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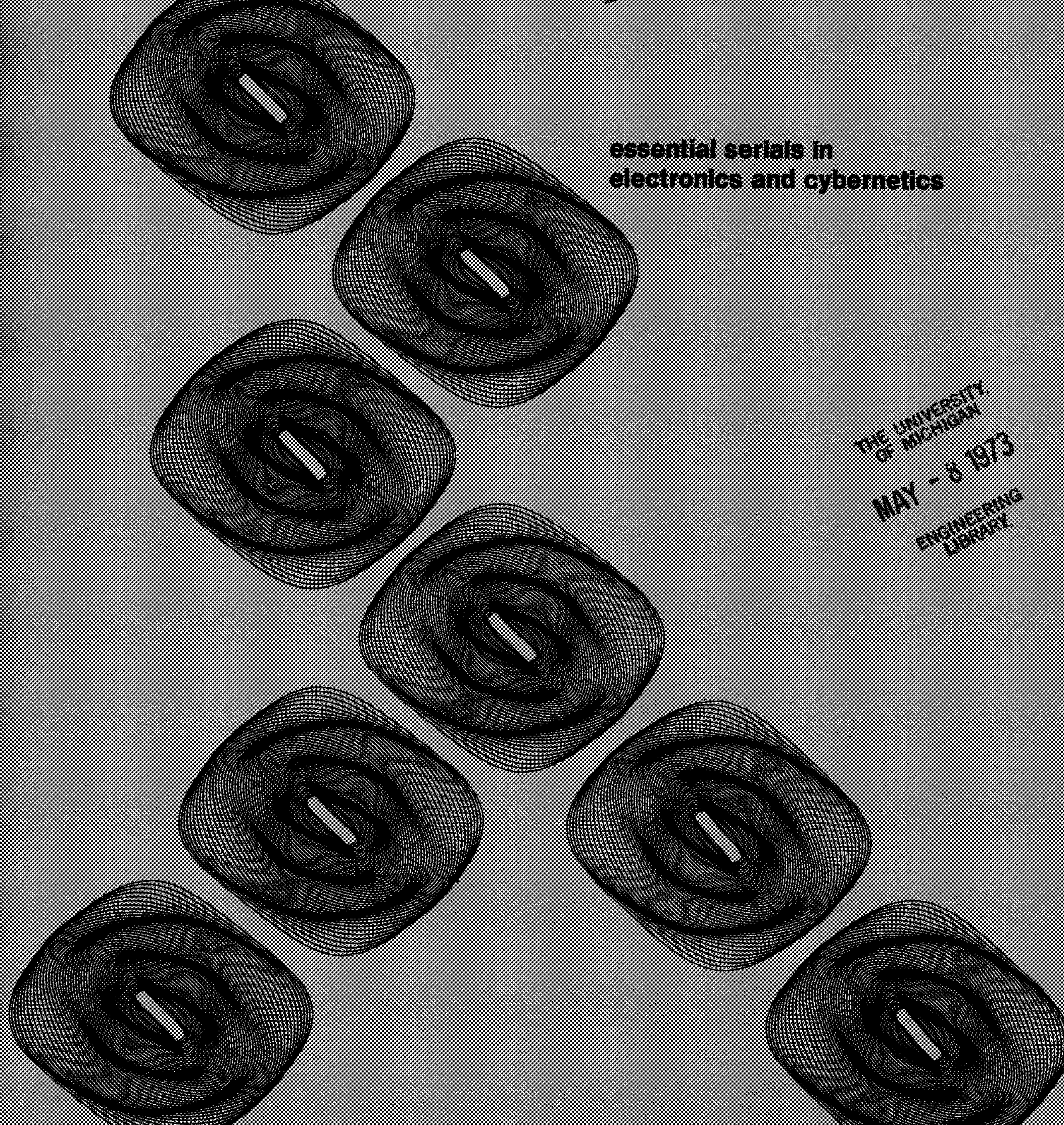

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REFERENCES

1. R. Faure, A Kaufmann and M. Denis-Papin, Modern Mathematics, Michel, Paris, 1963. (in Russian transl., Mir, 1966.)

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Finding the Maximum Cut in a Graph

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A method is proposed for finding the largest cut dividing a plane graph into two subgraphs. It is shown that such a cut in the original graph corresponds to the system of shortest chains pairwise connecting the vertices with odd powers in the dual graph. The total length of these chains must be minimum. We propose a method of curtailed sorting of the variants of the systems of chains that is analogous to the branch-and-bound method.

* * *

1. Let us consider an ordinary graph $G = (X, U)$, that is, an unoriented graph without loops such that every pair of vertices is connected by no more than one edge, where X is the set of vertices of the graph and U is the set of edges. Let X_1 and X_2 denote two disjoint subsets of the set of vertices of the graph such that $X_1 \cup X_2 = X$. The set of edges not included in the subgraphs generated by these sets of vertices is called a cut partitioning the graph into two subgraphs. Each of these subgraphs can consist of several connecting components. The number of edges of a cut is called its magnitude.

In a number of problems [1], we are required to find, out of all cuts of a given graph partitioning it into two subgraphs, the cut of greatest magnitude. If the graph has a large number of vertices, the number of possible variants of the cuts is extremely large, and direct sorting of them is inapplicable.

Let us solve this problem when the graph G is a plane graph. Obviously, the largest cut dividing the graph into two subgraphs is the union of the largest cuts of all the connecting components of the graph. Let us therefore look at one of the connecting components.

An edge whose removal increases the number of connecting components is called an isthmus.

Theorem 1. Every isthmus is included in the largest cut partitioning the graph into two subgraphs.

This is a direct consequence of the theorem proven in [1].

Theorem 2. A cut dividing the graph into two subgraphs has an even number (possibly zero) of edges in common with any cycle of that graph.

To see this, consider a cycle of the graph that contains edges of the cut. Beginning with any vertex of the set X_1, let us move along the cycle. Moving across the edges of the cut, we first enter the set of vertices X_1, then the set of vertices X_2. When the full cycle has been traversed, we return to the original vertex of the set X_1, and consequently pass through an even number of edges included in the cut. The converse is also true.

Theorem 3. If a nonempty set of edges has an even number of edges in common with any cycle of the graph, that set is a cut partitioning the graph into two subgraphs.

Let us look at a connected graph G without any isthmuses. Let us represent it in the plane. We assign to the graph G a graph Γ according to the following rule (see Fig. 1): Inside each face s of the graph G we put a vertex y of the graph Γ, and to each edge u of G we assign that edge v of Γ that connects the vertices y_1 and y_2 corresponding to the faces s and t on the two sides of the edge u; that is, edge v intersects edge u. Such a graph is called the dual of G. The graph Γ is a plane and connected graph. Since G does not have any isthmuses, Γ does not have any loops. Let us look at a cut in G that partitions it into two subgraphs, for example, the cut consisting of the edges (x_1, x_8), (x_2, x_5), and (x_3, x_4) in Fig. 1. Let us remove from Γ all edges corresponding to edges in G that are not included in that cut. We obtain a graph Γ'. (In Fig. 1, the edges $(y_1, y_2$, (y_2, y_5), and (y_5, y_1) belong to Γ'.)

Theorem 4. In the graph Γ', the powers of all the vertices are even.

To see this, let us look at an arbitrary face s in G. The cycle bounding this face has an even number of edges of the cut [the edges (x_1, x_8) and (x_2, x_5)]. Consequently, the corresponding vertex in Γ' will be of even power. A graph whose vertices have even power is called a quasicycle.

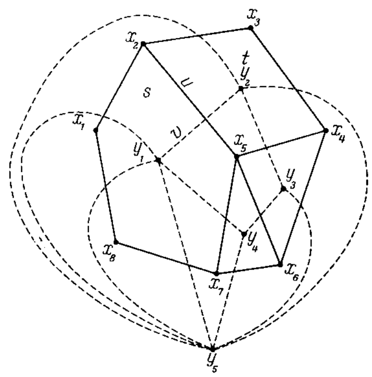

Fig. 1

<u>Theorem</u> 5. The set of edges of the graph G corresponding to the edges of an arbitrary nonempty quasicycle Γ' in Γ constitutes a cut partitioning G into two subgraphs.

We know that an arbitrary quasicycle can be represented in the form of a set of cycles that do not overlap along the edges [2]. Let us look at one of these cycles in Γ'. Let us single out in it the edges intersecting the edges of some cycle in G. There will be an even number of them, since they correspond to the points of intersection of two cycles in the plane, and there can be only an even number of such points. This means that, in the entire graph Γ', an even number of edges corresponds to the edges of any cycle in G. Consequently, in accordance with Theorem 3, the edges of G corresponding to the edges of Γ' constitute a cut in G.

Theorems 4 and 5 establish a one-to-one correspondence between the set of cuts in G partitioning it into two subgraphs, and the set of quasicycles Γ' in Γ. The number of edges of a cut, and the number of edges of the quasicycle corresponding to it, are the same. Thus, to find the greatest cut in G, we need to find in Γ the quasicycle with the greatest number of edges, or, what amounts to the same thing, to remove from Γ the smallest number of edges so that, in the remaining graph, the powers of all the vertices will be even. Suppose that there are 2s vertices of odd power (there will always be an even number of them) in Γ = (Y, V). We denote by O the set of vertices of odd powers.

<u>Theorem</u> 6. Edges in Γ = (Y, V) not included in the quasicycle Γ' = (Y, V') constitute a system of s chains, not intersecting along edges, between pairs of vertices of odd power in Γ.

<u>Proof:</u> Since all the vertices of Γ' = (Y, V') are of even power, the only vertices that are of odd power in the graph Γ'' = (Y, $V \setminus V'$) are those that are of odd power in Γ = (Y, V). But a graph with 2s odd vertices can be covered by a system of s chains, not intersecting along the edges, with ends at those vertices [3].

It is also obvious that, if from Γ we remove the edges constituting a system of s chains, not intersecting along the edges, between pairs of vertices of odd power, the remaining graph is a quasicycle. Let us denote such a system of chains by Q. Of course, the greatest quasicycle corresponds to the system Q with the smallest overall length of the chains. We shall call this system of chains minimal.

Suppose that the chain $C_0(x_1, x_2)$ is included in the minimal system of chains Q. Then it must be the shortest chain between these vertices, for otherwise, if the shortest chain $C(x_1, x_2)$ does not contain edges of other chains of the system, the chain $C_0(x_1, x_2)$ can be replaced with $C(x_1, x_2)$, and this will decrease the overall length of the system. If the shortest chain has common edges with any chain of the system $C(x_1, x_j)$, both these chains can be replaced with the chain $C(x_1, x_i)$ passing through the vertex a and the chain $C(x_2, x_j)$ passing through the vertex b, where a is the first common vertex of these chains as we move along the first chain from x_1, and b is the last (see Fig. 2). This will also decrease the overall length of the system. The sequence of analogous replacements can be carried out even in the case when the shortest chain $C(x_1, x_2)$ has common edges with several chains of the system Q in question. Let us partition the 2s vertices of odd power in Γ into pairs, and combine them with the shortest chains.

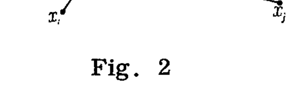

Fig. 2

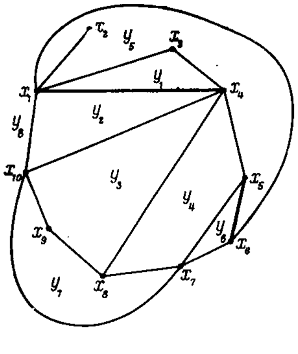

Fig. 3

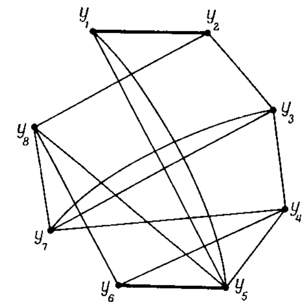

Fig. 4

Out of all possible partitions into pairs, let us find the one for which the overall length of these chains is smallest. When we have removed the edges of these chains, we obtain the quasicycle we are seeking. The dimension of the problem has been reduced, since only the shortest chains between vertices of odd powers are considered.

Let us list the steps in finding the greatest cut partitioning a plane graph into two subgraphs.

1. Remove the isthmuses from G (the edge (x_1, x_2) in Fig. 3).
2. Construct the graph Γ dual to it (see Fig. 4).
3. For each connecting component of Γ, find the vertices of odd powers (the vertices y_1, y_2, y_5, and y_6 in Fig. 4).
4. Find the minimal system of chains between pairs of these vertices (the edges (y_1, y_2) and (y_5, y_6) in Fig. 4). The edges included in these chains correspond to edges of the original graph G that are not included in the cut (the edges (x_1, x_4) and (x_5, x_6) in Fig. 3). The remaining edges constitute the greatest cut. The graph G (see Fig. 3) is divided into two subgraphs with sets of vertices $(x_2, x_3, x_5, x_6, x_8, x_{10})$ and (x_1, x_4, x_7, x_9).

2. Let us describe a method of sorting the various chains between vertices of odd power. We denote by (i, j) the shortest of the chains $C(y_i, y_j)$, where $y_i \in 0$ and $y_j \in 0$. Let $\alpha(i, j)$ denote the length of the chain (i, j), and define $\lambda(i) = \min\limits_{y_j \in 0,\ j \neq i} \alpha(i, j)$. Obviously $\alpha(i, j) \geqslant \lambda(i)$. Let us number the vertices of Y in such a way that vertices of odd power have smaller numbers than vertices of even power, and $\lambda(i) \geqslant \lambda(j)$ for $i < j$. Thus the vertices are numbered, beginning with zero, in decreasing order of $\lambda(i)$.

We shall call a sequence of r chains $Q_k^r = \{(1, q_2), (q_3, q_4), \ldots, (q_{2r-1}, q_{2r})\}$ an r-variant of the system. Here k denotes the number of the r-variant in the course of the solution. Corresponding to each r-variant Q_k^r is a sequence P_k^r of vertex numbers arranged in increasing order beginning with zero: $P_k^r = \{p^r_{k,1}, p^r_{k,2}, \ldots, p^r_{k,2s-2r}\}$, where $p^r_{k,i} \neq q_j$ (for i = 1, 2, …, 2r and j = 1, 2, …, …, 2s - 2r); that is, P_k^r includes the numbers representing those vertices that are not ends of the chains in Q_k^r. In correspondence with Q_k^r let us put the estimate

$$T_k^r = \sum_{i=1}^{r} \alpha(q_{2i-1}, q_{2i}) + \sum_{j=1}^{s-r} \lambda(p^r_{k,2j-1}).$$

We define the 0-variant Q_0^0 as the variant not containing a single chain. For it, $P_0^0 = \{1, 2, \ldots, 2s\}$ and $T_0^0 = \sum\limits_{j=1}^{s} \lambda(2j - 1)$. From the r-variants we construct successively the (r + 1)-variants. The variant Q_n^{r+1} is obtained from the variant Q_k^r already constructed by adjoining to the sequence of chains Q_k^r the chains $(p^r_{k,1}, m)$, where m is a given vertex number, $m \in P_k^r$, and $m \neq p^r_{k,1}$. Thus $Q_n^{r+1} = \{Q_k^r, (p^r_{k,1}, m)\}$.

Let us show that, with this construction, $T_n^{r+1} \geqslant T_k^r$. For $1 \leqslant i < m - 1$, we have $p^{r+1}_{n,i} = p^r_{k,i+1}$,

Number of vertices in the graph G	Number of vertices in the graph Γ	Number of variants constructed
10	11	8
20	18	54
45	41	76
50	49	81

and for $m - 1 < i \leqslant 2s - 2r - 1$, we have $p^{r+1}_{n,\, i-1} = p^{r}_{k,\, i+1}$. Consequently, for $2 \leqslant i \leqslant 2s - 2r - 1$ we have $p^{r+1}_{n,\, i-1} \leqslant p^{r}_{k,\, i+1}$, and

$$\lambda(p^{r+1}_{n,i-1}) \geqslant \lambda(p^{r}_{k,i+1}), \qquad (2.1)$$

$$T_n^{r+1} - T_k^{r} = \alpha(p^{r}_{k,1}, m) + \lambda(p^{r}_{k,1}) + \sum_{j=1}^{s-r-1} [\lambda(p^{r+1}_{n,2j-1}) + \lambda(p^{r}_{k,2j+1})].$$

It follows from (2.1) and $\lambda(p^{r}_{k,\, 1}) \leqslant \alpha(p^{r}_{k,\, 1}, m)$ that $T_n^{r+1} - T_k^{r} \geqslant 0$.

By definition, the s-variant Q_k^s consists of s chains between pairs of vertices from 0 and $T_k^s = \sum_{i=1}^{s} \alpha(q_{2i-1}, q_{2i})$; that is, T_k^s is the total length of a system of s chains. Thus the problem of finding the minimal system of chains Q is equivalent to finding the s-variant with minimal estimate T_k^s. To find such an s-variant, we use the idea of the method of branches and bounds [4].

Let us construct the variants sequentially. To each of the variants constructed at the first j steps we assign the set of numbers representing the vertices M_k^r from 0, where $m_i \in M_k^r(j)$, if the variant Q_k^r was filled out at each of these steps to form Q_n^{r+1} by adjoining the chain $(p^{r}_{k,\, 1}, m_i)$. We shall say that the variant Q_k^r $(r \neq s)$ at step j is closed if the sets of numbers representing the vertices p_k^r and $M_k^r(j)$ coincide. Otherwise we shall say that the variant Q_k^r is open.

At the first step, let us construct the 0-variant. At step j $(j > 1)$ we find, from among the open variants already constructed, those with values r and k for which the estimate T_k^r is minimal, and from among them select the variants with greatest r. Suppose that these are the l-variants $Q_a^l, Q_b^l, \ldots$. Let us expand them to $(l + 1)$-variants, adjoining to each Q_h^l, one by one, all the chains $(p^{l}_{h,\, 1}, p^{l}_{h,\, t})$, where $p^{l}_{h,\, 1} \in P_h^l$, $p^{l}_{h,\, t} \in P_h^l$, $p^{l}_{h,\, t} \overline{\not\in} M^{l}_{h,\, t}(j - 1)$, for which the length $\alpha(p^{l}_{h,\, 1}, p^{l}_{h,\, t})$ is minimal. To the Q_i^{l+1} that are constructed, we assign different subscripts i that do not coincide with any of the subscripts j of the Q_j^{l+1} constructed earlier. The process is terminated when we have obtained an s-variant Q_k^s with estimate T_k^s that is least among the estimates of the constructed variants Q_n^r (for $r = 0, 1, \ldots, \ldots, s$). The estimates of the variants in the construction process can only increase. Consequently, all the r-variants up to the instant of termination of the process can be extended to be s-variants with estimates less than T_k^s. Thus Q_k^s is the sought system of chains.

Obviously, for virtually all graphs, the number of variants that one must consider in solving the problem by the method proposed is an insignificant portion of the total number of variants of a system of s chains. The data shown in the table regarding some of the problems solved with this method testify to this.

Submitted December 28, 1970

REFERENCES

1. G. I. Orlova and Ya. G. Dorfman, The problem of optimal partitioning of a graph into several subgraphs, Izvestiya AN, Tekhnicheskaya Kibernetika (Engineering Cybernetics), 1972, No. 1.

2. A. A. Zykov, Teoriya Konechnykh Grafov (Theory of Finite Graphs), Nauka, 1969.
3. O. Ore, Theory of Graphs, IL, 1968 (in Russian transl.).
4. E. L. Lawler and D. E. Wood, Branch-and-bound methods: a survey, Operations Research, 1966, 14, No. 4.

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Construction of Optimum Schedules for Parallel Processors

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Using the branch-and-bound method we solve two problems of construction of optimum schedules for executing a semi-ordered set of operations in several parallel processors. One problem presupposes that all processors are identical, whereas in the other the processors differ in type in such a way that each processor can perform only the operations of corresponding type. The execution time of operations is determined in both problems by its type. An example is presented.

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1. INTRODUCTION

This paper is devoted to construction of shortest schedules, with constraints on the order of execution of operations in the processors. A processor can be the operational unit of a computer, an individual digital computer, a machine, etc., in other words, something that is capable of performing an operation and is characterized by the states "free" and "busy with the execution of an operation".

One of the problems is to find schedules for a set of processors that are identical, in the sense that each of them can be used for any operation of an algorithm, whereas in the other problem the processors differ in type, so that each processor can perform operations of only the corresponding type. In each problem, the execution time of an operation is determined by its type.

Such problems are encountered, for example, in compiling programs for a digital computer with parallel processors (general-purpose or special-purpose), or for a set of digital computers that operate in a system, and also in the planning of production, the assembly of parts, and network planning. The proposed algorithm for solution of the above problems can be used for obtaining long-time schedules (such as standard digital-computer programs, algorithms of special-purpose digital computers, etc).

2. STATEMENT OF THE PROBLEM

Let us consider a circuit-free oriented graph $G(X, U)$, in which the set of vertices $X = \{x_1, \ldots, \ldots, x_s\}$ represents the set of operations. Each arc $(x_i, x_j) = u_{i,j} \in U$ signifies that the operation x_i must be executed prior to the operation x_j. This will be denoted by $x_i < x_j$. We shall assume that X is partitioned into subsets $X_1, \ldots, X_w$. If $x_j \in X_i$, we shall say that x_j is an operation of type i.

Let us consider a set $F = \{f_1, \ldots, f_v\}$ and its partition into subsets $F_1, \ldots, F_w$. An element $f \in F$ is called a processor of type j if $f \in F_j$. Suppose we are also given a function τ: $X \to N$ (N being the set of natural numbers) that defines the execution time of an operation, and such that $\tau(x_j) = \tau_i$ if $x_j \in X_i$. A schedule R is a pair of functions $f(x)$ and $t(x)$ such that, for $x, y \in X$ (x and y being any operation belonging to X_i, $i = 1, \ldots, w$),

$$(x < y) \Rightarrow [(t(x) + \tau(x)) \leqslant t(y)], \tag{2.1}$$

$$[f(x) = f(y)] \Rightarrow [t(x) \neq t(y)],$$

$$[(t(x) < t(y)), \ (f(x) = f(y))] \Rightarrow [t(x) + \tau(x) \leqslant t(y)]. \tag{2.2}$$

In fact, $t(x)$ is the instant at which operation x begins, and $f(x)$ is the number of the processor in which it is executed. Condition (2.1) signifies that operation y can be executed only after operation x if $x < y$ in G, whereas condition (2.2) signifies that at each instant it is possible to execute only one operation in each processor. The length $L(R)$ of a schedule R is defined as $\max_{x \in X} [t(x) + \tau(x)] - \min_{x \in X} t(x)$. Let us formulate the following problems.