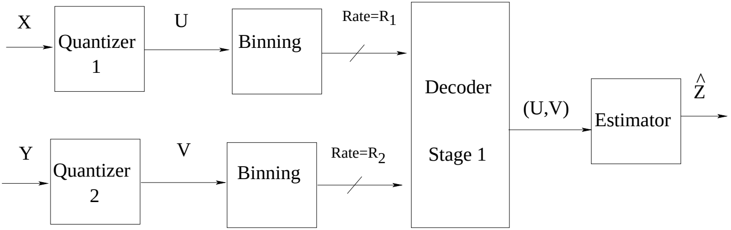


- Set of encoders observe different components of a vector source
- Central decoder receives quantized observations from the encoders
- Best known rate region - Berger-Tung based



- For given distortion D
- Quantize X to U and quantize Y to V

Berger-Tung Based Rate Region: Known Results



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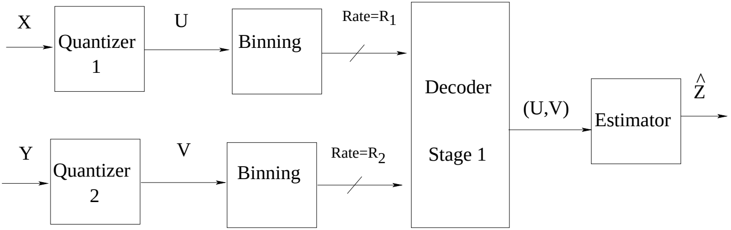

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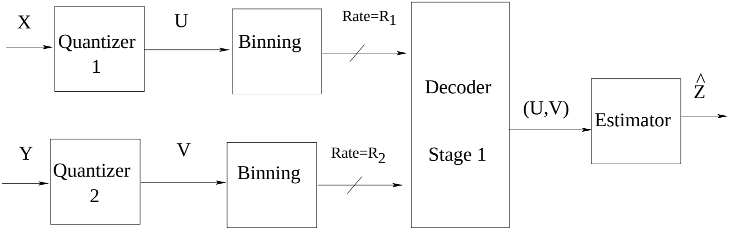

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- Rates incurred in quantization $= I(X; U)$ and $I(Y; V)$
- Rate rebate by exploiting correlation between U and $V = I(U; V)$

Berger-Tung Based Rate Region contd.

- Hence

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\begin{aligned}
R_1 &\geq I(X;U) - I(U;V) \\
R_2 &\geq I(Y;V) - I(U;V) \\
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- Observations:
  - Estimator $\hat{Z} = F(U,V)$ may be an information lossy transformation
  - Is it possible to reconstruct directly $\hat{Z}$ at the decoder?
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  - Can a joint design of quantizer and binning get better performance?

Example 1

- X, Y - 3 bit correlated binary sources, $d_H(X, Y) \leq 1$
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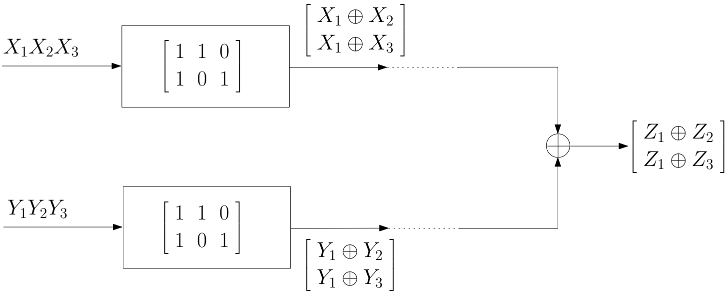
- Can we do better?

Example 1: Linear Coding Scheme



- $X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = \hat{Z}_1 \oplus \hat{Z}_2$
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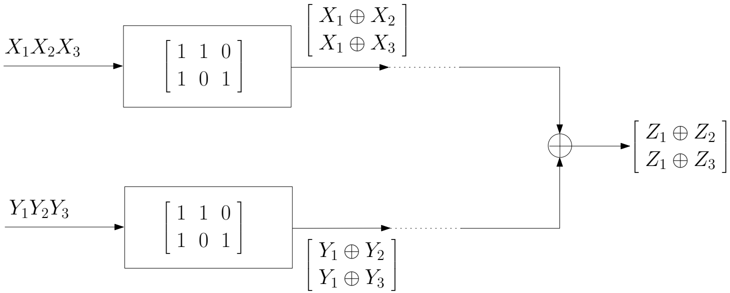

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- Sum rate: $2 + 2 = 4$ bits
- Significant features:
  - encoding function commutes with function $\hat{Z} = X \oplus Y$
  - Identical binning at both encoders
  - Linear codes

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01	01	00	11	10
10	10	11	00	01
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- Map binary sources into $\mathbb{F}_4$
- Encode sequentially one digit at a time
- Previously decoded digits $=$ side information at the decoder

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- Map binary sources into $\mathbb{F}_3$
- Construct linear codes over $\mathbb{F}_3$
- Can do better than Slepian-Wolf coding

Overview of our Coding Scheme

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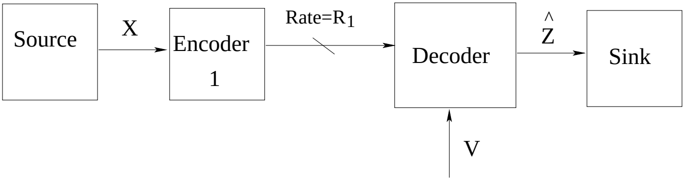
- We looked at linear codes for binning.
- How to do quantization using structured codes?

Berger-Tung Rate Region: Closer Look



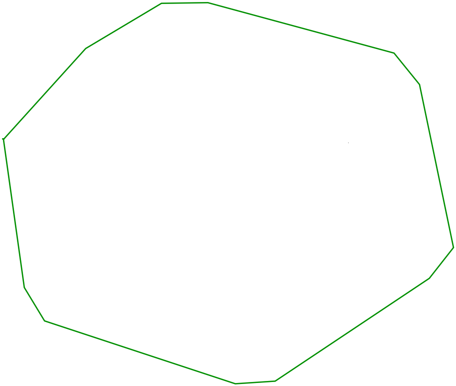
- $R_2 = I(Y;V)$, $\quad R_1 = I(X;U) - I(U;V)$
- Source X: alphabet $\mathcal{X}$, distribution p_X
- Side Information V: alphabet $\mathcal{V}$, distribution $p_{V|X}$.
- Reconstruction: alphabet $\hat{\mathcal{Z}}$.
- Compress X into bits to achieve a target distortion.

Berger-Tung Rate Region Contd.

Encoding: Quantization + Binning

- Quantize X to U with rate $I(X;U)$
- Partition the quantizer into bins of rate $I(U;V)$
- Each bin is a good channel code for the channel $p_{V|U}$.
- Send the bin index to the decoder
- Recover the quantizer codeword from the bin using V.

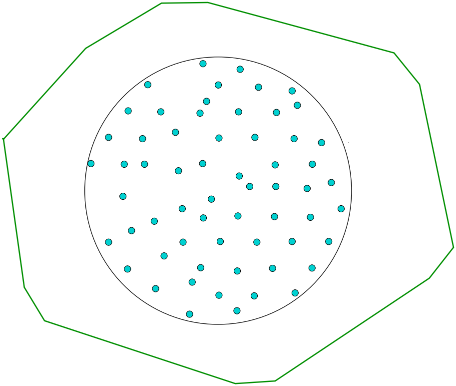
Space in which quantizer is built



- Space: $\mathcal{U}^n$
- Quantizer : X to U

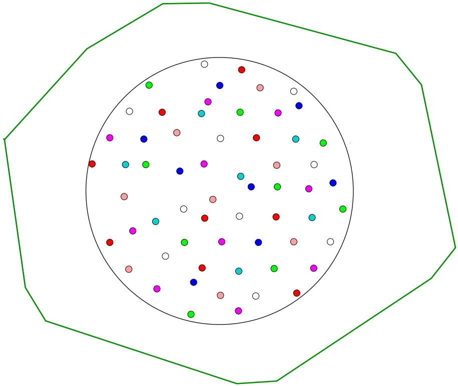
- Joint histogram of source word and its quantized version $\approx p_{XU}$

Good Quantizer



- Quantize X to U
- Must cover a specific region
- Typical set with respect to p_U.
- Rate: $I(X;U)$.
- Shannon source code

Quantizer Partition into bins



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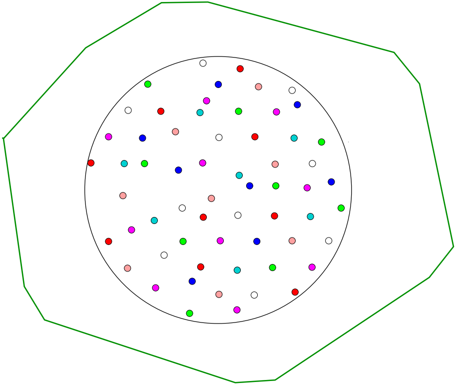

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- Bin = Good channel code
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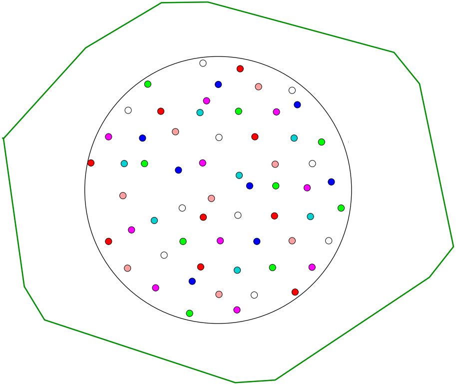

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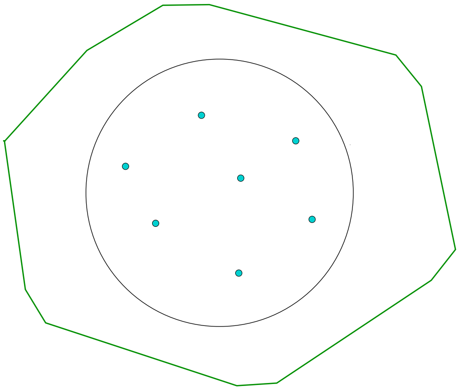
- Bin Rate: $I(U;V)$.
- Bin density rate:

  $= I(X;U) - I(U;V)$

Good Channel Code



- Each bin is a good channel code
- Pack codewords in the region
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Illustration of Encoding

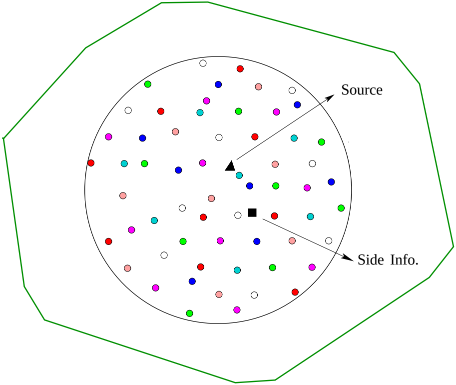
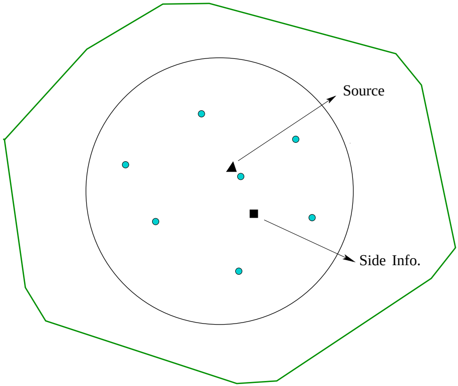


Illustration of Decoding



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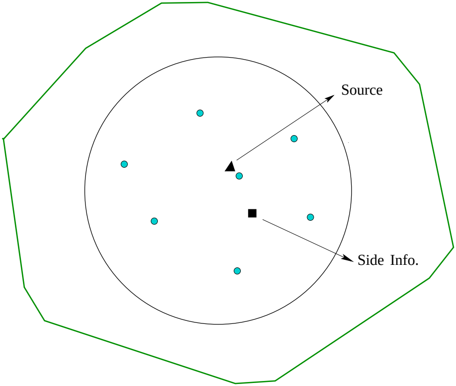

- None of the codes used here have any algebraic structure

Try this with linear codes

RECALL

- Linear Codes do not achieve
  - Shannon rate-distortion function
  - Shannon capacity-cost function
- Linear code do not achieve
  - the source rate $I(X;U)$
  - the channel rate $I(U;V)$

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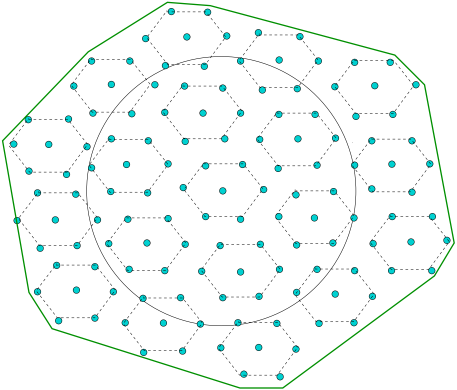
Q: How to achieve $I(X;U) - I(U;V)$ using linear codes?

Linear code as a Quantizer: Lemma 1



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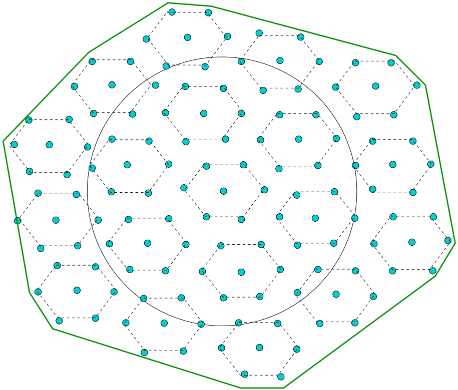

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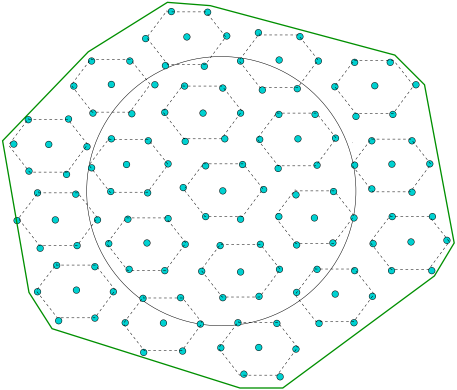

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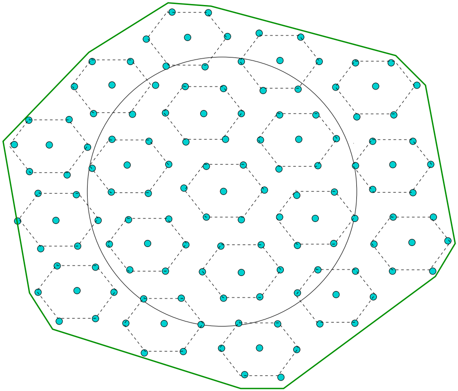

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- Refer: Good linear source code

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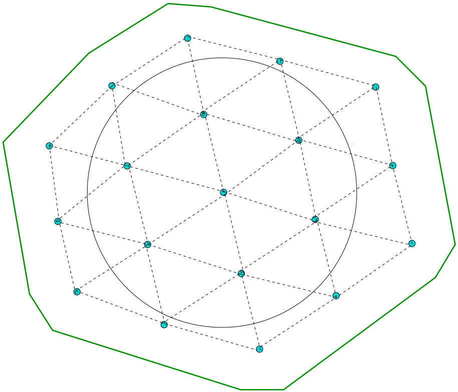

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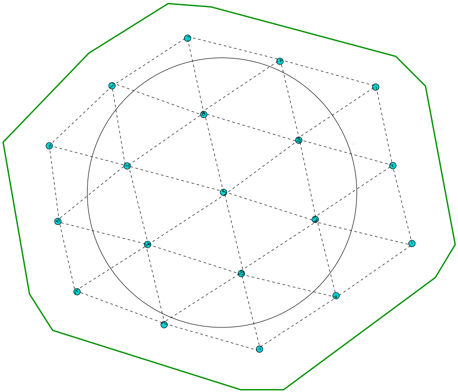

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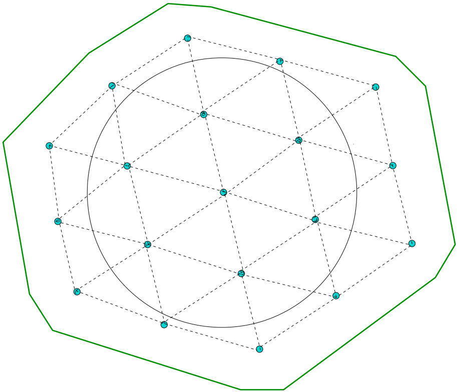

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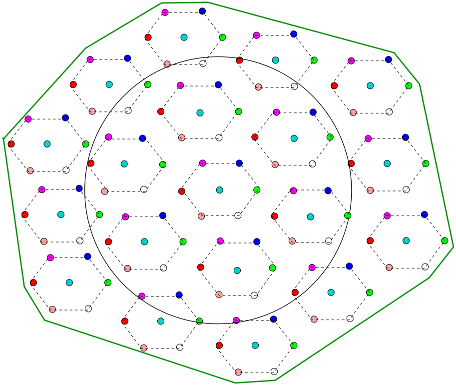

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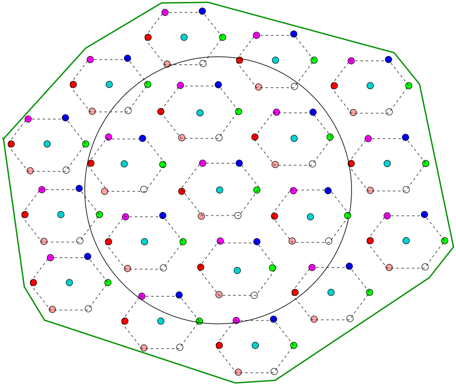

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- Coset density rate

$$= \log|\mathcal{U}| - H(U|X) - \log|\mathcal{U}| + H(U|V)$$
$$= I(X;U) - I(U;V)$$

Linear Codes: Upshot

- Built on Galois fields
- Nested linear codes can achieve Berger-Tung bound
- Good nested linear codes can achieve Shannon limit
  - Take V=constant
  - Source Code Rate: $\log|\mathcal{U}| - H(U|X)$
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- A specific form of non-linearity

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Next: extension to arbitrary abelian groups

Groups - An Introduction

- G - a finite abelian group of order n
- $G \cong \mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}} \cdots \times \mathbb{Z}_{p_k^{e_k}}$
- G isomorphic to direct product of possibly repeating primary cyclic groups

$$g \in G \Leftrightarrow g = (g_1, \ldots, g_k),\ \ g_i \in \mathbb{Z}_{p_i^{e_i}}$$

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- Call g_i as the ith digit of g
- Enough to prove coding theorems for primary cyclic groups
- Extension to arbitrary abelian groups through digit decomposition

Example 4:

- Let group size be 36
- $36 = 2^2 \times 3^2$
- Abelian groups of order 36:
  - $\mathbb{Z}_4 \times \mathbb{Z}_9$: Two digits
  - $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$: Three digits
  - $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$: Three digits
  - $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$: Four digits

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Example: $\hat{Z} = U \vee V$ can be embedded in $\mathbb{Z}_3$

Nested Group Codes - Motivation

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  - Cosets bin the entire space
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- Lossy coding: Need to quantize first
  - Dilute coset density - Nested group codes
  - Fine code - Quantizes the sources
  - Coarse code - Bins only the fine code

Nested Group Codes

- Group code over $\mathbb{Z}_{p^r}^n$: $\mathcal{C} < \mathbb{Z}_{p^r}^n$
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- $(\mathcal{C}_1, \mathcal{C}_2)$ nested if $\mathcal{C}_2 \subset \mathcal{C}_1$
- We need:
  - $\mathcal{C}_1 < \mathbb{Z}_{p^r}^n$: "good" source code
    - Can find $u^n \in \mathcal{C}_1$ jointly typical with source x^n
  - $\mathcal{C}_2 < \mathbb{Z}_{p^r}^n$: "good" channel code
    - Can distinguish between typical channel noise sequences

Good Group Source Codes

- Good group source code $\mathcal{C}_1$ for the triple $(\mathcal{X}, \mathcal{U}, P_{XU})$
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- Compare with optimal random code: $H(U) - H(U|X) = I(X;U)$
- Compare with linear code: $\log p^r - H(U|X)$
- Not good in Shannon sense
- Extra penalty for imposing group structure beyond linearity

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Proof Technique

- Matrix characterization of subgroups of direct product of a group
- random coding over subgroups
- Suen's inequality [1998]

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- Step 5: Nest the channel code inside the source codes
  - $\mathcal{C}_2(i) < \mathcal{C}_{11}(i)$ and $\mathcal{C}_2(i) < \mathcal{C}_{12}(i)$
  - Identical binning of quantizers

Encoding and Decoding

Encoders: at the ith stage

- Encode the sources X and Y to digits U_i and V_i sequentially
  - quantize + bin

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Decoder: at the ith stage

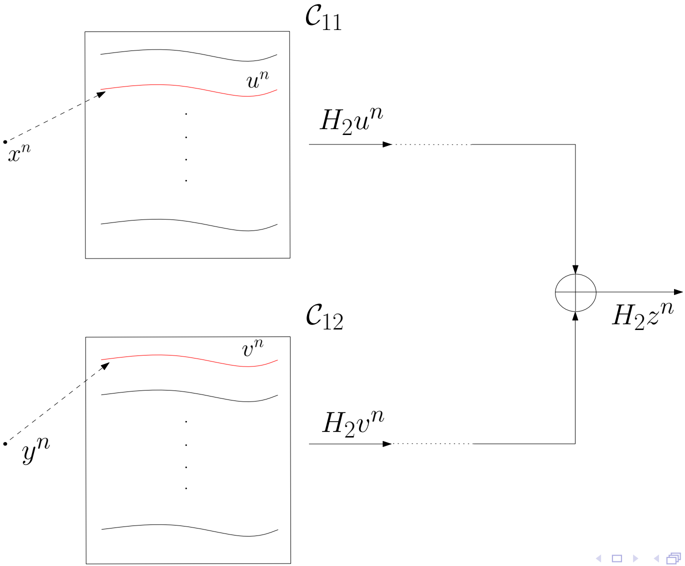
- Recover $\hat{Z}_i = U_i \oplus_{G_i} V_i$
- Use previously decoded digits as side information

Example 5

Suppose we embed $F(U, V)$ in $\mathbb{Z}_4 \times \mathbb{Z}_7$

- We have two digits: $(U_1, V_1, \hat{Z}_1)$ and $(U_2, V_2, \hat{Z}_2)$
- Two stages
- Stage 1: $\mathbb{Z}_4$ operation
- Stage 2: $\mathbb{Z}_7$ operation

Coding Strategy: Nested group codes $\mathcal{C}_2 < \mathcal{C}_{11}, \mathcal{C}_{12}$



Achievable Rates

Theorem

The set of tuples (R_1, R_2, D) *that satisfy*

$$R_1 \geq \max_{0 \leq i < r} \left(\frac{r}{r-i} \right) (H(\hat{Z}) - H([\hat{Z}]_i)) - \min\{H(U|X), r|H(U|X) - \log p^{r-1}|^+\}$$

$$R_2 \geq \max_{0 \leq i < r} \left(\frac{r}{r-i} \right) (H(\hat{Z}) - H([\hat{Z}]_i)) - \min\{H(V|Y), r|H(V|Y) - \log p^{r-1}|^+\}$$

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are achievable.

- More general rate region possible by
  - Embedding in general groups and using digit decomposition
  - Alternative coding strategy at ith stage - Encode (U_i, V_i) instead of $\hat{Z}_i$

Example 6: Lossless reconstruction of quaternary function

- X and Y take values in $\{0, 1, 2, 3\}$
- Lossless reconstruction of $\hat{Z} = X - Y \mod 4$

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- $\mathbb{Z}_4,\ \mathbb{Z}_7,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2,\ \mathbb{Z}_4 \oplus \mathbb{Z}_4$ give non-trivial performance

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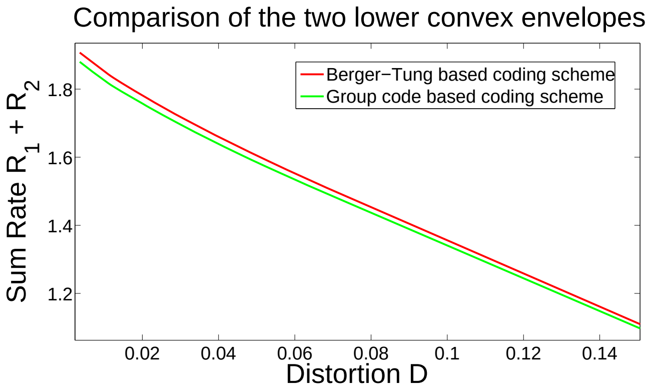
p_X	p_E	$R_{\mathbb{Z}_4}$	$R_{\mathbb{Z}_7}$	$R_{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2}$	$R_{\mathbb{Z}_4 \oplus \mathbb{Z}_4}$
$[\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}]$	$[\frac{1}{2}\ 0\ \frac{1}{4}\ \frac{1}{4}]$	3	3.9056	3.1887	3.5
$[\frac{3}{10}\ \frac{6}{10}\ \frac{1}{10}\ 0]$	$[0\ \frac{4}{5}\ \frac{1}{20}\ \frac{3}{20}]$	2.3911	2.0797	2.4529	2.1796
$[\frac{1}{3}\ \frac{1}{10}\ \frac{1}{2}\ \frac{1}{15}]$	$[\frac{3}{7}\ \frac{1}{7}\ \frac{1}{7}\ \frac{2}{7}]$	3.6847	4.5925	3.3495	3.4633
$[\frac{9}{10}\ \frac{1}{30}\ \frac{1}{30}\ \frac{1}{30}]$	$[\frac{3}{20}\ \frac{3}{4}\ \frac{1}{20}\ \frac{1}{20}]$	2.308	2.7065	1.9395	1.7815

Table: Example distributions for which embedding in a given group gives the lowest sum rate.

Example 7: Lossy Reconstruction of binary XOR

- Correlated binary sources (X, Y)
- Reconstruct $\hat{Z} = X \oplus_2 Y$ within Hamming distortion D
- U, V - binary auxiliary random variables
- $F(U, V)$ - one of 16 possibilities depending on $(p_{U|X}, p_{V|Y})$

Lossy Example contd.



- Rate gains over the Berger-Tung based scheme
- Implies Berger-Tung inner bound not tight for three-user case

Special cases

- Lossless compression using group codes - achievable rates
- Lossy compression for arbitrary sources and distortion measures using group codes
- Nested linear codes - Shannon rate-distortion bound for arbitrary sources and additive distortion measures

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- Lossless compression using group codes - achievable rates
- Lossy compression for arbitrary sources and distortion measures using group codes
- Nested linear codes - Shannon rate-distortion bound for arbitrary sources and additive distortion measures
- Recovers known rate regions (using nested linear codes) of
  - Berger-Tung problem
  - Wyner-Ziv problem, Wyner-Ahlswede-Korner problem
  - Yeung-Berger problem
  - Slepian-Wolf problem, Korner-Marton problem

Conclusions

- Presented a nested group codes based coding scheme
- Recovered known rate regions of several distributed source coding problems
- Offers rate gains over the Berger-Tung based coding scheme
- Extensions:
  - Codes over groups for multi-user channel coding problems
  - Codes over non-abelian groups