THE EFFICIENCY OF MATRICES FOR TAYLOR SERIES.

by (II), (III) and (IV) and therefore tends to 0 as $i \to \infty$. Thus the matrix $\{c_{in}\}$ also satisfies (ii). But

$$\sum_{n=1}^{\infty} |c_{in}| = (i+1) \sum_{n=1}^{\infty} |b(m_i, n)| + i \sum_{n=1}^{\infty} |b(w_i, n)| \geq \sum_{n=1}^{\infty} b(w_i, n) = i,$$

and hence $\{c_{in}\}$ does not satisfy (c) and is not a Toeplitz matrix.

References.
5. O. Toeplitz, "Über allgemeine lineare Mittelbildungen," *Prace mat. fiz.*, 22 (1911), 113–120.

St. John’s College,
Cambridge.

THE FACTORIZATION OF LINEAR GRAPHS

W. T. TUTTE*.

1. Introduction. We make use of the following definitions.

A graph is a finite simplicial 1-complex. The order of a graph is the number of its 0-simplexes. The degree of a 0-simplex in a graph is the number of 1-simplexes with which it is incident. If all the 0-simplexes in a graph $N$ have degree $\sigma$, $N$ is said to be regular and of the $\sigma$-th degree. A component of a graph is a connected part not contained in any larger connected part.

A subgraph of a graph $N$ is a graph consisting of all the 0-simplexes and some subset of the 1-simplexes of $N$. A factor is a regular subgraph of the first degree. If $N$ has no factor it is prime. Clearly all graphs of odd order are prime.

Let the 0-simplexes of a graph $N$ be enumerated as $a_1, a_2, \ldots, a_n$, and let $S = (a_{i_1}, a_{i_2}, \ldots, a_{i_r})$ be any subset of them. Then we denote by $N_S$ or $N_{i_1, \ldots, i_r}$ the graph obtained from $N$ by suppressing the 0-simplexes of $S$ and the 1-simplexes which are incident with members of $S$. We denote the number of members of $S$ by $f(S)$, the number of components of $N_S$ by $h(S)$ and the number of these components of odd order by $h_u(S)$. If $h(S) > 1$ we say that $S$ is an isthmoid of rank $f(S)$.

* Received 12 December, 1946; read 19 December, 1946.
In this paper, by a method based on the properties of skew-symmetric determinants, we show that a graph \( N \) is prime if and only if it contains an \( S \) such that \( h_u(S) > f(S) \). We deduce from this that if \( A \) is any \( 1 \)-simplex of a connected regular graph of even order and of degree \( \sigma \), having no isthmoid of rank < \( \sigma - 1 \), then \( N \) has a factor which contains \( A \). Those conversant with the theory of finite graphs will observe that this result contains Petersen's Theorem* as a special case.

2. Pfaffians. Let \( \Delta = |c_{ij}| \) be a skew-symmetric determinant in which the elements above the diagonal are independent indeterminates over the ring of rational integers. It is known† that if \( \Delta \) is of odd order it vanishes, but that if it is of even order \( 2m \) it is the square of a "Pfaffian". This Pfaffian \( P \) is given by

\[
P = \sum \epsilon c_{ij} c_{kl} \ldots c_{rs}
\]

in which the summation is over all partitions of the integers 1 to \( 2m \) into pairs \((i, j), (k, l), \ldots, (r, s)\), the order of the elements in each pair and the arrangement of the \( m \) pairs being immaterial. \( \epsilon \) is +1 or −1 according as the sequence \((i, j, k, l, \ldots, r, s)\) is an even or odd permutation of \((1, 2, \ldots, 2m)\). We assume hereafter that \( \Delta \) is of even order \( 2m \).

Let \( \Delta_{i_1 \ldots r} \) be the skew-symmetric determinant obtained from \( \Delta \) by striking out the \( i \)-th, \( j \)-th, \ldots, \( r \)-th rows and columns, and let \( P_{i_1 \ldots r} \) be the corresponding Pfaffian. Let \( C_{ij} \) be the cofactor of \( c_{ij} \) in \( \Delta \). By Jacobi's Theorem‡ we have

\[
\Delta \Delta_{ij} = \Delta_i \Delta_j - C_{ij} C_{ji} = C_{ij}^2
\]

since the \( \Delta \)'s are skew-symmetric and \( \Delta_i \) is of odd order.

By another application of Jacobi's Theorem we have

\[
\Delta^3 \Delta_{ijkl} = \begin{vmatrix}
0 & C_{ij} & C_{ik} & C_{il} \\
-C_{ij} & 0 & C_{jk} & C_{jl} \\
-C_{ik} & -C_{jk} & 0 & C_{kl} \\
-C_{il} & -C_{jl} & -C_{kl} & 0
\end{vmatrix} = (C_{ij} C_{kl} - C_{ik} C_{jl} + C_{il} C_{jk})^2.
\]

‡ See e.g. Aitken, "Determinants and matrices," (Edinburgh, 1939), p. 97.
THE FACTORIZATION OF LINEAR GRAPHS.

Hence, by (2),

$$PP_{ijkl} = \pm P_{ij}P_{kl} \pm P_{ik}P_{jl} \pm P_{il}P_{jk}.$$  \hspace{1cm} (3)

(We need not enquire into the values of the signs.)

Consider a graph $N$, of even order $2m$, whose 0-simplexes are enumerated as $a_1, a_2, ..., a_{2m}$. Let $P(N)$ be the Pfaffian derived from (1) by substituting 0 for each $c_{ij}$ for which $a_i$ and $a_j$ are not joined by a 1-simplex. We note that the substitution which changes $P$ into $P(N)$ also changes $P_{rs}$ into $P(N_{rs})$ (apart from sign). From (1) we have the

**LEMMA.** A graph $N$ of even order is prime if and only if its Pfaffian $P(N)$ vanishes.

3. **Prime graphs.** We define a singularity of a graph $N$ as a 0-simplex $a_i$ such that, for each $a_j \neq a_i$, $N_{ij}$ is prime.

**THEOREM I.** If $N$ is a prime graph of even order, and if $a_r, a_s$ are 0-simplexes of $N$ which can be joined in $N$ by a simple arc not having a singularity as an interior point, then $N_{rs}$ is prime.

First suppose $a_r, a_s$ to be joined by a 1-simplex $A_{rs}$. If there were a factor $F$ of $N_{rs}$, then $F \cup A_{rs}$ would be a factor of $N$, contrary to hypothesis.

Next, suppose there are distinct $a_i, a_j, a_k$, with $a_i$ not a singularity, such that $N_{ij}$ and $N_{ik}$ are prime. Then we can find $a_l$ such that $N_{il}$ is not prime. Using the lemma we have, by (3),

$$P(N_{ik})P(N_{il}) = 0,$$

where $P(N_{il}) \neq 0$. Hence $P(N_{ik}) = 0$ and so, by the lemma, $N_{ik}$ is prime. The theorem follows at once from these two results.

If, in a prime graph $N$, two 0-simplexes $a_r, a_s$ are joined by a 1-simplex whenever $N_{rs}$ is prime, we shall say that $N$ is hyperprime.

**THEOREM II.** If $N$ is a prime graph, we can construct a hyperprime graph $\bar{N}$ which contains $N$ as a subgraph.

If $N$ is hyperprime, there is nothing to prove. If not, there will be a pair of 0-simplexes $a_r, a_s$, not joined by a 1-simplex, such that $N_{rs}$ is prime. Add a new 1-simplex $A_{rs}$ joining them. The resulting graph is prime. For suppose it has a factor $F$. If $A_{rs} \in F$, then $F$ is a factor of $N$; if $A_{rs} \notin F$, then the intersection of $F$ with $N_{rs}$ is a factor of $N_{rs}$. In either case we have a contradiction.
If the resulting graph is not hyperprime we repeat the process, and so on. Since \( N \) is finite the process will eventually terminate in a hyperprime graph of which \( N \) is a subgraph.

**Theorem III.** Let \( \Sigma \) be the set of singularities of a hyperprime graph \( N \) of even order. Then \( h_u(\Sigma) > f(\Sigma) \).

By the definitions of a singularity and a hyperprime graph every pair of 0-simplexes of which one is in \( \Sigma \) is joined by a 1-simplex. Further, by Theorem I, every pair of 0-simplexes in the same component of \( N_x \) is joined by a 1-simplex.

If the theorem is false for some \( N \) we can, for each component \( Q_a \) of odd order of \( N_x \), select a 1-simplex joining a 0-simplex of \( Q_a \) to a 0-simplex \( D_8 \) of \( \Sigma \); and we can arrange that all the \( D_8 \) are distinct. For \( \Sigma \) and every component of \( N_x \) we can then select other 1-simplexes joining up the remaining even number of 0-simplexes in pairs. We thus obtain a factor of \( N \), contrary to its definition.

**Theorem IV.** A graph \( N \) is prime if and only if there is a subset \( S \) of its 0-simplexes such that \( h_u(S) > f(S) \).

The case in which the order of \( N \) is odd is trivial. (Take the null set as \( S \).)

Suppose \( N \) is of even order and that, for some \( S \), \( h_u(S) > f(S) \). Any factor \( F \) of \( N \) must evidently contain a 1-simplex joining a 0-simplex of a given component of \( N_s \) of odd order to a 0-simplex not in that component and therefore in \( S \). Hence there must be more 1-simplexes of \( F \) incident with members of \( S \) than there are members of \( S \), which is absurd since \( F \) is regular and of the first degree. Consequently \( N \) is prime.

Next, suppose \( N \) prime. Then we can construct a hyperprime graph \( \bar{N} \) of which \( N \) is a subgraph (Theorem II). Let \( \Sigma \) be the set of singularities of \( \bar{N} \). Then \( h_u(\Sigma) > f(\Sigma) \) is an inequality true for \( \bar{N} \) (Theorem III). Hence it is true also for \( N \), for each component of odd order of \( \bar{N}_x \) must contain at least one component of odd order of \( N_x \).

In virtue of the lemma it is easily seen that this theorem is equivalent to the following proposition.

**Let** \( M \) **be a skew-symmetric matrix in which, of the elements above the diagonal, some are zero and the others independent indeterminates. Then a necessary and sufficient condition for \( |M| \) to vanish is that \( M \) shall contain a diagonal submatrix \( M_0 \) which is a direct product of skew-symmetric matrices of which the number having odd order exceeds the difference of the orders of \( M \) and \( M_0 \).**
4. An existence theorem.

**Theorem V.** Let \( N \) be a connected graph of even order which is regular and of degree \( \sigma \). Suppose further that \( N \) has no isthmoid whose rank is less than \( \sigma - 1 \). Then at least one factor of \( N \) exists.

Let \( S \) be any isthmoid of \( N \), and let \( C \) be any component of \( N_S \). Let \( L(C) \) be the number of 1-simplexes having one end in \( C \) and the other in \( S \). If the order \( n(C) \) of \( C \) is odd, we have \( L(C) \geq \sigma \). For, since no isthmoid has rank less than \( \sigma - 1 \), the only other possibility is \( L(C) = \sigma - 1 \). In that case the number of 1-simplexes contained in \( C \) would be \( \frac{1}{2}[\sigma n(C) - \sigma + 1] \) which is not an integer. So, if \( k \) is the number of 1-simplexes having one end in \( S \) and the other in \( N_S \), we have

\[
\sigma h_u(S) \leq k \leq \sigma f(S).
\]

Thus for no \( S \) does \( h_u(S) \) exceed \( f(S) \), and so, by Theorem IV, \( N \) has a factor.

**Corollary.** Let \( A \) be any 1-simplex of \( N \). Then \( N \) has a factor which contains \( A \).

Let the vertices of \( A \) be \( a_r \) and \( a_s \).

Suppose that the corollary is false for some \( N \). Then \( N_{rs} \) is prime. So, by Theorem IV, there is an isthmoid \( S \) of \( N_{rs} \) such that \( h_u(S) > f(S) \) in \( N_{rs} \).

Let \( S' \) be the set formed by adding \( a_r \) and \( a_s \) to \( S \). Hereafter functions of \( S \) will refer to \( N_{rs} \), functions of \( S' \) to \( N \). Clearly

\[
f(S') = f(S) + 2 \quad (5)
\]

and

\[
h_u(S') = h_u(S) \quad (6)
\]

[for \((N_{rs})_S\) is the same as \( N_S \)]. Referring to the proof of the main theorem, we see that \( f(S') > h_u(S') \); for the second equality in (4) applies only if each 1-simplex incident with a member of \( S' \) is also incident with a 0-simplex of \( N_S \). This is not true of \( A \). But since \( N \) is of even order the numbers \( f(S') \) and \( h_u(S') \) must have the same parity. Hence

\[
f(S') \geq h_u(S') + 2,
\]

\[
f(S) \geq h_u(S) \quad \text{[by (5) and (6)].}
\]

This is contrary to the definition of \( S \).

Trinity College,
Cambridge.