New Achievable Rates for the Multiple-Access Channel with Feedback

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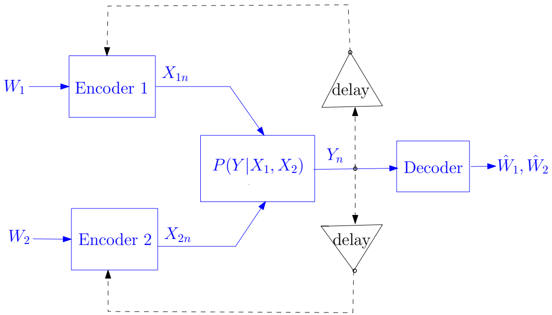
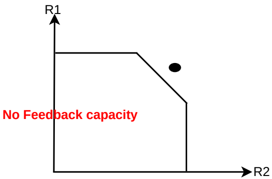
Multiple-access Channel (MAC)



Feedback

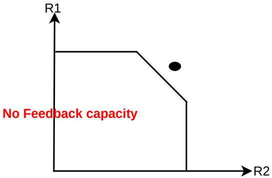
- X_{1n} function of (W_1, Y^{n-1})
- X_{2n} function of (W_2, Y^{n-1})
- Example of [Gaarder-Wolf '75] showed FB can increase capacity

Cover-Leung Scheme [1981]



Message graph before transmission

Cover-Leung Scheme [1981]



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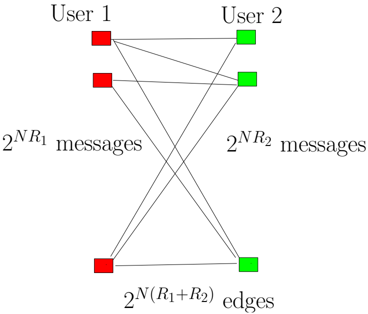


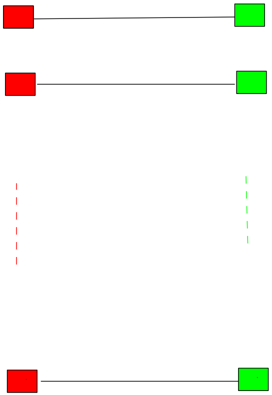
Typicality Graph

After receiving $\mathbf{Y}$

- LEFT vertices: $\mathbf{x_1}[w_1]$ j.typ with $\mathbf{Y}$
- RIGHT vertices: $\mathbf{x_2}[w_2]$ j.typ with $\mathbf{Y}$
- Edge: $\Rightarrow$ $(\mathbf{x_1}[w_1], \mathbf{x_2}[w_2])$ j.typ with $\mathbf{Y}$ - Possible message pair

Typicality graph

C-L impose conditions so that



$$2^{N(R_1+R_2)} \cdot 2^{-NI(X_1X_2;Y|U))} \text{ edges}$$

NEXT STAGE

- Y is common side information
- Point-to-point communication
- Source coding + channel coding
- Can be resolved by $\mathbf{U} \sim P_U$ if

number of edges $< 2^{NI(U;Y)}$

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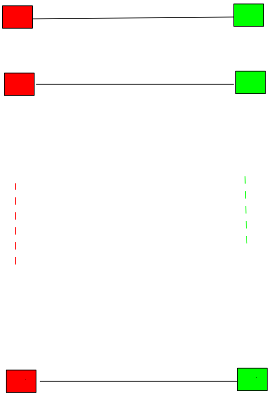

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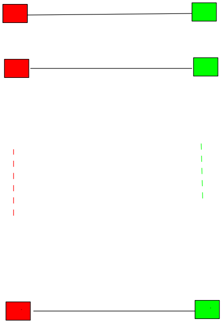
C-L Rates

How to achieve this graph?



$2^{N(R_1+R_2)} \cdot 2^{-NI(X_1X_2;Y|U))}$ edges

$$R_1 < I(X_1; Y|X_2U)$$

$$R_2 < I(X_2; Y|X_1U)$$

RESOLUTION with **U**: $R_1 + R_2 - I(X_1X_2; Y|U) < I(U; Y)$

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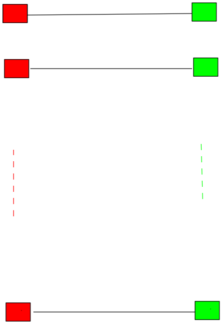

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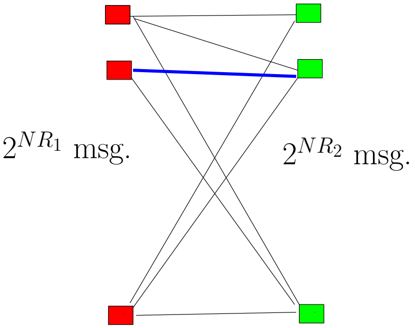
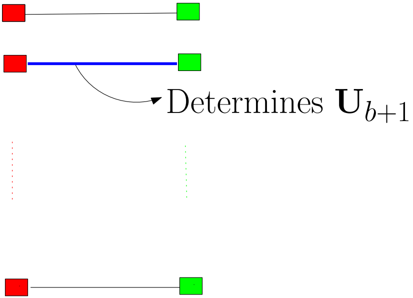
Block-Markov Superposition

Transmission in blocks $1, \ldots, L$

Message pair for block b



Decoder graph after $\mathbf{Y}_b$



- $P_U \cdot P_{X_1|U} \cdot P_{X_2|U} \cdot P_{Y|X_1X_2}$
- Block b: fresh info $\mathbf{X}_{1b}, \mathbf{X}_{2b}$ superposed on resolution info $\mathbf{U}_b$

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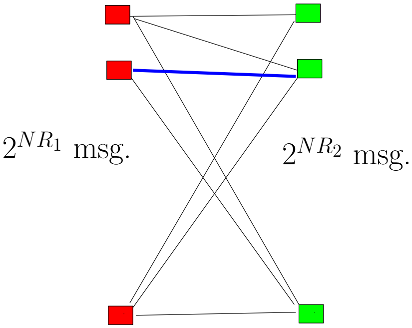

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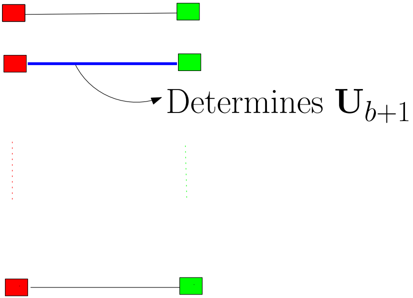

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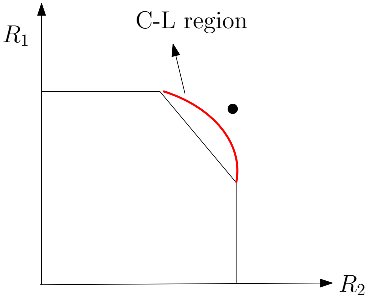
Optimality

- C-L scheme optimal for channels where

$$X_1 = f(X_2, Y) \text{ or } X_2 = g(X_1, Y)$$

- The channel ensures we always get the graph we want!
- Not optimal in general
  - white Gaussian MAC [Ozarow 84]

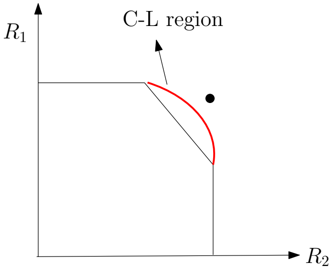
Outside the C-L region



Bross-Lapidoth '05

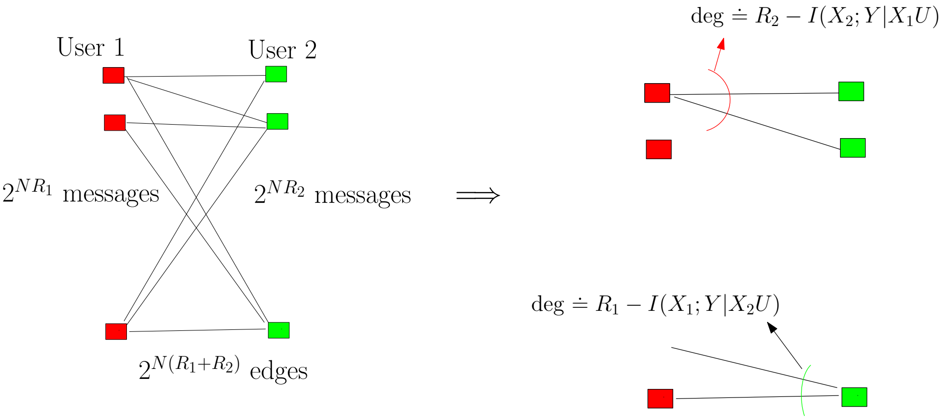
- C-L scheme: block length n
  - encoders now cannot decode using FB
- Spend EXTRA time $\eta \cdot n$ to exchange messages
- New block length $(1 + \eta)n$
- Can improve rates

Our approach



In terms of message graph ...

Graph at decoder



- Each encoder CANNOT decode message of the other using **Y**
- Degree of each vertex > 1
- But messages are *CORRELATED*

Correlated messages

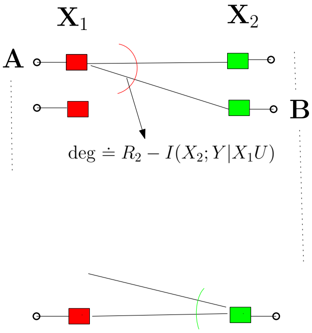
- Correlated messages on MAC (no FB)
  - [Cover, El G, Salehi 81], [Pradhan et al 07]
- Can achieve higher rates than independent messages.

First attempt . . .

Sending correlated messages

Common output two-way channel

Typicality graph given **Y**



For each $\mathbf{X}_1$, generate one $\mathbf{A} \sim P_{A|X_1Y}$

For each $\mathbf{X}_2$, generate one $\mathbf{B} \sim P_{B|X_2Y}$

- Enc. 1 sends **A** corresp. to LEFT vertex
- Enc. 2 sends **B** corresp. to RIGHT vertex
- **A** and **B** are correlated

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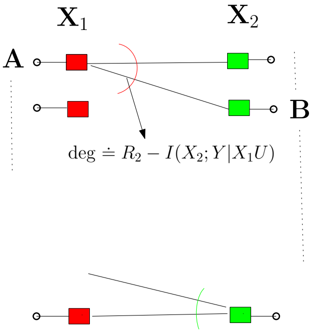

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Decoding

- Enc. 1 decodes **B**, enc. 2 decodes **A**
- Decoder cannot yet decode **A**, **B**

Decoder's typicality graph:

Decoding

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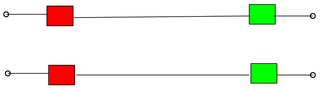

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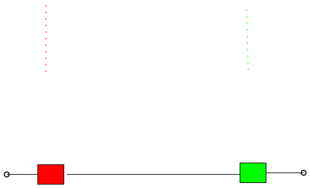
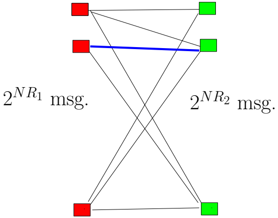
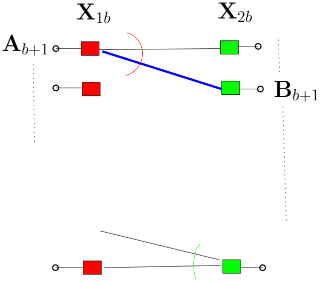


Block Markov Superposition

Message pair for block b



After receiving $\mathbf{Y}_b$



After receiving $\mathbf{Y}_{b+1}$:

Block Markov Superposition

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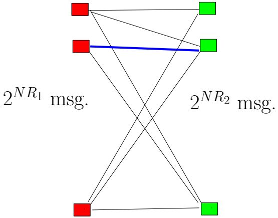

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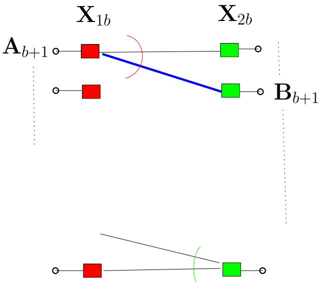

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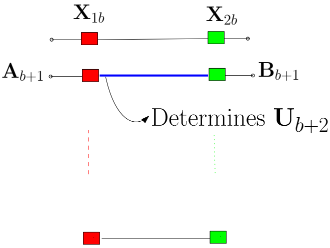

Issues

- L blocks of transmission
- For block b, $1 \leq b \leq L$, if we generate

$$\mathbf{A}_b \leftrightarrow (\mathbf{Y}_{b-1}, \mathbf{X}_{1(b-1)}), \qquad \mathbf{B}_b \leftrightarrow (\mathbf{Y}_{b-1}, \mathbf{X}_{2(b-1)})$$

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Does not yield 'single-letter' characterization

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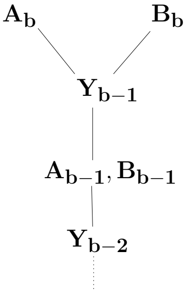

Does not yield 'single-letter' characterization

Fix

WANT

Seqs. in each block $\sim P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_{Y|X_1X_2}$

- Define fn. $f : \mathcal{U} \times \mathcal{A} \times \mathcal{B} \times \mathcal{Y} \rightarrow \mathcal{W}$

$$\mathbf{W}_b = f((\mathbf{U}, \mathbf{A}, \mathbf{B}, \mathbf{Y})_{b-1})$$

- W summarizes info common to both encoders at end of prev. block - e.g. $\mathbf{W}_b = \mathbf{Y}_{b-1}$

CONSISTENCY COND. 1

GOAL: Ensure $\mathbf{W}_b \sim P_W$ in each block:

$$P_W(w) = \sum P(u, a, b, y)\mathbf{1}(f(u, a, b, y) = w)$$

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CONSISTENCY COND. 2

- $\mathbf{A}_b$ gen. from $(\mathbf{U}, \mathbf{A}, \mathbf{B}, \mathbf{Y}, \mathbf{X_1})_{b-1} \sim Q^1$
- $\mathbf{B}_b$ gen. from $(\mathbf{U}, \mathbf{A}, \mathbf{B}, \mathbf{Y}, \mathbf{X_2})_{b-1} \sim Q^2$

Given $\mathbf{W}_b$, ensure $\mathbf{A}_b, \mathbf{B}_b \sim P_{AB|W}$ in each block

$\forall(u, a, b, y)$, need:

$$\sum_{x_1, x_2} Q^1(a_b|u, a, b, y, x_1) \cdot Q^1(b_b|u, a, b, y, x_2) \cdot P(x_1, x_2|u, a, b, y)$$

$$= P_{AB|W}(a_b, b_b|f(u, a, b, y))$$

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Block-Markov Scheme

Assume conditions satisfied $P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_{Y|X_1X_2}$

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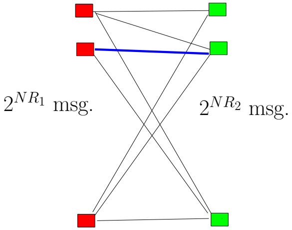

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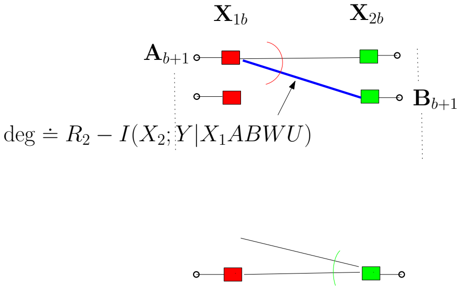

Encoder 1 needs to decode $\mathbf{B}_{b+1}$ from $\mathbf{Y}_{b+1}$:

$$R_2 - I(X_2;Y|X_1ABWU) < I(B;Y|X_1AWU)$$

or $R_2 < I(X_2;Y|X_1AWU)$

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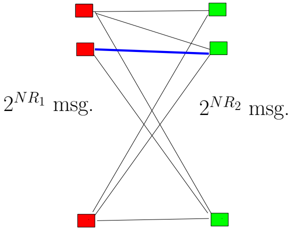

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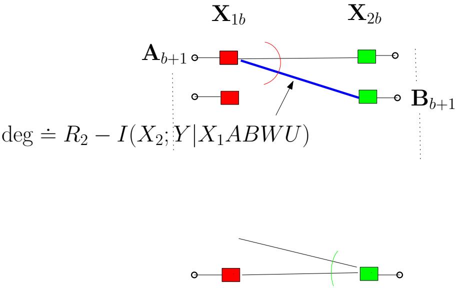

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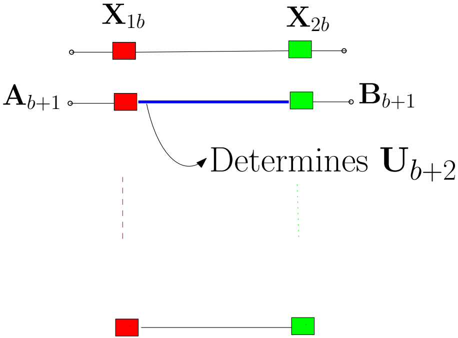
Upon receiving $\mathbf{Y}_{b+1}$, decoder's graph:



For $\mathbf{U}_{b+2}$ to resolve this:

$$\text{Number of edges } < I(U;Y)$$

Result

Theorem

For the MAC $P_{Y|X_1X_2}$, let $P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_{Y|X_1X_2}$ be any joint distribution that satisfies the consistency conditions. Then the following rate pairs (R_1, R_2) are achievable.

$$R_1 < I(X_1; Y|X_2BWU) - [I(A; X_2|YBWU) - I(U; Y)]^+$$

$$R_2 < I(X_2; Y|X_1AWU) - [I(B; X_1|YAWU) - I(U; Y)]^+$$

$$R_1 + R_2 < I(X_1X_2; Y|UW) + I(U; Y)$$

$W = A = B = \phi$ recovers Cover-Leung region

Binary MAC [Bross-Lapidoth '05]:

$$P_{Y|X_1X_2}(1|01) = P_{Y|X_1X_2}(1|10) = q$$
$$P_{Y|X_1X_2}(1|11) = 2q$$
$$P_{Y|X_1X_2}(1|00) = 0$$

- Capacity $\rightarrow 0$ as $q \rightarrow 0$
- $\frac{\text{Max. sum rate}}{q}$ as $q \rightarrow 0$

C-L region: 0.499

Bross-Lapidoth: 0.553

Our region: 0.651

Summary

- Exploiting feedback $\Rightarrow$ thinning of graph of typical messages
  - Then cooperate to help the decoder
- C-L scheme thins graph in 1 block
  - we achieved gains by thinning over 2 blocks
- Potential gains by gradually thinning over $3, 4, \ldots$ blocks
  - More auxiliary random variables