On Nonlinear Forbidden 0-1 Matrices: A Refutation of a Füredi-Hajnal Conjecture

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Abstract

A 0-1 matrix A is said to *avoid* a forbidden 0-1 matrix (or pattern) P if no submatrix of A matches P, where a 0 in P matches either 0 or 1 in A. The theory of forbidden matrices subsumes many extremal problems in combinatorics and graph theory such as bounding the length of Davenport-Schinzel sequences and their generalizations, Stanley and Wilf's permutation avoidance problem, and Turán-type subgraph avoidance problems. In addition, forbidden matrix theory has proved to be a powerful tool in discrete geometry and the analysis of both geometric and non-geometric algorithms.

Clearly a 0-1 matrix can be interpreted as the incidence matrix of a bipartite graph in which vertices on each side of the partition are *ordered*. Our primary contribution is a refutation of a conjecture of Füredi and Hajnal: that if P corresponds to an acyclic graph then the maximum number of 1s in an $n \times n$ matrix avoiding P is $O(n \log n)$. In addition, we give a simpler proof that there are infinitely many minimal nonlinear patterns and give tight bounds on the extremal functions for several small forbidden patterns.

1 Introduction

A 0-1 matrix A is said to *avoid* a $k \times l$ 0-1 matrix P if no submatrix A' induced by k rows and l columns of A is 1 in those locations where P is 1, i.e., a 0 in P matches either a 0 or 1 in A'. Let $\mathrm{Ex}(\mathcal{P}, n)$ be the maximum weight (i.e., number of 1s) of an $n \times n$ 0-1 matrix avoiding all patterns in the set $\mathcal{P}$, and let $\mathrm{Ex}(P, n)$ be short for $\mathrm{Ex}(\{P\}, n)$ for a single pattern P. Forbidden submatrix theory arose in the early 1990s to address two specific geometric problems and has since found many applications in discrete geometry, computational geometry, and (non-geometric) data structures. The *forbidden submatrix method* is striking in both its simplicity and diverse applicability. In one of the first applications of the method, Füredi [11] showed that the number of unit distances between points in a convex n-gon is upper-bounded by $\mathrm{Ex}(P_1, n)$; see Figure 1. Furthermore, he showed $\mathrm{Ex}(P_1, n) = \Theta(n \log n)$. At about the same time Bienstock and Györi [4] bounded the running time of Mitchell's algorithm [25], which finds shortest paths avoiding n-vertex obstacles in the plane, as a function of $\mathrm{Ex}(P_2, n) = \Theta(n \log n)$ [4, 33]. In subsequent

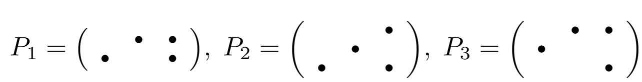

Figure 1: Following a common convention [33, 17, 15] we write forbidden 0-1 matrices with bullets for 1s and blanks for 0s.

years the method has been applied to several other geometric problems. Pach and Sharir [27] bounded the number of pairs of vertically visible line segments[1] in terms of $\mathrm{Ex}(P_1, n)$. Pach and Tardos [28] showed that the number of so-called *critical placements* of an n-gon in a hippodrome[2] is on the order of $\mathrm{Ex}(P_3, n)$, which Tardos [33] proved was $O(n)$. This result implied an upper bound of $O(n \log^3 n \log\log n)$ on the running time of Efrat and Sharir's [8] *segment center* algorithm. Pach and Tardos [28] used the forbidden submatrix framework to obtain a new proof that there are at most $O(n^{4/3})$ unit distances among n points in the plane, matching the best known upper bound [31, 23, 32, 1]. Very recently the author [30] has shown that numerous data structures based on path compression and binary search trees can be analyzed in a simple, uniform way using the forbidden submatrix method.

After the original applications of forbidden patterns P_1 and P_2 [11, 4, 27], Füredi and Hajnal [14] began a campaign to categorize all small forbidden patterns by their extremal function and to understand the prop-

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[1](whose left and/or right endpoints are anchored at a common vertical line)

[2]A hippodrome is a set of points equidistant from a line segment. A critical placement puts 3 vertices on the hipprodrome.

erties for forbidden patterns that influence their extremal functions. They made the important but simple observations that the forbidden submatrix framework essentially subsumes extremal problems in Davenport-Schinzel sequences and Turán-type (unordered) subgraph avoidance.[3] These observations immediately implied tight bounds on $\mathrm{Ex}(P_4, n) = \Theta(n\alpha(n))$ and $\mathrm{Ex}(P_5, n) = \Theta(n^{3/2})$, where α is the inverse-Ackermann function. (See, e.g., [16, 26, 34, 7].)

$$P_4 = \begin{pmatrix} & \bullet & & \bullet \\ \bullet & & \bullet & \end{pmatrix} \qquad P_5 = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$$

Here $\mathrm{Ex}(P_4, n)$ corresponds to the maximum length of an *ababa*-free sequence over an n-letter alphabet and $\mathrm{Ex}(P_5, n)$ to the maximum number of edges in an $n \times n$ bipartite graph avoiding 4-cycles. In other words, the forbidden submatrix framework had actually been used for decades under different guises.

Following [11, 4], Füredi and Hajnal [14], and Tardos [33] managed to categorize the growth of $\mathrm{Ex}(P, n)$ for every weight-4 pattern P and Tardos [33] bounded the growth of $\mathrm{Ex}(\mathcal{P}, n)$ for most sets $\mathcal{P}$ of weight-4 patterns. However, our current understanding of weight-5 and larger forbidden patterns is incomplete [33, 28, 10, 17, 15]. Moreover, we have no characterization of patterns with linear, near-linear, or polynomial extremal functions.

The Füredi-Hajnal Conjectures. Füredi and Hajnal concluded their paper with several conjectures, the most well known of which concerns permutation matrices, in which exactly one 1 appears in each row and column.

Conjecture 1. *[14] For any 0-1 permutation matrix P, $\mathrm{Ex}(P, n) = O(n)$.*

Obviously a 0-1 matrix can be interpreted as the incidence matrix of an ordered bipartite graph. Conjecture 1 is then equivalent to asking how large an ordered bipartite graph can be while avoiding an ordered matching. Let $H(P)$ be the *unordered* (bipartite) graph corresponding to matrix P and let $\mathrm{Ex}_{Tu}(H, n)$ be the Turán-number of a graph H, i.e., the maximum number of edges in an n-vertex graph avoiding subgraphs isomorphic to H. At a high level the growth of $\mathrm{Ex}_{Tu}(H, n)$ is understood very well; it is $\Theta(n^2)$ if H is not bipartite, $O(n)$ if H is a forest, and $\Omega(n^{1+c_1})$ and $O(n^{1+c_2})$ in all other cases, for constants $0 < c_1 \leq c_2 < 1$ [7, 13, 22, 2, 6, 5]. Füredi and Hajnal conjectured [14] that the extremal functions for 0-1 matrix avoidance and unordered subgraph avoidance never differ by more than a logarithmic factor:

Conjecture 2. *[14] For all 0-1 patterns P, $\mathrm{Ex}(P, n) = O(\mathrm{Ex}_{Tu}(H(P), n) \log n)$.*

Recognizing that this statement may be too strong, Füredi and Hajnal conjectured specifically whether Conjecture 2 held at least for patterns P for which $H(P)$ is a forest.

Conjecture 3. *[14] For all acyclic 0-1 patterns P, $\mathrm{Ex}(P, n) = O(n \log n)$.*

Conjecture 3 is a special case of Conjecture 2 since $\mathrm{Ex}_{Tu}(H, n) = O(n)$ for any forest H.

Status of the Conjectures. Marcus and Tardos [24] proved Conjecture 1 with a startlingly simple proof. An earlier result of Klazar [19] showed that Conjecture 1 would imply the Stanley-Wilf conjecture, which says that the number of n-permutations avoiding a fixed k-permutation is at most c_k^n for some constant c_k. Geneson [15] recently generalized the Marcus-Tardos proof [24] to show that double permutation matrices[4] are also linear.

Pach and Tardos [28] disproved Conjecture 2 by showing that for each $k \geq 2$, there is an ordered $2k$-cycle O_k for which $\mathrm{Ex}(O_k, n) = \Omega(n^{4/3})$. For $k \geq 4$ this bound differs sharply from the well known upper bound of $O(n^{1+1/k})$ on $\mathrm{Ex}_{Tu}(H(O_k), n)$.

Our Results. In Section 2 we refute Conjecture 3 by exhibiting a class of 0-1 matrices with weight $\Theta(n \log n \log\log n)$ that avoids a relatively small acyclic pattern with weight 8. Although the forbidden pattern is probably not of any particular interest, our method for constructing matrices of density $\Theta(\log n \log\log n)$ uses two generic composition procedures on 0-1 matrices that might be of interest to anyone exploring the landscape of forbidden 0-1 matrices or their applications. In Section 3 we give a substantially simpler proof [17, 15] that there are infinitely many minimal nonlinear patterns with respect to containment. Our technique lets us prove that patterns in Keszegh's class [17] are nonlinear, as well as several previously unclassified ones.

Related Results on 0-1 Matrices. Call a 0-1 matrix *light* if it contains exactly one 1 in each column and call a function *quasilinear* if it is of the form $O(n2^{\alpha^{O(1)}(n)})$, where $\alpha(n)$ is the inverse-Ackermann function. The quasilinear bounds on generalized Davenport-Schinzel sequences[5] [18, 26] imply that $\mathrm{Ex}(P, n)$ is quasilinear for all light P.

Pach and Tardos [28] considered a slight generalization of the problem considered in this paper, where the

[3] The order requirements can effectively be removed by forbidding all possible orders.

[4] A $k \times 2k$ double permutation matrix is derived from a $k \times k$ permutation matrix by immediately repeating every column.

[5] Here *generalized* means that the forbidden subsequence is arbitrary, not necessarily of the form $ababab\cdots$.

forbidden ordered graph H and host ordered graph are not necessarily bipartite. Let $\mathrm{Ex}_{PT}(H, n)$ be the extremal function for this variant. If H does not have interval chromatic number two[6] then $\mathrm{Ex}_{PT}(H, n)$ is trivially $\Theta(n^2)$. They proved that when H does have interval chromatic number two it always holds that $\mathrm{Ex}_{PT}(H, n) = O(\mathrm{Ex}(M(H), n) \log n)$, where $M(H)$ is the obvious representation of H as a 0-1 matrix; cf. Conjecture 2.

When the forbidden pattern is an all-1 $k \times l$ matrix $K_{k,l}$ there is no asymptotic difference between $\mathrm{Ex}(K_{k,l}, n)$ and $\mathrm{Ex}_{Tu}(H(K_{k,l}), n)$ since any graph contains a bipartite graph with more than half the edges. (Determining $\mathrm{Ex}(K_{k,l}, n)$ is sometimes called *Zarankiewicz's problem* [7, 13, 22, 2].) Assume without loss of generality that $l \geq k$. Kövari, Sós, and Turán [22] proved that $\mathrm{Ex}_{Tu}(K_{k,l}) = O(n^{2-1/k})$ and it is widely believed that this is the correct bound for fixed k and l. However, the upper bound has only been proved tight when $k \in \{1, 2, 3\}$ (with ever sharper bounds on the leading constants and lower order terms [9, 7, 13, 12, 2]) or if $k \geq 4$ and $l \geq (k-1)! + 1$ [2, 21]. In other cases the best lower bound is given by the probabilistic method: $\mathrm{Ex}_{Tu}(K_{k,l}, n) = \Omega(n^{2-\frac{k+l-2}{kl-1}})$.

Forbidden submatrix problems can obviously be generalized to higher dimensions. In graph terminology a d-dimensional 0-1 matrix corresponds to a d-partitite ordered hypergraph in which each edge is incident to exactly d vertices. Klazar and Marcus [20] (see also [3]) showed the Marcus-Tardos [24] result extends to higher dimensions. In particular, the maximum weight of a matrix in $\{0, 1\}^{n^d}$ avoiding any fixed d-dimensional permutation matrix is $\Theta(n^{d-1})$. Little is known about other forbidden patterns in dimensions greater than 2.

1.1 Notation and Basic Results

In this paper all matrices are indexed starting from zero. A row/column index prefixed with '$-$', say $-i$, indicates the row/column i from the last row/column of the matrix. For example, in an $m \times n$ matrix M, $M(0, 0)$ and $M(-0, -0) = M(m-1, n-1)$ are the northwest and southeast corners of M, respectively.

Lemmas 1.1–1.3 bound the extremal function of forbidden matrices relative to their submatrices. The first two lemmas are trivial. Pach and Tardos [28] proved the third, as well as a couple of similar lemmas that we do not need in the present paper.

Lemma 1.1. (Füredi-Hajnal [14]) *If P' is contained in P, where P, P' are 0-1 matrices, then* $\mathrm{Ex}(P', n) \leq \mathrm{Ex}(P, n)$.

[6] I.e., if it is not a bipartite graph in which vertices on one side of the bipartition precede those on the other side.

Lemma 1.2. (Füredi-Hajnal [14]) *Let $P \in \{0, 1\}^{k \times l}$ be a forbidden matrix where $P(i, l-1) = 1$ (i.e., a 1 in the last column of P) and let $P' \in \{0, 1\}^{k \times (l+1)}$ be identical to P in the first l columns and where $P'(i, l) = 1$, $P'(i', l) = 0$ for $i' \neq i$. Then* $\mathrm{Ex}(P', m, n) \leq \mathrm{Ex}(P, m, n) + n$.

Lemma 1.3. (Pach-Tardos [28]) *Let $P \in \{0, 1\}^{k \times l}$ be a forbidden matrix with a single 1 in the last column and let $P' \in \{0, 1\}^{k \times (l-1)}$ be P with the last column removed. Then* $\mathrm{Ex}(P, n) = O(n + \mathrm{Ex}(P', n) \log n)$ *and if* $\mathrm{Ex}(P', n) = n^{1+\Omega(1)}$ *then* $\mathrm{Ex}(P, n) = \Theta(\mathrm{Ex}(P', n))$

Since $\mathrm{Ex}(P, n)$ is invariant with respect to rotation and reflection of P, one can obviously apply Lemmas 1.2 and 1.3 to rows rather than columns. Lemmas 1.1 and 1.2 can be used in tandem to *stretch* a 0-1 matrix without changing its weight. Using the terminology from Lemma 1.1, let P' be derived from P with $P(i, l-1) = 1$ by adding a weight-1 column with $P'(i, l) = 1$ and setting $P'(i, l-1) = 0$. We call P' a stretched version of P. Obviously if P' is contained in P or is a stretched version of P, the nonlinearity of $\mathrm{Ex}(P', n)$ bears witness to the nonlinearity of $\mathrm{Ex}(P, n)$. For example, all nonlinear weight-4 matrices can be reduced to P_6 via zero or more stretching operations [14, 33]. Since $\mathrm{Ex}(P_6, n) = \Theta(n\alpha(n))$ is nonlinear [14], it represents the *sole* cause of nonlinearity among weight-4 matrices.

$$P_6 = \begin{pmatrix} & \bullet & & \bullet \\ & & \bullet & \\ \bullet & & & \end{pmatrix}$$

2 The Füredi-Hajnal Conjecture for Acyclic Forbidden Patterns

We first recall a standard construction of matrices avoiding the weight-4 patterns Q_1, Q_1', and Q_2:

$$Q_1 = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \bullet \end{pmatrix}, \quad Q_1' = \begin{pmatrix} \bullet & \bullet \\ & \bullet \\ \bullet & \end{pmatrix}, \quad Q_2 = \begin{pmatrix} & \bullet & \bullet \\ & & \bullet \\ \bullet & & \end{pmatrix}$$

Let D_q be a $2^q \times 2^q$ matrix with 1s on the diagonals that are powers of two and zero elsewhere; see Figure 2 for an example. The index q may be omitted if implied or irrelevant.

$$D_q(i, j) = \begin{cases} 1 & \text{if } j - i = 2^k, \text{ for some } k \in [0, q) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.1. $\mathrm{Ex}(\{Q_1, Q_1', Q_2\}, n) = \Omega(n \log n)$. *In particular D avoids Q_1, Q_1', and Q_2.*

Proof. Let $n = 2^q$. One can see that D_q has weight $(q-1)2^q + 1 = \Omega(n \log n)$. Consider an occurrence of $R = \begin{pmatrix} \bullet & \bullet \\ & \bullet \end{pmatrix}$ in D_q and let $(i, j'), (i, j)$, and (i', j) be the locations in D_q corresponding to $R(0,0), R(0,1)$, and $R(1,1)$. If $j-i=2^k$ then $j' \leq j-2^{k-1}$ and $i' \geq i+2^{k-1}$, which implies $D_q(i',j')$ lies on or below the main diagonal since $j'-i' \leq (j-i)-2^k = 0$. Since D_q contains no 1s on or below the main diagonal it must avoid Q_1, Q_1', and Q_2, as well as $\begin{pmatrix}\bullet & \bullet\\ \bullet & \bullet\end{pmatrix}$.

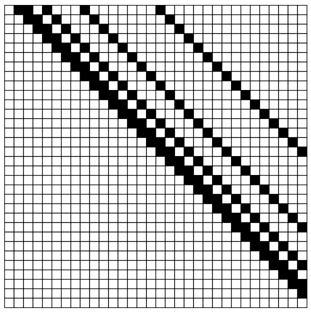

Figure 2: A depiction of D_5, with 0s and 1s indicated by white and black, respectively.

Theorem 2.1 gives a specific counterexample to the Füredi-Hajnal conjecture, which we prove in the remainder of this section.

Theorem 2.1. *There exists an acyclic forbidden matrix X for which $\mathrm{Ex}(X,n)=\omega(n\log n)$. Specifically, $\mathrm{Ex}(X,n)=\Omega(n\log n\log\log n)$ where:*

$$X = \begin{pmatrix} & \bullet & & \bullet & \bullet \\ & & \bullet & & \bullet \\ & \bullet & & & \\ \bullet & & & & \bullet \end{pmatrix}$$

A $2l$-bit number $i=i_1 2^l + i_2$ may be written $\langle i_1, i_2\rangle$. Let $n=2^{2^{k'+1}}$ for some integer k' and let $k=2^{k'}$ and $K=2^k=\sqrt{n}$. We will show the following $n\times n$ matrix A with weight $\Theta(k'kK^2)=\Theta(n\log n\log\log n)$ avoids X. The matrix A is a sparser version of a simpler matrix $\tilde{A}$ with weight $\Theta(k^2K^2)=\Theta(n\log^2 n)$. For much of the proof we consider $\tilde{A}$ rather than A.

$$A(\langle i_1,i_2\rangle, \langle j_1,j_2\rangle) = \begin{cases} 1 & \text{if } j_1-i_1=2^{k_1},\ \ j_2-i_2=2^{k_2}, \\ & \text{and } k_1+k_2-(k-1)=2^{k_3}, \\ & \text{for } k_1,k_2\in[0,k) \text{ and } k_3\in[0,k') \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{A}(\langle i_1,i_2\rangle, \langle j_1,j_2\rangle) = \begin{cases} 1 & \text{if } j_1-i_1=2^{k_1} \text{ and } j_2-i_2=2^{k_2}, \\ & \text{for } k_1,k_2\in[0,k) \\ 0 & \text{otherwise} \end{cases}$$

A *block* of $\tilde{A}$ (or A) consists of all entries $(\langle i_1,i_2\rangle, \langle j_1,j_2\rangle)$ with common i_1 and j_1 coordinates. The block matrix of $\tilde{A}$ (or A) is a $K\times K$ matrix whose entries are 0 and 1 if the corresponding block in $\tilde{A}$ (or A) is 0 or non-zero, respectively. One can view $\tilde{A}$ as the composition of D_k with itself. Note that if a given matrix has polylogarithmic density then composing it with itself roughly squares the density. This operation alone is not very useful for building matrices avoiding some submatrices: composing a matrix with density $\omega(1)$ with itself gives rise to a matrix with arbitrarily large all-1 submatrices.

Observation 1. *The block matrix of $\tilde{A}$ and every non-zero block in $\tilde{A}$ are exactly D_k.*

One can view A as being derived from $\tilde{A}$ by a different type of composition operation. Roughly speaking, we partition the 1s in $\tilde{A}$ into a collection of all-1 submatrices and replace each such submatrix with a copy (or, more accurately, a fragment of a copy) of $D_{k'}$. (This composition is effected by the '$k_1+k_2-(k-1)=2^{k_3}$' condition in the definition of A.) Sparsifying the matrix $\tilde{A}$ in this way reduces the density by a factor $\Theta(k/k') \approx \log n/\log\log n$.

As we noted above, X and every other fixed submatrix appears in $\tilde{A}$. However, Lemma 2.2 shows that the ways in which X can appear in $\tilde{A}$ are rather restricted.

Lemma 2.2. *Consider an occurrence of X in $\tilde{A}$ and let the locations in $\tilde{A}$ identified with $X(0,1)$, $X(0,4)$, $X(1,4)$, $X(3,4)$ be (i,j'), (i,j), (i',j), and (i'',j), respectively. If we write $x=\langle x_1,x_2\rangle$ for $x\in\{i,i',i'',j,j'\}$ then all of the following must be true:*

1. *Either $j_1'=j_1$ or $i_1=i_1''$ but not both.*
2. *If $j_1'=j_1$ then $i_1\neq i_1'$ and $i_2=i_2'$.*
3. *Similarly, if $i_1=i_1''$ then $j_1'\neq j_1$ and $j_2'=j_2$.*

Proof. Below is X, with rows and columns labeled:

$$X = \begin{pmatrix} & & j' & & & j \\ i & & \bullet & & \bullet & \bullet \\ i' & & & \bullet & & \bullet \\ & & \bullet & & & \\ i'' & \bullet & & & & \bullet \end{pmatrix}$$

For part (1), if $j_1' \neq j_1$ and $i_1 \neq i_1''$ then $X(0,1), X(0,4), X(3,0), X(3,4)$ (forming an instance of Q_1) must lie in separate blocks of $\tilde{A}$. By Observation 1 the block matrix of $\tilde{A}$ is exactly D_k, which does not contain Q_1. If, on the other hand, $j_1'=j_1$ and $i_1=i_1''$ then $X(0,3), X(0,4), X(1,2), X(1,4)$ lie in the same block (i.e., in D_k) and form an instance of Q_1, a contradiction. Turning to part (2), if $j_1'=j_1$ and $i_1=i_1'$ then the first two rows of X lie in the same block and contain Q_1, a contradiction. If $j_1' = j_1, i_1 < i_1'$, and $i_2 \neq i_2'$ then the first two rows of X lie in different blocks and different rows within their respective blocks. Depending on whether i_2 is greater or less than i_2', this implies that D_k contains either:

$$\begin{pmatrix} \bullet & & \bullet & \bullet \\ & \bullet & & \bullet \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} & \bullet & & \bullet \\ \bullet & & \bullet & \bullet \end{pmatrix}.$$

Both of these matrices contain Q_1, contradicting the fact that D_k excludes Q_1. Part (3) follows the same lines as part (2). If columns 1 and 4 of X were in the same block then that block (D_k) would include Q_1', a contradiction; if they are in different blocks and $j_2' \neq j_2$ then, depending on which of j_2' and j_2 is larger, D_k would include either:

$$\begin{pmatrix} \bullet & \bullet \\ & \bullet \\ \bullet & \\ & \bullet \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \bullet & \bullet \\ \bullet & \\ & \bullet \\ \bullet & \end{pmatrix},$$

both of which include Q_1', a contradiction that concludes the proof.

The cth *block column* consists of all entries $(\langle i_1, i_2\rangle, \langle j_1, j_2\rangle)$ in $\tilde{A}$ with $j_1 = c$; similarly, the rth *block row* consists of all entries with $i_1 = r$. We define the $k \times k$ matrix $\tilde{C}_{c,r}$, where $c \in [1, K-1], r \in [0, K-2]$ to be the submatrix of $\tilde{A}$ obtained by selecting the rth row in each non-zero block in block column c, and the columns in block column c that contain 1s in the selected rows. There may not be k such rows and columns; if there are fewer then the selected rows and columns will be packed into the southwest corner of $\tilde{C}_{c,r}$. The matrix $\tilde{R}_{r,c}$ is defined analogously with respect to block row $r \in [0, K-2]$ and column $c \in [1, K-1]$. More formally, $\tilde{C}$ and $\tilde{R}$ are defined as follows:

$$\tilde{C}_{c,r}(-i, j) = \begin{cases} \tilde{A}\left(\langle c - 2^i, r\rangle, \langle c, r + 2^j\rangle\right) & \text{for valid } i, j \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{R}_{r,c}(-i, j) = \begin{cases} \tilde{A}\left(\langle r, c - 2^i\rangle, \langle r + 2^j, c\rangle\right) & \text{for valid } i, j \\ 0 & \text{otherwise} \end{cases}$$

where i and j are valid if $i \in [0, \lfloor \log c \rfloor]$, and $j \in [0, \lfloor \log(K - r - 1) \rfloor]$. The matrices $C_{c,r}$ and $R_{r,c}$ select the same locations out of A that $\tilde{C}_{c,r}$ and $\tilde{R}_{r,c}$ select out of $\tilde{A}$. Figure 3 illustrates how $\tilde{R}_{r,c}$ is selected.

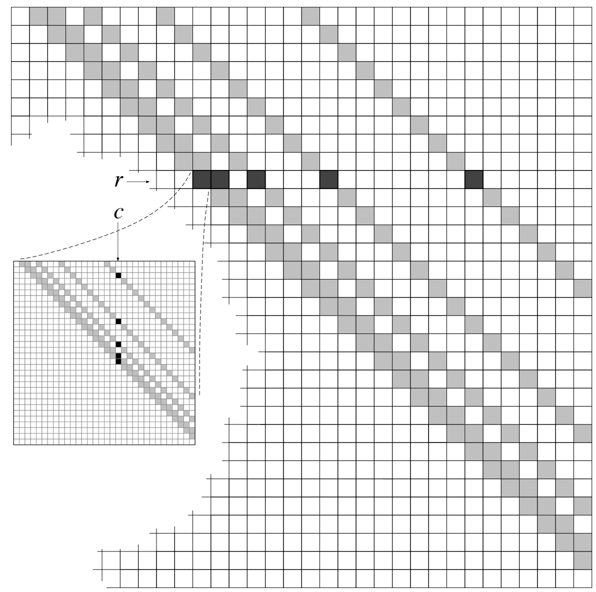

Figure 3: In this diagram $\tilde{A}$ is a $2^{10} \times 2^{10}$ matrix derived by composing D_5 with itself. $\tilde{R}_{r,c}$ is a 5×5 matrix obtained by selecting the cth column in each of the non-zero blocks in block row r and the rows in block row r in which the selected columns are 1. (Clearly $2^{10} = n$ is not of the form $2^{2^{k'+1}}$. The definition of $\tilde{R}_{r,c}$ does not depend on n being of this form.)

LEMMA 2.3. *For $c \in [1, K-1], r \in [0, K-2]$, $C_{c,r} = \tilde{C}_{c,r} \wedge D_{k'}$ and $R_{r,c} = \tilde{R}_{r,c} \wedge D_{k'}$, where $\wedge$ is the element-wise conjunction operator that interprets 0 and 1 as false and true, respectively.*

Proof. First observe that for $c \in [1, K-1], r \in [0, K-2]$, both $\tilde{C}_{c,r}$ and $\tilde{R}_{r,c}$ contain 1s in the $\lfloor 1 + \log c \rfloor \times \lfloor 1 + \log(K - r - 1) \rfloor$ contiguous submatrix at the southwest corner and 0s everywhere else. These entries were taken from $\tilde{A}$ and are all 1 by the definition of $\tilde{A}$. We now need to show that for $p \in [0, \lfloor \log c \rfloor]$ and $q \in [0, \lfloor \log(K - r - 1) \rfloor]$, $C_{c,r}(-p, q) = C_{c,r}(k - p - 1, q) = 1$ (and $R_{r,c}(-p, q) = R_{r,c}(k - p - 1, q) = 1$) if and only if $D_{k'}(k - p - 1, q) = 1$. Let $(i, j) = (\langle i_1, i_2\rangle, \langle j_1, j_2\rangle)$ be the location in A corresponding to $C_{c,r}(-p, q)$. It follows from the definition of $C_{c,r}$ that $j_1 - i_1 = 2^p$ and $j_2 - i_2 = 2^q$. By the definition of A, $A(i, j) = 1$ if and only if $p + q - (k - 1)$ is a power of 2. The criterion for $D_{k'}(k - p - 1, q) = 1$ is precisely the same: that $q - (k - p - 1)$ be a power of two. The case of $R_{r,c}(-p, q)$ follows the same lines. If $(i, j) = (\langle i_1, i_2\rangle, \langle j_1, j_2\rangle)$ is the location in A corresponding to $R_{r,c}(-p, q)$ then $j_1 - i_1 = 2^q$ and $j_2 - i_2 = 2^p$. Then $A(i, j) = 1$ iff $p + q - (k - 1)$ is a power of 2, which is precisely the same criterion for $D_{k'}(k - q - 1, p) = 1$: that

$p-(k-q-1) = p+q-(k-1)$ be a power of 2.

LEMMA 2.4. *The pattern X does not appear in A.*

Proof. Let i, i', i'', j, j' be as in Lemma 2.2. Further, let $(i, j'''), (i', j'')$, and (i''', j') be the locations in A corresponding to positions $X(0,3), X(1,2)$, and $X(2,1)$. Below is X, with rows and columns labeled:

$$X = \begin{pmatrix} & & j' & j'' & j''' & j \\ i & & \bullet & & \bullet & \bullet \\ i' & & & \bullet & & \bullet \\ i''' & & \bullet & & & \\ i'' & \bullet & & & & \bullet \end{pmatrix}$$

If X appears in A, Lemma 2.2(1) implies that either (a) columns 1–4 of X are mapped to one block column in A, or (b) rows 0–3 of X are mapped to one block row in A.

In case (a), Lemma 2.2(2) further states that $i_1 < i'_1$ and $i_2 = i'_2$, i.e., rows 0 and 1 of X appear in different blocks but the same row in their respective blocks. However, this implies that the submatrix C_{j_1,i_2} represents rows i, i' and columns j', j'', j''', j of A; in particular, C_{j_1,i_2} contains:

$$\begin{pmatrix} \bullet & & \bullet & \bullet \\ & \bullet & & \bullet \end{pmatrix}.$$

This is a contradiction since, by Lemma 2.3, C_{j_1,i_2} is contained in $D_{k'}$, which avoids Q_1.

Case (b) is symmetric. Lemma 2.2(3) states that $j'_1 < j_1$ but $j'_2 = j_2$, i.e., columns 1 and 4 of X appear in different blocks but the same column in their respective blocks. However, this implies that the submatrix R_{i_1,j_2} represents rows i, i', i''', i'' and columns j', j of A; in particular, R_{i_1,j_2} contains:

$$\begin{pmatrix} \bullet & \bullet \\ & \bullet \\ \bullet & \\ & \bullet \end{pmatrix}$$

contradicting the fact that $D_{k'}$ (containing R_{i_1,j_2}, by Lemma 2.3) avoids Q'_1.

This concludes the proof of Theorem 2.1.

3 More Nonlinear Matrices

In this section we give tight or nearly tight bounds on some low weight matrices and reprove a result due to Keszegh [17] and Geneson [15] that there are infinitely many minimal nonlinear matrices with respect to containment and stretching. Like their proof, ours is nonconstructive. The only known minimal nonlinear matrices with respect to these operations are:

$$\begin{pmatrix} & \bullet & & \bullet \\ & & \bullet & \\ \bullet & & & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & \bullet & \bullet & \\ & & & \bullet \\ & & & \bullet \\ \bullet & & & \end{pmatrix},$$

with extremal functions $\Theta(n\alpha(n))$ [14] and $\Theta(n \log n)$ [17, 28], respectively.

With one exception all our lower bounds are based on the following recursive construction of matrices with weight $\Theta(n \log n)$.

$$R_1 = \begin{pmatrix} \bullet & \bullet \\ & \bullet \end{pmatrix}, \quad R_q = \left(\begin{array}{c|c} R_{q-1} & \pi \\ \hline & R_{q-1} \end{array}\right),$$
where π is a permutation matrix

This construction is a slight generalization of one from Füredi and Hajnal [14], who simply fixed π to be the identity matrix. We use R_q to denote any matrix that could be generated in this way (over some choices of permutation matrices π) and let $R_q^{\backslash}$ and $R_q^{/}$ be the matrices when π is always chosen to be, respectively, the identity matrix and 90-degree rotation of the identity. The subscript q is omitted whenever the dimension of the matrix is not relevant. Clearly R_q is a $2^q \times 2^q$ matrix with more than $q2^{q-1}$ 1s.

OBSERVATION 2. *If a matrix S does not appear in R_{q-1}, S can only appear in R_q if it can be divided into quadrants:*

$$S = \left(\begin{array}{c|c} S_{nw} & S_{ne} \\ \hline S_{sw} & S_{se} \end{array}\right)$$

such that S_{ne} can appear in the permutation π selected by R_q and S_{sw} is either empty or 0.

Observation 2 implies that Q_1, Q'_1, and Q_2 do not appear in R, for any choice of permutation matrices.[7] In Theorems 3.1 and 3.2 we argue that a number of other matrices do not appear in at least one of $R, R^{/}$, or $R^{\backslash}$ using the following proof template. Call a matrix NW/SE (northwest/southeast) separable if it is possible to divide it into *nonempty* quadrants such that the NW and SE quadrants are non-zero while the NE and SW quadrants are 0. If J is our forbidden submatrix, consider every subset $\hat{J}$ of 1s in J that could appear in the NE permutation quadrant of R (or $R^{/}$ or $R^{\backslash}$) and let $J\backslash\hat{J}$ be J after flipping all 1s in $\hat{J}$ to 0. (In particular, $\hat{J}$ must form a permutation and there cannot be any 1s in $J\backslash\hat{J}$ to the north/east/northeast of any 1 in $\hat{J}$.) We say that $\hat{J}$ *separates* J if it is *not* the case that all 1s in $J\backslash\hat{J}$ appear south of all 1s in $\hat{J}$ or west of all 1s in $\hat{J}$. (In other words, putting $\hat{J}$ in the NE quadrant of R forces at least one 1 of J to be put in the NW and SE quadrants of R.) We can now say that J does not appear in R if, for every $\hat{J}$, $\hat{J}$ separates J and $J\backslash\hat{J}$ is

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[7] We note that there are $(n/2)!(n/4)!^2 \cdots (n/2^i)!^{2^{i-1}} \cdots$ ways to generate R, which is $2^{\Theta(n \log^2 n)}$ and on par with the $\binom{n^2}{n \log n} = 2^{\Theta(n \log^2 n)}$ matrices with weight $n \log n$. Previous constructions [11, 14, 33] implied (trivially) that there were $2^{\Theta(n \log n)}$ matrices with weight $\Theta(n \log n)$ avoiding Q_1, Q'_1, and Q_2.

not NW/SE separable. In other words, putting $\hat{J}$ in the NE quadrant of R forces at least one 1 to appear in each of the other quadrants of R, a contradiction. Let us restate this observation as a lemma:

Lemma 3.1. *A forbidden submatrix J cannot appear in $\tilde{R} \in \{R, R^{/}, R^{\backslash}\}$ if, for every subset $\hat{J}$ of the 1s in J that could appear in the NE permutation quadrant of $\tilde{R}$, no 1 in $J \backslash \hat{J}$ is north, east, or northeast of any 1 in $\hat{J}$, $\hat{J}$ separates J, and $J \backslash \hat{J}$ is not NW/SE separable.*

Lemma 3.1 implies that H, H', H'', and H''', defined in Theorem 3.1, do not appear in at least one of $R, R^{/}$, or $R^{\backslash}$. Keszegh [17] proved that $\mathrm{Ex}(H, n) = \Omega(n \log n)$. The nonlinear bounds on H', H'', H''' are new.[8]

Theorem 3.1. *For H, H', H'', H''' as defined below, $\mathrm{Ex}(H, n), \mathrm{Ex}(H', n), \mathrm{Ex}(H'', n)$ are all $\Theta(n \log n)$ and $\mathrm{Ex}(H''', n)$ is $\Omega(n \log n)$ and $O(n \log n 2^{\alpha(n)^{O(1)}})$.*

$$H = \begin{pmatrix} & \bullet & \bullet & \\ & & & \bullet \\ & & & \bullet \\ \bullet & & & \end{pmatrix} \qquad H'' = \begin{pmatrix} & \bullet & & \bullet & & \\ & & & & \bullet & \\ \bullet & & & & & \\ & & & & & \bullet \\ & & & & & \bullet \\ & & \bullet & & & \end{pmatrix}$$

$$H' = \begin{pmatrix} & \bullet & & \bullet & \\ & & & & \bullet \\ \bullet & & & & \\ & & & & \bullet \\ & & \bullet & & \end{pmatrix} \qquad H''' = \begin{pmatrix} & \bullet & & & \bullet & \\ & & & \bullet & & \\ \bullet & & & & & \\ & & & & & \bullet \\ & & & & & \bullet \\ & & \bullet & & & \end{pmatrix}$$

Proof. For the lower bounds we show that R avoids H, $R^{/}$ avoids H' and H'', and $R^{\backslash}$ avoids H'''. If H is to appear in R, one or both of $H(0,2)$ and $H(1,3)$ must appear in the NE quadrant of R, The set $\hat{H} = \{(0,2),(1,3)\}$ separates H and $H \backslash \hat{H}$ is not NW/SE separable; by Lemma 3.1, H cannot appear in R. If H' appears in $R^{/}$ then exactly one of $H'(0,3)$ or $H'(1,4)$ must appear in the NE permutation quadrant; these two cases are symmetric. The set $\hat{H}' = \{(0,3)\}$ separates H' and $H' \backslash \hat{H}'$ is not NW/SE separable, implying that $R^{/}$ avoids H'. If H'' appears in $R^{/}$ then exactly one of $H''(0,3), H''(1,4), H''(3,5)$ must be in the NE quadrant of $R^{/}$; let $\hat{H}''$ be one such entry. Clearly $\hat{H}''$ separates H'' and $H'' \backslash \hat{H}''$ is not NW/SE separable, implying that H'' does not appear in $R^{/}$. For H''', either or both of $H'''(0,4)$ and $H'''(3,5)$ must appear in the NE permutation quadrant of $R^{\backslash}$. The set $\hat{H}''' = \{(0,4),(3,5)\}$ separates H''' and $H''' \backslash \hat{H}'''$ is not NW/SE separable.

Turning to the upper bounds, for H, one application of Lemma 1.3 (to the first column) and several applications of Lemmas 1.1 and 1.2 show $\mathrm{Ex}(H, n) = O(n \log n)$. For H', one application of Lemma 1.3 to the first column leaves a matrix known[9] to be linear [33, 17]. For H'', one application of Lemma 1.3 and Lemma 1.2 (to the bottom two rows) yields a submatrix of a double permutation matrix, all of which are known to be linear [15]. For H''', one application of Lemma 1.3 and Lemma 1.2 (to the bottom two rows) yields a light matrix with a single 1 in each column. All such matrices are quasilinear [18, 26].

The argument employed in Lemma 3.1 and Theorem 3.1 allows us to give a significantly simpler proof that there are infinitely many minimally nonlinear matrices with respect to containment and stretching [17, 15]. Definition 1 gives an infinite set of matrices $\{H_s^\flat, H_s^\sharp\}_{s \geq 0}$, which includes all of Keszegh's matrices [17], and Theorem 3.2 proves that all have extremal function $\Theta(n \log n)$.

Definition 1. *For $s \geq 0$, $H_s = \bar{H}$ is a $(3s+4) \times (3s+4)$ matrix in which $\bar{H}(0,1) = \bar{H}(0,2) = \bar{H}(3,0) = \bar{H}(3s+1, 3s+3) = \bar{H}(3s+2, 3s+3) = 1$ and for $t \in [1, s]$, $\bar{H}(3t-2, 3t+1) = \bar{H}(3t-1, 3t+2) = \flat$, $\bar{H}(3t-1, 3t+1) = \bar{H}(3t-2, 3t+2) = \sharp$, and $\bar{H}(3t+3, 3t) = 1$. All other entries of $\bar{H}$ are zero. Let $H_s^\flat$ (respectively, $H_s^\sharp$) be H_s after substituting 1 for all $\flat$s (resp., $\sharp$s) and 0 for all $\sharp$s (resp., $\flat$s). Let $\mathcal{H} = \{H_s^\flat, H_s^\sharp\}_{s \geq 0}$ be the set of all such matrices.*

$$H_1 = \begin{pmatrix} & \bullet & \bullet & & & & \\ & & & & \flat & \sharp & \\ & & & & \sharp & \flat & \\ \bullet & & & & & & \\ & & & & & & \bullet \\ & & & & & & \bullet \\ & & & \bullet & & & \end{pmatrix}$$

$$H_2 = \begin{pmatrix} & \bullet & \bullet & & & & & & & \\ & & & & \flat & \sharp & & & & \\ & & & & \sharp & \flat & & & & \\ \bullet & & & & & & & & & \\ & & & & & & & \flat & \sharp & \\ & & & & & & & \sharp & \flat & \\ & & & \bullet & & & & & & \\ & & & & & & & & & \bullet \\ & & & & & & & & & \bullet \\ & & & & & & \bullet & & & \end{pmatrix}$$

Note that $H = H_0^\flat = H_0^\sharp$ is the smallest member of $\mathcal{H}$. Keszegh [17] proved that $\mathrm{Ex}(H_s^\sharp, n) = \Omega(n \log n)$ by showing that $H_s^\sharp$ is not contained in the 0-1 matrix K for which $K(i,j) = 1$ if and only if $j - i = 3^k$ for some integer k. Needless to say, his proof is delicate inasmuch as it needs K to be defined with respect to powers of 3

[8] Actually, H''' is trivially nonlinear because it contains $\begin{pmatrix} & \bullet & & \bullet \\ & & \bullet & \\ \bullet & & & \end{pmatrix}$, though the lower bound $\mathrm{Ex}(H''', n) = \Omega(n \log n)$ is new.

[9] To be more specific, one takes P_3, defined in the introduction and shown to be linear by Tardos [33], then applies Keszegh's [17] operation, which preserves the extremal function. The resulting matrix contains $H' \backslash \{(2,0)\}$.

rather than 2. We show that $H_s^\sharp$ and $H_s^\flat$ do not appear in $R^\backslash$ and $R^/$ respectively, implying that $\mathrm{Ex}(H_s^\sharp, n)$ and $\mathrm{Ex}(H_s^\flat, n)$ are $\Omega(n \log n)$. In fact, by Lemma 1.3 and [15] this bound is tight.

Theorem 3.2. *For all $s \geq 0$, $R^\backslash$ avoids $H_s^\sharp$ and $R^/$ avoids $H_s^\flat$. Consequently, $\mathrm{Ex}(H_s^\flat, n)$ and $\mathrm{Ex}(H_s^\sharp, n)$ are $\Omega(n \log n)$. Furthermore, this bound is tight.*

Proof. Let $\tilde{H} = H_s^\sharp$ and suppose that $\tilde{H}$ appears in $R^\backslash$. It follows that some subset of $\hat{H} = \{\tilde{H}(0,2), \cdots, \tilde{H}(3t-2, 3t+2), \cdots, \tilde{H}(3s+1, 3s+3)\}$ (for $t \in [1, s]$) appears in the NE permutation quadrant of $\tilde{R}$. Since $\tilde{H}(0,1)$ lies to the west/northwest of every member of $\hat{H}$ and $\tilde{H}(3s+2, 3s+3)$ appears to the south/southeast of every member of $\hat{H}$, $\hat{H}$ separates $\tilde{H}$. If we can show that $\tilde{H} \backslash \hat{H}$ is not NW/SE separable then, by Lemma 3.1, $R^\backslash$ avoids $\tilde{H}$. Consider the following zigzagging list L of positions in $\tilde{H} \backslash \hat{H}$ containing 1s:

$$L = (0,1), (3,0), (2,4), \ldots, (3t+3, 3t), (3t+2, 3t+4), \ldots, (3s+3, 3s), (3s+2, 3s+3)$$

The coordinates at even positions in L appear to the southwest of the preceding coordinate and the coordinates at odd positions in L appear to the northwest of the preceding coordinate. Since $\tilde{H}(0,1)$ must lie in the NW quadrant of $R^\backslash$ this implies that *all* coordinates of L lie in the NW quadrant of $R^\backslash$, which contradicts the fact that $\tilde{H}(3s+2, 3s+3)$ lies in the SE quadrant. Thus $\tilde{H} \backslash \hat{H}$ is not NW/SE separable and $\tilde{H}$ cannot appear in $R^\backslash$.

The proof for $\tilde{H} = H_s^\flat$ is rather similar. If $\tilde{H}$ appears in $R^/$ it follows that exactly one 1 among $\tilde{H}(0,2), \cdots, \tilde{H}(3t-2, 3t+1), \tilde{H}(3t-1, 3t+2), \ldots, \tilde{H}(3s+1, 3s+3)$, for $t \in [1, s]$, appears in the NE quadrant of $R^/$. Let $\hat{H}$ be the coordinate of this 1. By the same reasoning as above, $\tilde{H}(0,1)$ and $\tilde{H}(3s+2, 3s+3)$ must lie in the NW and SE quadrants of $R^/$, which implies that $\hat{H}$ separates $\tilde{H}$. There is a zigzagging list L' (analogous to L) from $(0,1)$ to $(3s+2, 3s+3)$ that demonstrates that all of L' lies in the NW quadrant of $R^/$, contradicting the fact that $(3s+2, 3s+3)$ is in the SE quadrant.

For the upper bound, one application of Lemma 1.3 and Lemma 1.2, to the bottom two rows, yields a submatrix of a double permutation. The extremal functions for such matrices are linear [24, 15].

Each member of $\mathcal{H}$ obviously contains a minimally nonlinear matrix but it is plausible that the only minimally nonlinear member of $\mathcal{H}$ is $H = H_0^\flat = H_0^\sharp$. Nonetheless, we can conclude that there are infinitely many minimal nonlinear matrices. Theorem 3.3 was proved in [17, 15]. We give a slightly simpler presentation.

Theorem 3.3. *(Keszegh [17], Geneson [15]) There are infinitely many minimal nonlinear forbidden submatrices with respect to containment and stretching.*

Proof. Stretching any $H_s^\flat$ or $H_s^\sharp$ yields a submatrix of a double permutation, which is linear [15]. We make use of results of Keszegh and Geneson [17, 15], who showed that if P and P' are double permutation matrices (or rotations thereof) and P'' is the matrix that contains P in the NW quadrant, P' in the SE quadrant, and 0 elsewhere, then $\mathrm{Ex}(P'', n) = O(n)$.[10] Call this operation *linking* P and P'. Consider a minimal nonlinear submatrix $\tilde{H}$ of $H_s^\sharp$. Clearly $\tilde{H}$ must be 1 in positions $(0,1), (0,2), (3s+1, 3s+3), (3s+2, 3s+3)$; otherwise it would be contained in a double permutation. It also must be 1 in positions of the form $(3t+3, 3t)$ and must be 1 in at least on of the two positions $(3t-1, 3t+1), (3t-2, 3t+2)$; otherwise $\tilde{H}$ would be contained in a matrix formed by linking two double permutations. Thus, $\tilde{H}$ could be 0 at positions of the form $(3t-2, 3t+2)$ but must otherwise be identical to $H_s^\sharp$. In other words, $\tilde{H}$ is contained in $H_s^\sharp$ and may be contained in $H_s^\flat$ (if it is 0 in all the positions identified above) but it cannot be contained in any $H_{s'}^\sharp$, or $H_{s'}^\flat$, for $s' \neq s$. The number of minimal nonlinear matrices is therefore infinite.

Tardos [33] defined a matrix very similar to $R^\backslash$ where the rows appear in the same order but the columns are shuffled. He showed this class of matrices avoids the pattern T_0, defined in Theorem 3.4. We show that his class of matrices also avoids an infinite number of generalizations of T_0.

Theorem 3.4. *Let T_k be a $(k+3) \times (k+3)$ pattern matrix in which $T(0,0) = T(0,2) = T(k+1, 1) = T(k+2, k+2) = 1$ and for $1 \leq i \leq k$, $T(i, i+1) = T(i, i+2) = 1$; in all other locations T_k is 0. The first few patterns in this set are as follows:*

$$T_0 = \begin{pmatrix} \bullet & & \bullet \\ & \bullet & \\ & & \bullet \end{pmatrix}, \quad T_1 = \begin{pmatrix} \bullet & & \bullet & \\ & & \bullet & \bullet \\ & \bullet & & \\ & & & \bullet \end{pmatrix},$$

$$T_2 = \begin{pmatrix} \bullet & & \bullet & & \\ & & \underline{\bullet} & \underline{\bullet} & \\ & & & \underline{\bullet} & \bullet \\ & \underline{\bullet} & & & \\ & & & & \bullet \end{pmatrix}$$

[10] This is actually a corollary of Keszegh's theorem [17]. Let P be a matrix with a 1 in the SE corner, P' be a matrix with a 1 in the NW corner, and P'' be the matrix that contains copies of P and P' that only overlap in their SE and NW corners respectively. Then $\mathrm{Ex}(P'', n) \leq \mathrm{Ex}(P, n) + \mathrm{Ex}(P', n)$. If P and P' are double permutations but not of the requisite form, each can simply be enlarged to yield new double permutations with 1s in the proper corners.

Then $\mathrm{Ex}(\{T_k\}_{k\geq 0}, n) = \Theta(n \log n)$.

Before proving Theorem 3.4 we make a couple observations. First, for $k \geq 2$, T_k contains the pattern Q_2 (underlined above), implying that $\mathrm{Ex}(T_k, n) = \Omega(n \log n)$ by Lemma 2.1. However, this does not imply that, for example, $\mathrm{Ex}(\{T_0, T_2\}, n) = \Omega(n \log n)$ because the $n \log n$-weight matrices avoiding T_0 and Q_2 are very different. It is an easy exercise to show that $\mathrm{Ex}(\{T_0, Q_2\}, n) = O(n)$. Second, we cannot prove that $\mathrm{Ex}(T_1, n) = \Omega(n \log n)$ by the method of Theorems 3.1 and 3.2 since setting $T_1(0, 2)$ or $T_1(1, 3)$ to be zero leaves a NW/SE separable matrix.

Proof. (Theorem 3.4) Let $\bar{A}$ be a $2^K \times 2^K$ matrix whose rows and columns are associated with K-bit strings or equivalently, K-bit integers. Let $\mathrm{rev}(i)$ be the integer obtained by reversing the bit-string representation of i, e.g., if $K = 4$, $\mathrm{rev}(12) = \mathrm{rev}(1100_2) = 0011_2 = 3$. Let $i <^* j$ if $\mathrm{rev}(i) < \mathrm{rev}(j)$. The rows of $\bar{A}$ are sorted according to $<$ and the columns according to $<^*$.

$$\bar{A}(i, j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ differ in one bit and } i < j \\ 0 & \text{otherwise} \end{cases}$$

Tardos [33] proved that $\bar{A}$ avoids T_0. We will only prove that $\bar{A}$ avoids T_1, giving the lower bound $\mathrm{Ex}(T_1, n) = \Omega(n \log n)$. It will be clear that the proof can be extended to any T_k.

Suppose that there exist rows $x < y < z < w$ and columns $i <^* j <^* k <^* l$ in $\bar{A}$ containing an occurrence of T_1. Let $a, b, c, d, e \in [0, K-1]$ be the indices for which $x_a = 0, i_a = 1; x_b = 0, k_b = 1; y_c = 0, k_c = 1; y_d = 0, l_d = 1$; and $w_e = 0, l_e = 1$. Since i and k only differ from x in bit positions a and b, respectively, we have $i_b = x_b = 0$ and $k_a = x_a = 0$. From the ordering $i <^* k$ it follows that $a < b$. Similarly, x and y only differ in bit positions b and c, where $x_c = k_c = 1$ and $y_b = k_b = 1$; from the ordering $x < y$ it follows that $b < c$. The same reasoning shows that $c < d < e$. See Figure 4. From the ordering $y < z < w$ and the fact that y and w agree at indices 0 through $d-1$, it follows from the row ordering according to $<$ that z agrees with y, w at those indices. In particular $z_c = 0$. Similarly, the ordering $i <^* j <^* k$ implies that i, j, and k are equal at indices $b+1$ through $K-1$, and, in particular, that $j_c = 1$. Obviously c is the single bit position where z and j differ. This implies that y and z agree at positions $c+1$ through $K-1$ since y, k, and j agree on those as well. Thus $z = y$, a contradiction. Similarly, j agrees with k at bit position c, and, since k, y and z agree at positions 0 through $c-1$ we have $j = k$, another contradiction.

Turning to the upper bound, one application of Lemma 1.3, to the bottom row, and another application

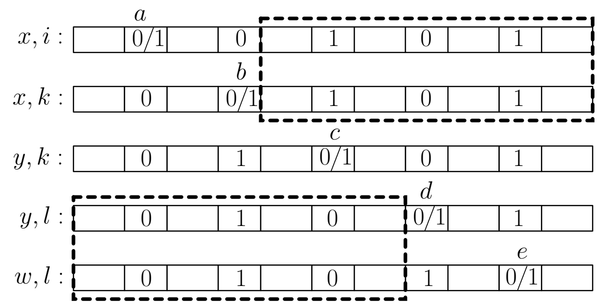

Figure 4: The bit-string representations of x, y, w, i, k, and l are identical except in positions a, b, c, d, e. The bit-string of j is identical to i and k after position b and the bit-string of z is identical to y and w before position d.

of Lemma 1.2, to the right column, yields the matrix T_1':

$$T_1' = \begin{pmatrix} \bullet & & \bullet \\ & & \bullet \\ & \bullet & \end{pmatrix}$$

Tardos [33] proved that $\mathrm{Ex}(T_1', n) = O(n)$.

4 Conclusions and Open Problems

We have exhibited an acyclic forbidden 0-1 pattern with extremal function $\omega(n \log n)$, thereby disproving a conjecture of Füredi and Hajnal [14]. However, our result does not imply or suggest any general upper bound on acyclic patterns. It is plausible that our composition technique could be generalized, but a straightforward generalization would only get us additional $\mathrm{poly}(\log \log n)$ factors in the extremal function.[11] Pach and Tardos conjectured [28] that all acyclic patterns have extremal functions $n(\log n)^{O(1)}$ and verified this claim for all but two patterns with weight at most six. We believe the Pach-Tardos conjecture is true but might be too weak. Is there some fixed c such that all acyclic patterns have extremal functions in $O(n \log^c n)$?

It would be desirable to identify more minimal nonlinear matrices, since, at present, we know of only two: P_6 and H defined in Sections 1 and 3, respectively. Is it true that a light pattern is nonlinear if and only if it contains P_6 or an equivalent? One might begin to answer this question by analyzing the weight-5 patterns P_7 and P_8, defined below. Theorem 3.2 demonstrates that there exist infinitely many minimal non*quasi*linear patterns with respect to containment

[11] In the terminology of Section 2, A was derived by composing D with itself and replacing all-1 submatrices with copies of D. If it were possible to compose A with itself in this way (which we have not verified is possible), the resulting $n \times n$ matrix would, at best, have density $\Theta(\log n(\log \log n)^2 \log \log \log n)$.

and stretching. However, we are aware of only four such patterns: P_2, Q_2, H, and H', defined in Sections 1–3. To show other patterns are minimal nonquasilinear — H'' and T_1 are two likely candidates — we need to better understand several other weight-5 patterns. For example, are P_9 and P_{10} quasilinear?

$$P_7 = \begin{pmatrix} & & \bullet & & \\ \bullet & & & & \bullet \\ & \bullet & & \bullet & \end{pmatrix} \qquad P_8 = \begin{pmatrix} & & & \bullet & \\ & \bullet & & & \\ & & \bullet & & \bullet \\ \bullet & & & & \end{pmatrix}$$

$$P_9 = \begin{pmatrix} \bullet & & \bullet & \\ & & & \bullet \\ & \bullet & & \\ & & & \bullet \end{pmatrix} \qquad P_{10} = \begin{pmatrix} \bullet & & \bullet & \\ & & \bullet & \\ & \bullet & & \\ & & & \bullet \end{pmatrix}$$

Results of Keszegh and Geneson [17, 15] prove the existence of infinitely many minimal nonlinear forbidden matrices *with respect to containment and stretching.* However, we think this is the wrong answer to the loosely defined question of whether there are infinitely many different causes of nonlinearity. As an analogy, there are infinitely many minimal nonplanar graphs with respect to subgraph containment but these should not be considered distinct causes of nonplanarity. The infinite sets of matrices $\{H_s^\sharp\}_{s\geq 0}, \{H_s^\flat\}_{s\geq 0}$, and $\{T_s\}_{s\geq 0}$, defined in Section 3, were constructed by applying a "daisy chaining" operation s times to a base matrix. This type of daisy chaining operation has also been used in the context of generalized Davenport-Schinzel sequences [29] in order to generate an infinite set of nonlinear forbidden subsequences. Is there an infinite set of minimally nonlinear forbidden matrices that does not use daisy chaining[12] or a similarly mechanical operation?

[12] Rather than precisely define what daisy chaining is, we will take our cue from Justice Potter Stewart and trust that the reader knows it when he sees it.

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