New Achievable Rates for the Multiple-Access Channel with Feedback

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Multiple-access Channel (MAC)

Feedback

- $X_{1n}$ function of $(W_1, Y^{n-1})$
- $X_{2n}$ function of $(W_2, Y^{n-1})$
- Example of [Gaarder-Wolf ’75] showed FB can increase capacity
Message graph before transmission

No Feedback capacity
Cover-Leung Scheme [1981]

Message graph before transmission

User 1

User 2

$2^{NR_1}$ messages

$2^{NR_2}$ messages

$2^N(R_1 + R_2)$ edges
After receiving $\Upsilon$

- LEFT vertices: $x_1[w_1]$ j.typ with $\Upsilon$
- RIGHT vertices: $x_2[w_2]$ j.typ with $\Upsilon$
- Edge: $\Rightarrow (x_1[w_1], x_2[w_2])$ j.typ with $\Upsilon$ - Possible message pair
C-L impose conditions so that

\[ 2^{N(R_1+R_2)} \cdot 2^{-NI(X_1X_2;Y|U)} \] edges

NEXT STAGE

- \( Y \) is common side information
- Point-to-point communication
- Source coding + channel coding
- Can be resolved by \( U \sim P_U \) if
  
  number of edges < \( 2^{NI(U;Y)} \)
C-L impose conditions so that

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\[ 2^{NI(U;Y)} \]

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- Can be resolved by \( U \sim P_U \) if

\[ \text{number of edges} < 2^{NI(U;Y)} \]
C-L Rates

How to achieve this graph?

\[ R_1 < I(X_1; Y|X_2 U) \]

\[ R_2 < I(X_2; Y|X_1 U) \]

\[ 2^{N(R_1+R_2)} \cdot 2^{-NI(X_1X_2;Y|U)} \text{ edges} \]

RESOLUTION with \( U \): \( R_1 + R_2 - I(X_1X_2; Y|U) < I(U; Y) \)
How to achieve this graph?

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RESOLUTION with \( U \):

\[ R_1 + R_2 - I(X_1X_2; Y|U) < I(U; Y) \]
Block-Markov Superposition

Transmission in blocks 1, \ldots, L

**Message pair for block** $b$

- $2^{NR_1}$ msg.
- $2^{NR_2}$ msg.

**Decoder graph after** $Y_b$

- Determines $U_{b+1}$

- $P_U \cdot P_{X_1|U} \cdot P_{X_2|U} \cdot P_{Y|X_1X_2}$

- Block $b$: fresh info $X_{1b}, X_{2b}$ superposed on resolution info $U_b$
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- $2^{NR_1}$ msg.
- $2^{NR_2}$ msg.

- $P_U \cdot P_{X_1|U} \cdot P_{X_2|U} \cdot P_Y|X_1X_2$

- Block $b$: fresh info $X_{1b}, X_{2b}$ superposed on resolution info $U_b$
Optimality

- C-L scheme optimal for channels where

\[ X_1 = f(X_2, Y) \text{ or } X_2 = g(X_1, Y) \]

- The channel ensures we always get the graph we want!

- Not optimal in general
  - white Gaussian MAC [Ozarow 84]
Outside the C-L region

Bross-Lapidoth '05

- C-L scheme: block length $n$
  - Encoders now cannot decode using FB
- Spend EXTRA time $\eta \cdot n$ to exchange messages
- New block length $(1 + \eta)n$
- Can improve rates
Our approach

In terms of message graph ...
Each encoder CANNOT decode message of the other using $Y$

- Degree of each vertex $> 1$
- But messages are CORRELATED
Correlated messages on MAC (no FB)
- [Cover, ElG, Salehi 81], [Pradhan et al 07]

Can achieve higher rates than independent messages.

First attempt . . .
Sending correlated messages

Common output two-way channel

Typicality graph given $\mathbf{Y}$

For each $\mathbf{X}_1$, generate one $\mathbf{A} \sim P_{A|X_1\mathbf{Y}}$

For each $\mathbf{X}_2$, generate one $\mathbf{B} \sim P_{B|X_2\mathbf{Y}}$

- Enc. 1 sends $\mathbf{A}$ corresp. to LEFT vertex
- Enc. 2 sends $\mathbf{B}$ corresp. to RIGHT vertex
- $\mathbf{A}$ and $\mathbf{B}$ are correlated

\[ \text{deg} = R_2 - I(X_2; Y|X_1U) \]
Sending correlated messages

Common output two-way channel

Typicality graph given $\mathbf{Y}$

For each $X_1$, generate one $A \sim P_{A|X_1 Y}$

For each $X_2$, generate one $B \sim P_{B|X_2 Y}$

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- Enc. 2 sends $B$ corresp. to RIGHT vertex
- $A$ and $B$ are correlated
Decoding

- Enc. 1 decodes $B$, enc. 2 decodes $A$
- Decoder cannot yet decode $A, B$

Decoder’s typicality graph:
Decoding

- Enc. 1 decodes B, enc. 2 decodes A
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Decoder’s typicality graph:
Block Markov Superposition

Message pair for block $b$

After receiving $Y_b$: $X_{1b}$ $X_{2b}$ $A_{b+1}$ $B_{b+1}$

After receiving $Y_{b+1}$:
Block Markov Superposition

Message pair for block $b$

After receiving $\mathbf{Y}_b$

After receiving $\mathbf{Y}_{b+1}$:
Issues

- $L$ blocks of transmission
- For block $b$, $1 \leq b \leq L$, if we generate

\[ A_b \leftrightarrow (Y_{b-1}, X_{1(b-1)}) \]
\[ B_b \leftrightarrow (Y_{b-1}, X_{2(b-1)}) \]

- Dependence ripples across blocks!

Does not yield 'single-letter' characterization.
Issues

- $L$ blocks of transmission

- For block $b$, $1 \leq b \leq L$, if we generate

  $$A_b \leftrightarrow (Y_{b-1}, X_{1(b-1)}), \quad B_b \leftrightarrow (Y_{b-1}, X_{2(b-1)})$$

- Dependence ripples across blocks!

  ![Diagram]

  Does not yield ‘single-letter’ characterization
Fix

**WANT**

Seqs. in each block $\sim P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_{Y|X_1X_2}$

- Define fn. $f : U \times A \times B \times Y \to W$

  $$W_b = f((U, A, B, Y)_{b-1})$$

- $W$ summarizes info common to both encoders at end of prev. block
  - e.g. $W_b = Y_{b-1}$

**CONSISTENCY COND. 1**

**GOAL:** Ensure $W_b \sim P_W$ in each block:

$$P_W(w) = \sum P(u, a, b, y)1(f(u, a, b, y) = w)$$
Fix

**WANT**

Seqs. in each block \( \sim P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_{Y|X_1X_2} \)

- Define fn. \( f : U \times A \times B \times Y \rightarrow W \)
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  W_b = f((U, A, B, Y)_{b-1})
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**CONSISTENCY COND. 1**

**GOAL:** Ensure \( W_b \sim P_W \) in each block:

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P_W(w) = \sum P(u, a, b, y)1(f(u, a, b, y) = w)
\]
Conditions ctd. Want seqs. to be $\sim P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_Y|X_1X_2$

**CONSISTENCY COND. 2**

- $A_b$ gen. from $(U, A, B, Y, X_1)_{b-1} \sim Q^1$
- $B_b$ gen. from $(U, A, B, Y, X_2)_{b-1} \sim Q^2$

Given $W_b$, ensure $A_b, B_b \sim P_{AB|W}$ in each block

\[ \forall (u, a, b, y), \text{ need:} \]

\[ \sum_{x_1, x_2} Q^1(a_b|u, a, b, y, x_1) \cdot Q^1(b_b|u, a, b, y, x_2) \cdot P(x_1, x_2|u, a, b, y) \]

\[ = P_{AB|W}(a_b, b_b|f(u, a, b, y)) \]
Conditions ctd. Want seqs. to be $\sim P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_{Y|X_1X_2}$

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Given $W_b$, ensure $A_b, B_b \sim P_{AB|W}$ in each block

$\forall (u, a, b, y)$, need:

$$
\sum_{x_1, x_2} Q^1(a_b|u, a, b, y, x_1) \cdot Q^1(b_b|u, a, b, y, x_2) \cdot P(x_1, x_2|u, a, b, y)
= P_{AB|W}(a_b, b_b|f(u, a, b, y))
$$
Block-Markov Scheme

Assume conditions satisfied $P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_{Y|X_1X_2}$

Message pair for block $b$

After receiving $Y_b$:

Encoder 1 needs to decode $B_{b+1}$ from $Y_{b+1}$:

$$R_2 - I(X_2; Y|X_1ABWU) < I(B; Y|X_1AWU)$$

or $R_2 < I(X_2; Y|X_1AWU)$
Block-Markov Scheme

Assume conditions satisfied $P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_{Y|X_1X_2}$

Message pair for block $b$

After receiving $Y_b$:

$$deg = R_2 - I(X_2; Y|X_1ABWU)$$

Encoder 1 needs to decode $B_{b+1}$ from $Y_{b+1}$:

$$R_2 - I(X_2; Y|X_1ABWU) < I(B; Y|X_1AWU)$$

or $R_2 < I(X_2; Y|X_1AWU)$
Upon receiving $Y_{b+1}$, decoder’s graph:

For $U_{b+2}$ to resolve this:

Number of edges $< I(U; Y)$
Result

**Theorem**

For the MAC $P_{Y|X_1X_2}$, let $P_U \cdot P_{WAB} \cdot P_{X_1|UA} \cdot P_{X_2|UB} \cdot P_{Y|X_1X_2}$ be any joint distribution that satisfies the consistency conditions. Then the following rate pairs $(R_1, R_2)$ are achievable.

\begin{align*}
R_1 &< I(X_1; Y|X_2BWU) - [I(A; X_2|YBWU) - I(U; Y)]^+ \\
R_2 &< I(X_2; Y|X_1AWU) - [I(B; X_1|YAWU) - I(U; Y)]^+ \\
R_1 + R_2 &< I(X_1X_2; Y|UW) + I(U; Y)
\end{align*}

$W = A = B = \emptyset$ recovers Cover-Leung region
Binary MAC [Bross-Lapidoth ’05]:

\[
P_{Y|X_1X_2}(1|01) = P_{Y|X_1X_2}(1|10) = q \\
P_{Y|X_1X_2}(1|11) = 2q \\
P_{Y|X_1X_2}(1|00) = 0
\]

- Capacity → 0 as \( q \) → 0
- Max. sum rate \( \frac{\text{sum rate}}{q} \) as \( q \) → 0

C-L region: 0.499
Bross-Lapidoth: 0.553
Our region: 0.651
Summary

- Exploiting feedback $\Rightarrow$ thinning of graph of typical messages
  - Then cooperate to help the decoder

- C-L scheme thins graph in 1 block
  - we achieved gains by thinning over 2 blocks

- Potential gains by gradually thinning over 3, 4, ... blocks
  - More auxiliary random variables