

Computing sum of sources over an arbitrary multiple access channel

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Abstract—The problem of computing sum of sources over a multiple access channel (MAC) is considered. Building on the technique of linear computation coding (LCC) proposed by Nazer and Gastpar [1], we employ the ensemble of nested coset codes to derive a new set of sufficient conditions for computing sum of sources over an *arbitrary* MAC. The optimality of nested coset codes [2] enables this technique outperform LCC even for linear MAC with a structural match. Examples of non-additive MAC for which the technique proposed herein outperforms separation and systematic based computation are also presented. Finally, this technique is enhanced by incorporating separation based strategy, leading to a new set of sufficient conditions for computing sum over a MAC.

I. INTRODUCTION

Consider a scenario wherein a centralized receiver is interested in evaluating a multi-variate function, the arguments of which are available to spatially distributed transmitters. Traditionally, the technique of computing functions at a centralized receiver is based on its decoding of the arguments in its entirety. Solutions based on this technique have been proven optimal for particular instances of distributed source coding. Moreover, this technique lends itself naturally for communication based on separation. Buoyed by this partial success and ease of implementation, the de facto framework for computing at a centralized receiver is by enabling the decoder decode the arguments of the function in its entirety.

The problem of computing mod-2 sum of distributed binary sources has proved to be an exception. Studied in the context of a source coding problem, Körner and Marton [3] propose an ingenious technique based on linear codes, that circumvent the need to communicate sources to the decoder, and thereby perform strictly better for a class of source distributions. In fact, as proposed in [3], the decoder needs only sum of message indices put out by the source encoder. This fact has been further exploited by Nazer and Gastpar [1] in developing a channel coding technique for a *linear* MAC, henceforth referred to as linear computation coding (LCC), that enables the decoder reconstruct the sum of the message indices input to the channel encoder. Since the decoder does not need to disambiguate individual message indices, this technique, when applicable, outperforms earlier known techniques.

LCC [1] is built around employing the same linear code as a channel code at both encoders. The message indices output by the Körner-Martón (KM) source code is linearly

mapped into channel codewords. Since a linear MAC first computes a sum of the transmitted codewords, it is as if the codeword corresponding to the sum of messages was input to the ensuing channel. The first question that comes to mind is the following. If the MAC is not linear, would it be possible to decode sum of message indices without having to decode the individual codewords? In other words, what would be the generalization of LCC for an arbitrary MAC?¹ If there exist such a generalization, how efficient would it be?

In this article, we answer the above question in the affirmative. Firstly, we recognize that in order to decode the sum of transmitted codewords, it is most efficient to employ channel codes that are closed under addition, of which a linear code employed in LCC is the simplest example. Closure under addition contains the range of the sum of transmitted codewords and thereby support a larger range for individual messages. Secondly, typical set decoding circumvents need for the MAC to be linear. Since nested coset codes have been proven to achieve capacity of arbitrary point-to-point channels [2] and are closed under addition, we employ this ensemble for generalizing the technique of LCC. As illustrated by examples 1,2 in section III, the generalization we propose (i) outperforms separation based technique for an arbitrary MAC and moreover (ii) outperforms LCC even for examples with a structural match.² We remark that analysis of typical set decoding of a function of transmitted codewords with nested coset codes that contain statistically dependent codewords contains new elements detailed in [4].

Even in the case of a structural match, separation based schemes might outperform LCC [1, Example 4]. This raises the following question. What then would be a unified scheme for computing over an arbitrary MAC? Is there such a scheme that reduces to (i) separation when the desired function and MAC are not matched and (ii) LCC when appropriately matched? We recognize that KM technique is indeed suboptimal for a class of source distributions. For such sources, it is more efficient to communicate sources as is. We therefore take the approach of Ahlswede and Han [5, Section VI], where in a

¹The technique of systematic computation coding (SCC) [1] may not be considered as a generalization of LCC. Indeed SCC does not reduce to LCC for a linear MAC.

²This is expected since linear codes achieve only symmetric capacity and nested coset codes can achieve capacity of arbitrary point-to-point channels.

two layer source code accomplishes distributed compression. The first layer generates message indices of those parts that are best reconstructed as is, and the second employs a KM technique. In section IV, we propose a two layer channel code for MAC that is compatible with the above two layer source code. The first layer of the MAC channel code communicates the message indices as is, while the second enables the decoder to decode the sum of second layer message indices, and thereby develop a unifying strategy that subsumes separation and LCC.

We highlight the significance of our contribution. Firstly, we propose a strategy based on nested coset codes and derive a set of sufficient conditions for the problem of computing sum of sources over an *arbitrary* MAC. The proposed strategy subsumes all current known strategies and performs strictly better for certain examples (section III). Secondly, our findings highlight the utility of nested coset codes [2] as a generic ensemble of structured codes for communicating over arbitrary multi-terminal communication problems. Thirdly, and perhaps more importantly, our findings hint at a general theory of structured codes. Linear and nested linear codes have been employed to derive communication strategies for particular symmetric additive source and channel coding problems that outperform all classical unstructured-code based techniques. However the question remains whether these structured code based techniques can be generalized to arbitrary multi-terminal communication problems. Our findings indicate that strategies based on structured codes can be employed to analyze more intelligent encoding and decoding techniques for an arbitrary multi-terminal communication problem.

II. PRELIMINARIES AND PROBLEM STATEMENT

Following remarks on notation (II-A) and problem statement (II-B), we briefly describe LCC for a linear MAC (II-C) and set the stage for its generalization.

A. Notation

We employ notation that is now widely adopted in the information theory literature supplemented by the following. We let \mathcal{F}_q denote a finite field of cardinality q . When the finite field is clear from context, we let \oplus denote addition in the same. When ambiguous, or to enhance clarity, we specify addition in \mathcal{F}_q using \oplus_q . In this article, we repeatedly refer to pairs of objects of similar type. To reduce clutter in notation, we use an underline to refer to aggregates of similar type. For example, (i) \underline{S} abbreviates (S_1, S_2) , (ii) if $\mathcal{X}_1, \mathcal{X}_2$ are finite alphabet sets, we let $\underline{\mathcal{X}}$ either denote the Cartesian product $\mathcal{X}_1 \times \mathcal{X}_2$ or abbreviate the pair $\mathcal{X}_1, \mathcal{X}_2$ of sets. More non trivially, if $e_j : \mathcal{S}^n \rightarrow \mathcal{X}_j^n : j = 1, 2$ are a pair of maps, we let $\underline{e}(\underline{s}^n)$ abbreviate $(e_1(s_1^n), e_2(s_2^n))$.

B. Problem statement

Consider a pair (S_1, S_2) of information sources each taking values over a finite field \mathcal{S} of cardinality q . We assume outcome $(S_{1,t}, S_{2,t})$ of the sources at time $t \in \mathbb{N}$, is independent and identically distributed across time, with distribution $W_{\underline{S}}$. We let $(\mathcal{S}, W_{\underline{S}})$ denote this pair of sources. S_j is observed

by encoder j that has access to input j of a two user discrete memoryless multiple access channel (MAC) that is used without feedback. Let $\mathcal{X}_1, \mathcal{X}_2$ be the finite input alphabet sets and \mathcal{Y} the finite output alphabet set of MAC. Let $W_{Y|X_1 X_2}(y|x_1, x_2)$ denote MAC transition probabilities. We refer to this as MAC $(\underline{\mathcal{X}}, \mathcal{Y}, W_{Y|\underline{\mathcal{X}}})$. The objective of the decoder is to compute $S_1 \oplus S_2$. In this article, we provide a characterization of a sufficient condition for computing $S_1 \oplus S_2$ with arbitrary small probability of error. The relevant notions are made precise in the following definitions.

Definition 1: A computation code (n, \underline{e}, d) for computing sum of sources $(\mathcal{S}, W_{\underline{S}})$ over the MAC $(\underline{\mathcal{X}}, \mathcal{Y}, W_{Y|\underline{\mathcal{X}}})$ consists of (i) two encoder maps $e_j : \mathcal{S}^n \rightarrow \mathcal{X}_j^n : j = 1, 2$ and (ii) a decoder map $d : \mathcal{Y}^n \rightarrow \mathcal{S}^n$.

Definition 2: The average error probability $\bar{\xi}(\underline{e}, d)$ of a computation code (n, \underline{e}, d) is

$$\sum_{\underline{s} \in \mathcal{S}^n} \sum_{\substack{y^n : d(y^n) \neq \\ s_1^n \oplus s_2^n}} W_{Y^n|\underline{X}^n}(y^n|\underline{e}(\underline{s}^n)) W_{\underline{S}^n}(\underline{s}^n).$$

Definition 3: The sum of sources $(\mathcal{S}, W_{\underline{S}})$ is computable over MAC $(\underline{\mathcal{X}}, \mathcal{Y}, W_{Y|\underline{\mathcal{X}}})$ if for all $\eta > 0$, there exists an $N(\eta) \in \mathbb{N}$ such that for all $n > N(\eta)$, there exists an $(n, \underline{e}^{(n)}, d^{(n)})$ computation code such that $\bar{\xi}(\underline{e}^{(n)}, d^{(n)}) \leq \eta$.

The main objective in this article is to provide a sufficient condition for computability of sum of sources over a MAC.

C. Linear Computation Coding

We describe the technique of LCC in a simple setting and highlight the key aspects. Consider binary sources and a binary additive MAC, i.e., $\mathcal{S} = \mathcal{X}_1 = \mathcal{X}_2 = \{0, 1\}$ and $Y = X_1 \oplus X_2 \oplus N$, where N is independent of the inputs and $P(N = 1) = q$. Furthermore assume sources are symmetric, uniform, i.e., $P(\underline{S} = (0, 0)) = \frac{1-p}{2} = P(\underline{S} = (1, 1))$ and $P(\underline{S} = (0, 1)) = P(\underline{S} = (1, 0)) = \frac{p}{2}$ such that $h_b(p) < 1 - h_b(q)$.

By employing a KM source code, the two message indices at rate $h_b(p)$ can be employed to decode $S_1 \oplus S_2$. Let $h \in \mathcal{S}^{k \times n}$ denote a parity check matrix for the KM source code, with $\frac{k}{n}$ arbitrarily close to $h_b(p)$. Nazer and Gastpar observe that the decoder only requires the sum $h(S_1^n \oplus S_2^n) = h(S_1^n) \oplus h(S_2^n)$ of message indices. If the map from message indices to channel code is linear, then the decoder can infer $h(S_1^n) \oplus h(S_2^n)$ by decoding the codeword corresponding to sum of transmitted codewords. Since sum of transmitted codewords passes through a BSC(q), they employ a capacity achieving linear code of rate arbitrarily close to $1 - h_b(q)$ with generator matrix $g \in \mathcal{X}_1^{l \times n}$. Each encoder employs the same linear code and transmits $x_j^n := h(S_j^n)g$. The decoder receives Y^n and decodes as if the channel is a BSC(q). It ends up decoding message corresponding to $x_1^n \oplus x_2^n$ which was precisely what it was looking for.

We note that a separation based scheme will require the sum capacity of the MAC to be greater than $2h_b(p)$ and hence LCC is more efficient. What are key aspects of LCC? Note that (i) the channel code is designed for the $X_1 \oplus X_2$ to Y channel, i.e., the BSC(q) and (ii) both encoders employ the same

linear channel code, thereby ensuring their codes are closed under addition. This contains range of the sum of transmitted codewords to a rate $1 - h_b(q)$. It is instructive to analyze the case when the two users are provided two linear codes of rates R_1 and R_2 spanning disjoint subspaces. Since the range of sum of transmitted codewords is $R_1 + R_2$, the same decoding rule will impose the constraint $R_1 + R_2 < 1 - h_b(q)$ resulting in the constraint $2h_b(p) \leq 1 - h_b(q)$ which is strictly suboptimal. *We conclude that the two users' channel codes being closed under addition is crucial to the optimality of LCC for this problem.* Furthermore, the coupling of (i) a linear map of KM message indices to the channel code at the encoder and (ii) decoding of the sum of transmitted codewords, is central to LCC.

In the following section, we make use of the above observations to propose a generalization of LCC for computing sum of sources over an arbitrary MAC.

III. NESTED COSET CODES FOR COMPUTING SUM OF SOURCES OVER A MAC

In this section, we propose a technique for computing $S_1 \oplus S_2$ over an *arbitrary* MAC using the ensemble of nested coset codes [2], and derive a set of sufficient conditions under which, sum of sources $(\mathcal{S}, W_{\underline{S}})$ can be computed over a MAC $(\mathcal{X}, \mathcal{Y}, W_{Y|\underline{X}})$. Definitions 4 and theorem 1 state these sufficient conditions. This is followed by examples that illustrate significance of theorem 1.

Definition 4: Let $\mathbb{D}(W_{Y|\underline{X}})$ be collection of distributions $p_{V_1 V_2 X_1 X_2 Y}$ defined over $\mathcal{S}^2 \times \mathcal{X} \times \mathcal{Y}$ such that (i) $p_{V_1 X_1 V_2 X_2} = p_{V_1 X_1} p_{V_2 X_2}$, (ii) $p_{Y|\underline{XV}} = p_{Y|\underline{X}} = W_{Y|\underline{X}}$. For $p_{V\underline{X}Y} \in \mathbb{D}(W_{Y|\underline{X}})$, let $\alpha(p_{V\underline{X}Y})$ be defined as

$$\{R \geq 0 : R \leq \min\{H(V_1), H(V_2)\} - H(V_1 \oplus V_2|Y)\}, \text{ and} \\ \alpha(W_{Y|\underline{X}}) := \sup_{p_{V\underline{X}Y} \in \mathbb{D}(W_{Y|\underline{X}})} \alpha(p_{V\underline{X}Y}).$$

Theorem 1: The sum of sources $(\mathcal{S}, W_{\underline{S}})$ is computable over a MAC $(\mathcal{X}, \mathcal{Y}, W_{Y|\underline{X}})$ if $H(S_1 \oplus S_2) \leq \alpha(W_{Y|\underline{X}})$.

In the interest of brevity, we briefly discuss the coding strategy and indicate how we attain the rates promised above. The reader is referred to [4] for a complete proof of theorem 1.

We begin with a description of the encoding rule. Encoder j employs a KM source code to compress the observed source. Let $M_j^l := hS_j^n$ denote corresponding message index, where $h \in \mathcal{S}^{l \times n}$ is a KM parity check matrix of rate $\frac{l}{n} \approx H(S_1 \oplus S_2)$. Each encoder is provided with a common nested linear code taking values over \mathcal{S} . The nested linear code is described through a pair of generator matrices $g_I \in \mathcal{S}^{k \times n}$ and $g_{O/I} \in \mathcal{S}^{l \times n}$, where g_I and $\begin{bmatrix} g_I^T & g_{O/I}^T \end{bmatrix}^T$ are the generator matrices of the inner (sparser) code and complete (finer) codes respectively, where

$$\frac{k}{n} \stackrel{(a)}{\geq} 1 - \frac{\min\left\{\frac{H(V_1)}{H(V_2)}\right\}}{\log |\mathcal{S}|}, \quad \frac{k+l}{n} \stackrel{(b)}{\leq} 1 - \frac{H(V_1 \oplus V_2)}{\log |\mathcal{S}|}. \quad (1)$$

Encoder j picks a codeword in coset $(a^k g_I \oplus M_j^l g_{O/I} : a^k \in \mathcal{S}^k)$ indexed by M_j^l that is typical with respect to p_{V_j} . Based on this chosen codeword X^n is generated according to $p_{X_j|V_j}$ and transmitted.

The decoder is provided with the same nested linear code. Having received Y^n it lists all codewords that are jointly typical with Y^n with respect to distribution $p_{V_1 \oplus V_2, Y}$. If it finds all such codewords in a unique coset, say $(a^k g_I \oplus m^l g_{O/I} : a^k \in \mathcal{S}^k)$, then it declares m^l to be the sum of KM message indices and employs KM decoder to decode the sum of sources. Otherwise, it declares an error.

We derive an upper bound on probability of error by averaging the error probability over the ensemble of nested linear codes. For the purpose of proof, we consider user codebooks to be cosets of nested linear codes.³ We average uniformly over the entire ensemble of nested *coset* codes. Lower bound (1(a)) ensures the encoders find a typical codeword in the particular coset. Upper bound (1(b)) enables us derive an upper bound on the probability of decoding error. From (1), it can be verified that if $H(S_1 \oplus S_2) \approx \frac{l}{n} \leq \min\{H(V_1), H(V_2)\} - H(V_1 \oplus V_2|Y)$ then the decoder can reconstruct the sum of sources with arbitrarily small probability of error.

How does nesting of linear codes enable attain non-uniform distributions?⁴ As against to a linear code, nesting of linear codes provides the encoder with a coset to choose the codeword from. The vectors in the coset being uniformly distributed, it contains at least one vector typical with respect to p_{V_j} with high probability, if the coset is of rate at least $1 - \frac{H(V_j)}{\log |\mathcal{S}|}$. By choosing such a vector, the encoder induces a non-uniform distribution on the input space. Therefore, constraint (1(a)) enables achieve non-uniform input distributions.

Since the codebooks employed by the encoders are uniformly and independently distributed cosets of a common random linear code, the sum of transmitted codewords also lies in a codebook that is a uniformly distributed coset of the same linear code. Any vector in this codebook is uniformly distributed over it's entire range. Therefore, a vector in this codebook other than the legitimate sum of transmitted codewords is jointly typical with the received vector with probability at most $|\mathcal{S}|^{n(H(V_1 \oplus V_2|Y) - 1)}$.⁵ Employing a union bound, it can be argued that the probability of decoding error decays exponentially if (1(b)) holds.

Since the ensemble of codebooks contain statistically dependent codewords and moreover user codebooks are closely related, deriving an upper bound on the probability of error involves new elements. The informed reader will recognize that in particular, deriving an upper bound on the probability of decoding error will involve proving statistical independence of the pair of cosets indexed by KM indices (M_1^l, M_2^l) and any codeword in a coset corresponding to $\hat{m}^l \neq M_1^l \oplus M_2^l$. The statistical dependence of the codebooks results in new elements to the proof. The reader is encouraged to peruse details in [4, Proof of Theorem1].

It can be verified that the rate region presented in theorem 1 subsumes that presented in [1, Theorem1, Corollary

³This is analogous to the use of cosets of a linear code to prove achievability of symmetric capacity over point-to-point channels.

⁴Note that linear codes only achieve mutual information with respect to uniform input distributions.

⁵Here, the logarithm is taken with respect to base $|\mathcal{S}|$.

2]. This follows by substituting a uniform distribution for V_1, V_2 . Therefore examples presented in [1] carry over as examples of rates achievable using nested coset codes. One might visualize a generalization of LCC for arbitrary MAC through the modulo-lattice transformation (MLT) [6, Section IV]. Since the map for KM source code message indices to the channel code has to be linear, the virtual input alphabets of the transformed channel are restricted to be source alphabets as in definition 4. It can now be verified that any virtual channel, specified through maps from (i) virtual to actual inputs, (ii) output to the estimate of the linear combination, identifies a corresponding test channel in $\mathbb{D}(W_{Y|X})$. Hence, the technique proposed herein subsumes MLT. Moreover, while MLT is restricted to employing uniform distributions over the auxiliary inputs, nested coset codes can induce arbitrary distributions.

We now present a sample of examples to illustrate significance of theorem 1. The reader is referred to [4] for additional examples that illustrate utility of the coding technique proposed herein. As was noted in [1, Example 4] a uniform distribution induced by a linear code maybe suboptimal even for computing functions over a MAC with a structural match. The following example, closely related to the former, demonstrates the ability of nested coset codes to achieve a nonuniform distribution and thus exploit the structural match better.

Example 1: Let S_1 and S_2 be a pair of independent and uniformly distributed sources taking values over the field \mathcal{F}_5 of five elements. The decoder wishes to reconstruct $S_1 \oplus_5 S_2$. The two user MAC channel input alphabets $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{F}_5$ and output alphabet $\mathcal{Y} = \{0, 2, 4\}$. The output Y is obtained by passing $W = X_1 \oplus_5 X_2$ through an asymmetric channel whose transition probabilities are given by $p_{Y|W}(y|1) = p_{Y|W}(y|3) = \frac{1}{3}$ for each $y \in \mathcal{Y}$ and $p_{Y|W}(0|0) = p_{Y|W}(2|2) = p_{Y|W}(4|4) = 1$. Let the number of source digits output per channel use be λ . We wish to compute the range of values of λ for which the decoder can reconstruct the sum of sources. This is termed as computation rate in [1].

It can be verified that the decoder can reconstruct $S_1 \oplus_5 S_2$ using the technique of LCC if $\lambda \leq \frac{3 \log_2(3)}{5 \log_2(5)} = 0.4096$. A separation based scheme enables the decoder reconstruct the sum if $\lambda \leq \frac{1 \log_2(3)}{2 \log_2(5)} = 0.3413$. We now explore the use of nested coset codes. It may be verified that pmf

$$p_{VXY}(v, x, x_1 \oplus_5 x_2) = \begin{cases} \frac{1}{4} & \text{if } v_1=x_1, v_2=x_2 \\ & \text{and } v_1, v_2 \in \{0, 2\} \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

defined on $\mathcal{F}_5 \times \mathcal{F}_5$ satisfies (i), (ii) of definition 4 and moreover $\alpha(p_{VXY}) = \{R \geq 0 : R \leq 1\}$. Thus nested coset codes enable reconstructing $S_1 \oplus_5 S_2$ at the decoder if $\lambda \leq \frac{1}{\log_2 5} = .43067$. The above example illustrates the need for nesting codes in order to achieve nonuniform distributions. However, for the above example, a suitable modification of LCC is optimal. Instead of building codes over \mathcal{F}_5 , let each user employ the linear code of rate 1^6 built on \mathcal{F}_2 . The map $\mathcal{F}_2 \rightarrow \mathcal{X}_j : j = 1, 2$ defined as $0 \rightarrow 0$ and $1 \rightarrow 2$ induces a code over \mathcal{F}_5 and it can

be verified that LCC achieves the rate achievable using nested coset codes. However, the following example precludes such a modification of LCC.

Example 2: The source is assumed to be the same as in example 1. The two user MAC input and output alphabets are also assumed the same, i.e., $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{F}_5$ and output alphabet $\mathcal{Y} = \{0, 2, 4\}$. The output Y is obtained by passing $W = X_1 \oplus_5 X_2$ through an asymmetric channel whose transition probabilities are given by $p_{Y|W}(y|1) = p_{Y|W}(y|3) = \frac{1}{3}$ for each $y \in \mathcal{Y}$ and $p_{Y|W}(0|0) = p_{Y|W}(2|2) = p_{Y|W}(4|4) = 0.90$, $p_{Y|W}(2|0) = p_{Y|W}(4|0) = p_{Y|W}(0|2) = p_{Y|W}(4|2) = p_{Y|W}(0|4) = p_{Y|W}(2|4) = 0.05$.

The technique of LCC builds a linear code over \mathcal{F}_5 . It can be verified that the symmetric capacity for the $X_1 \oplus_5 X_2 (= W) - Y$ channel is 0.6096 and therefore LCC enables decoder reconstruct the sum if $\lambda \leq \frac{0.6096}{\log_2 5} = 0.2625$. A separation based scheme necessitates communicating each of the sources to the decoder and this can be done only if $\lambda \leq \frac{1 \log_2 3}{2 \log_2 5} = 0.3413$. The achievable rate region of the test channel in (2) is $\alpha(p_{VXY}) = \{R \geq 0 : R \leq 0.91168\}$ and therefore nested coset codes enable decoder reconstruct the sum if $\lambda \leq \frac{0.91168}{\log_2 5} = 0.3926$.

Example 3: Let S_1 and S_2 be independent sources distributed uniformly over $\{0, 1, 2\}$. The input alphabets $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{F}_3$ is the ternary field and the output alphabet $\mathcal{Y} = \mathcal{F}_2$ is the binary field. Let $W = 1_{\{X_1 \neq X_2\}}$ and output Y is obtained by passing W through a BSC with crossover probability 0.1. The decoder is interested in reconstructing W . As noted in [1, Example 8], W is 0 if and only if $S_1 \oplus_3 2S_2 = 0$. Therefore, it suffices for the decoder to reconstruct $S_1 \oplus_3 2S_2$. Following the arguments in proof of theorem 1 it can be proved that $S_1 \oplus_3 2S_2$ can be reconstructed using nested coset codes if there exists a pmf $p_{VXY} \in \mathbb{D}(W_{Y|X})$ such that $H(S_1 \oplus_3 2S_2) \leq \min\{H(V_1), H(V_2)\} - H(V_1 \oplus_3 2V_2|Y)$. It can be verified that for pmf p_{VXY} wherein V_1, V_2 are independently and uniformly distributed over \mathcal{F}_3 , $X_1 = V_1$, $X_2 = V_2$, the achievable rate region is $\alpha(p_{VXY}) = \{R : R \leq 0.4790\}$. The computation rate achievable using SCC and separation technique are 0.194 and 0.168 respectively. The computation rate achievable using nested coset codes is $\frac{0.4790}{\log_2 3} = 0.3022$.

Example 4: Let S_1 and S_2 be independent and uniformly distributed binary sources and the decoder is interested in reconstructing the binary sum. The MAC is binary, i.e. $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \mathcal{F}_2$ with transition probabilities $P(Y = 0|X_1 = x_1, X_2 = x_2) = 0.1$ if $x_1 \neq x_2$, $P(Y = 0|X_1 = X_2 = 0) = 0.8$ and $P(Y = 0|X_1 = X_2 = 1) = 0.9$. It can be easily verified that the channel is not linear, i.e., $X - X_1 \oplus X_2 - Y$ is NOT a Markov chain. This restricts current known techniques to either separation based coding or SCC [1, Section V]. SCC yields a computation rate of 0.3291. The achievable rate region for the test channel p_{VXY} where in V_1 and V_2 are independent and uniformly distributed binary sources, $X_1 = V_1$, $X_2 = V_2$ is given by $\{R : R \leq 0.4648\}$.

We conclude by recognizing that example 4 is indeed a family of examples. As long as the MAC is close to additive we can expect nested coset codes to outperform separation and SCC.

⁶This would be the set of all binary n -length vectors

IV. GENERAL TECHNIQUE FOR COMPUTING SUM OF SOURCES OVER A MAC

In this section, we propose a general technique for computing sum of sources over a MAC that subsumes separation and computation. The architecture of the code we propose is built on the principle that techniques based on structured coding are not in lieu of their counterparts based on unstructured coding. Indeed, the KM technique is outperformed by the Berger-Tung [7] strategy for a class of source distributions. A general strategy must therefore incorporate both.

We take the approach of Ahlswede and Han [5, Section VI], where in a two layer source code is proposed. Each source encoder j generates two message indices M_{j1}, M_{j2} . M_{j1} is an index to a Berger-Tung source code and M_{j2} is an index to a KM source code. The source decoder therefore needs M_{11}, M_{21} and $M_{12} \oplus M_{22}$ to reconstruct the quantizations and thus the sum of sources. We propose a two layer MAC channel code that is compatible with the above source code. The first layer of this code is a standard MAC channel code based on unstructured codes. The messages input to this layer are communicated as is to the decoder. The second layer employs nested coset codes and is identical to the one proposed in theorem 1. A function of the codewords selected from each layer is input to the channel. The decoder decodes a triple - the pair of codewords selected from the first layer and a sum of codewords selected from the second layer - and thus reconstructs the required messages. The following characterization specifies rates of layers 1 and 2 separately and therefore differs slightly from [5, Theorem 10].

Definition 5: Let $\mathbb{D}_{\text{AH}}(W_{\underline{S}})$ be collection of distributions $p_{T_1 T_2 S_1 S_2}$ defined over $\mathcal{T}_1 \times \mathcal{T}_2 \times \mathcal{S}^2$ such that (a) $\mathcal{T}_1, \mathcal{T}_2$ are finite sets, (b) $p_{S_1 S_2} = W_{\underline{S}}$, (c) $T_1 - S_1 - S_2 - T_2$ is a Markov chain. For $p_{\underline{T}\underline{S}} \in \mathbb{D}_{\text{AH}}(W_{\underline{S}})$, let

$$\beta_S(p_{\underline{T}\underline{S}}) := \left\{ \begin{array}{l} (R_{11}, R_{12}, R_2) \in \mathbb{R}^3 : R_{11} \geq I(T_1; S_1 | T_2), \\ R_{12} \geq I(T_2; S_2 | T_1), R_2 \geq H(S_1 \oplus S_2 | \underline{T}), \\ R_{11} + R_{12} \geq I(\underline{T}; \underline{S}) \end{array} \right\}.$$

Let $\beta_S(W_{\underline{S}})$ denote convex closure of the union $\beta_S(p_{\underline{T}\underline{S}})$ over $p_{\underline{T}\underline{S}} \in \mathbb{D}_{\text{AH}}(W_{\underline{S}})$

We now characterize achievable rate region for communicating these indices over a MAC. We begin with a definition of test channels and the corresponding rate region.

Definition 6: Let \mathbb{D}_{G} be collection of distributions $p_{U_1 U_2 V_1 V_2 X_1 X_2 Y}$ defined on $\mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{S} \times \mathcal{S} \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}$ such that (i) $p_{UVX} = p_{U_1 V_1 X_1} p_{U_2 V_2 X_2}$, (ii) $p_{Y|XUV} = p_{Y|\underline{X}} = W_{Y|\underline{X}}$. For $p_{UVXY} \in \mathbb{D}_{\text{G}}$, let $\beta_C(p_{UVXY})$ be defined as

$$\left\{ \begin{array}{l} (R_{11}, R_{12}, R_2) \in \mathbb{R}^3 : 0 \leq R_{11} \leq I(U_1; Y, U_2, V_1 \oplus V_2), \\ 0 \leq R_{12} \leq I(U_2; Y, U_1, V_1 \oplus V_2), R_{11} + R_{12} \leq I(\underline{U}; Y, V_1 \oplus V_2) \\ R_2 \leq \mathcal{H}_{\min}(V|U) - H(V_1 \oplus V_2 | Y, \underline{U}) \\ R_{11} + R_2 \leq \mathcal{H}_{\min}(V|U) + H(U_1) - H(V_1 \oplus V_2, U_1 | Y, U_2) \\ R_{12} + R_2 \leq \mathcal{H}_{\min}(V|U) + H(U_2) - H(V_1 \oplus V_2, U_2 | Y, U_1) \\ R_{11} + R_{12} + R_2 \leq \mathcal{H}_{\min}(V|U) + H(U_1) + H(U_2) - H(V_1 \oplus V_2, \underline{U} | Y) \end{array} \right\}.$$

where $\mathcal{H}_{\min}(V|U) := \min\{H(V_1|U_1), H(V_2|U_2)\}$ and define $\beta_C(W_{Y|\underline{X}})$ as the convex closure of the union

$\beta_C(p_{UVXY})$ over $p_{UVXY} \in \mathbb{D}_{\text{G}}(W_{Y|\underline{X}})$.

Theorem 2: The sum of sources $(\mathcal{S}, W_{\underline{S}})$ is computable over MAC $(\mathcal{X}, \mathcal{Y}, W_{Y|\underline{X}})$ if $\beta_S(W_{\underline{S}}) \cap \beta_C(W_{Y|\underline{X}}) \neq \emptyset$.

Remark 1: It is immediate that the general strategy subsumes separation and computation based techniques. Indeed, substituting $\underline{T}, \underline{U}$ to be degenerate yields the conditions provided in theorem 1. Substituting \underline{V} to be degenerate yields separation based technique.

V. CONCLUDING REMARKS

Having decoded the sum of sources, we ask whether it would be possible to decode an arbitrary function of the sources using the above techniques? The answer is yes and the technique involves ‘embedding’. Example 3 illustrates embedding and a framework is proposed in [4]. This leads us to the following fundamental question. The central element of the technique presented above was to decode the *sum* of transmitted codewords and use that to decode sum of KM message indices. If the MAC is ‘far from additive’, is it possible to decode a different bivariate function of transmitted codewords and use that to decode the desired function of the sources? The answer to the first question is yes. Indeed, the elegance of joint typical encoding and decoding enables us reconstruct other ‘well behaved’ functions of transmitted codewords. We recognize that if codebooks take values over a finite field and were closed under addition, it was natural and more efficient to decode the sum. On the other hand, if the codebooks were taking values over an algebraic object, for example a group, and were closed with respect to group multiplication, it would be natural and efficient to decode the product of transmitted codewords. Since, we did not require the MAC to be linear in order to compute the sum of transmitted codewords, we will not require it to multiply in order for us to decode the product of transmitted codewords. We elaborate on this in [4].

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