AN APPROXIMATIVE ALGORITHM FOR THE FIXED-CHARGES TRANSPORTATION PROBLEM*

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ABSTRACT

A computationally simple method for approximating the optimal solution to transportation problems is described. Also, a method to deal with fixed charges is proposed. The computational algorithms have been designed primarily to handle very large numbers of transportation problems each involving a small number of origins and destinations as is frequently the situation in the Navy supply system. The results of empirical tests of the effectiveness of the methods are summarized.

INTRODUCTION

In July of 1959 our research group was authorized to proceed on a research project for the Bureau of Supplies and Accounts of the United States Navy. The project was given the title "Modified Linear Programming." Specifications for the project, as prepared by the Bureau, read in part as follows:

"This project task is directed toward development of more efficient rules for the distribution and redistribution of material in the Navy supply system, through modification of the techniques of solution of the transportation problem to represent adequately and minimize the total costs of alternative allocation and redistribution decision patterns for Navy material. In particular, the project seeks to discover feasible and workable approximating rules for distribution decisions where a fixed cost of shipment is postulated for each 'channel' in addition to the customary variable cost, linear with respect to quantity with fixed costs for a shipment approximately the same for all activities, with linear or piecewise linear variable costs based on distance and transportation rate data... ."

APPROXIMATING AN OPTIMAL SOLUTION

Our study indicated that approximative techniques are indeed appropriate for the handling of the problem. The number of transportation routing calculations in the Navy supply system is tremendous. One division of the Navy supply system alone handles over 150,000 items, of which some 20-30,000 require review of their stock position about every three weeks.

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In the average review about 7000 line items require a transportation routing decision. This profusion of transportation calculations suggests that a full-scale simplex method or network linear-programming calculation is likely to be impractical. Even with a relatively efficient and speedy program, the amount of time required may add up rapidly. A recent study at one Navy supply installation suggested that as much as 40 hours of computer time per review period might be required by an ordinary transportation calculation which took into account no complications such as fixed charges.

Very little work has been done on approximate methods prior to this study. (The work of Houthakker [1] is almost unique.) This is not surprising in view of the satisfactory state of the available exact methods and the fact that very few nonmilitary organizations have a quantity of problems so large as to necessitate approximations. The natural place to start a search for approximative techniques was to examine feasible solutions for the simplex method. These are extremely simple to compute and, as will be seen in the discussion of "The Approximation Methods Tested," provide a degree of approximation that is quite satisfactory for the problem at hand.

But first we record one relevant theoretical result which has, however, not been incorporated in the calculations. The reason that it has not been used is that, although it is fairly effective in reducing the transportation costs, it has the unfortunate effect of sometimes increasing the number of shipments. In common parlance, the result states: "If costs are proportionate to distance shipped, never use routes which intersect!" This statement is based on an explicit solution of the following 2-by-2 transportation problem:

$$\begin{array}{c|cc|} & R_c & R_d \\ \cline{2-3} E_a & c_{ac} & c_{ad} \\ E_b & c_{bc} & c_{bd} \\ \cline{2-3} \end{array}$$

where E_a and E_b are the excesses available for shipment out of surplus installations A and B, R_c and R_d are the quantities required at two deficit installations C and D, and the c_{ij} are unit transportation costs.

Now suppose we happen to have

$$c_{ac} + c_{bd} < c_{bc} + c_{ad} \tag{1}$$

and that $E_a \leq R_c$. Then it can be proved that the following distribution is the <u>unique</u> solution.

$$\begin{bmatrix} x_{ac} & x_{ad} \\ x_{bc} & x_{bd} \end{bmatrix} = \begin{bmatrix} E_a & 0 \\ R_c - E_a & R_d \end{bmatrix}. \tag{2}$$

PROOF: This distribution is clearly feasible. Moreover, note that for any feasible distribution (x_{ij}) and for any values of the (dual) variables u_a, u_b, v_c, and v_d satisfying the constraints of the dual problem

$$u_i + v_j \leqq c_{ij} \tag{3}$$

the inequality

$$\Sigma\, u_i E_i + \Sigma\, v_j R_j \leqq \Sigma\, c_{ij} x_{ij} \tag{4}$$

follows immediately by the multiplication of each inequality (3) by x_{ij} and summing. If equality holds in (3) for each route on which material is shipped (i.e., for which $x_{ij} > 0$) then equality holds in (4). Moreover, if values of the x_{ij} can be found for which equality holds in (4) these values must constitute an optimal solution, for by (4) $\Sigma\, c_{ij} x_{ij}$ must then be minimal.

Now, consider the particular values

$$u_a = -c_{bc} \qquad v_c = c_{ac} + c_{bc}$$

$$u_b = -c_{ac} \qquad v_d = c_{ac} + c_{bd}\,.$$

Clearly, the constraints (3) are satisfied and, indeed, equality holds for each positive shipment of the distribution (x_{ij}) proposed as a solution. Therefore equality holds in (4); since the right-hand side is total transportation cost, the proposed distribution is optimal. Note further that $u_a + v_d < c_{ad}$; this implies $x_{ad} = 0$ in all distributions which achieve the minimum, and the distribution is thereby determined uniquely.

To derive our prohibition against "cross-hauling," we note that the sum of the lengths of the diagonals of any quadrilateral is always greater than the sum of the lengths of the opposite sides (because the sum of the lengths of any two sides of a triangle, e.g., $\overline{AV} + \overline{VC}$ is always greater than the length of the remaining side, $\overline{AC}$). Now let Figure 1 represent the location of our four installations, A, B, C, and D, and the routes between any pair of them. Then, by the theorem on quadrilaterals which was just given, we must have $\overline{AC} + \overline{BD} < \overline{AD} + \overline{BC}$.

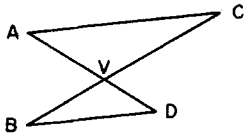

Figure 1

If transportation costs vary monotonically with distances, condition (1) must therefore be satisfied. Now let the amounts shipped in Figure 1 be represented by

$$\begin{bmatrix} x_{ac} & x_{ad} \\ x_{bc} & x_{cd} \end{bmatrix}$$

Suppose that a cross-haul has been made. This means that $x_{ad} > 0$ and $x_{bc} > 0$. But we may assume that $E_a = x_{ac} + x_{ad} \leqq x_{ac} + x_{bc} = R_c$ without loss of generality. Since the costs satisfy assumption (1), the unique minimum cost solution (2) involves $x_{ad} = 0$, which is a contradiction.

THE FIXED CHARGES PROBLEM

In addition to the problem of finding a good approximation technique, a second and more difficult problem is that which arises out of the presence of fixed charges—charges which are incurred whenever a redistribution action is taken but which do not vary with the amount of material involved in a particular shipment. A prime example of this sort of cost arises from the preparation of some of the papers which are required in the course of such an action. Thus, if a shipment is eliminated altogether, the cost of invoice preparation is avoided. But once a shipment is undertaken, the cost of making out the invoice is not substantially affected by a decision to send 200 cases rather than 10 cases of the item.

Since information on the magnitude of these fixed charges is still rather limited, it is difficult to assess their importance to the Navy supply system. It is clear, however, that they can arise in at least two different ways.

1. Fixed costs arise out of the clerical and administrative work associated with any shipment. It has been pointed out, however, that in the short run the elimination of this sort of fixed cost may result in relatively little cash saving to the Navy. If 1 hour of clerical time is saved per day per activity, it is very unlikely that any personnel reduction will occur; but in the long run a sufficient accumulation of such savings can lead to a decrease in clerical outlays by reducing the number of clerks who must be hired as replacements for or additions to existing staff.

2. A second type of fixed cost which has been identified derives from the fact that an increase in the number of shipments can result in a slowing down of commodity movements. If paperwork is a bottleneck, an additional redistribution action can reduce the speed with which others can be processed. This reduction in speed may then be dependent on the number of such actions rather than on the magnitude of the shipments involved. Therefore, consumer waiting time may very well be a fixed charge.

The difficulty of the fixed-charge computation is well known. It is considerably simplified, however, if there is a reasonable presumption that all fixed charges are approximately the same. In this case, there is a theorem which states that unless the problem is degenerate the ordinary linear programming solution will in fact be optimal, i.e., the solution will be the same whether or not fixed charges are present.[1]

Clearly then, if fixed charges are approximately equal for all shipping routes, the only hope for cost reduction lies in degeneracy. Only degeneracy will permit fixed charge savings by making it possible for a reduction in the number of routes employed where, in a transportation problem, degeneracy is defined to mean that some subset of the activity requirements adds

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[1] Outline of a proof: In the absence of degeneracy, the solution to our fixed charges programming problem must have at least as many nonzero elements as there are independent constraints. Any distribution that is feasible for the fixed charges problem is also feasible for the underlying transportation problem. This problem involves mn variables (the number of routes) constrained by $m + n$ equations, of which $m + n - 1$ are independent. Hence, non-degeneracy means no feasible distribution for either problem can have fewer than $m + n - 1$ nonzero variables.

If the fixed-charge constants K_{ij} in our cost function are all equal to the same number K, the number of these fixed charges must be equal to the number of nonzero shipments (the number of positive x's), i.e., the fixed charges must add up to at least $(m + n - 1)K$.

Now, the objective function of our problem is $\Sigma(K + c_{ij}x_{ij})$, while the objective function of the ordinary linear-programming problem (in the absence of fixed charges) is $\Sigma\, c_{ij}x_{ij}$. It is well known that the linear-programming problem has an optimal solution, x^*, which contains exactly $m + n - 1$ nonzero elements. Hence, for these values of the x's we will have $\Sigma\, K = (m + n - 1)K$; i.e., both $\Sigma\, K$ and $\Sigma\, c_{ij}x_{ij}$ will be at their minima. Thus the linear-programming problem solution x^* must also be the solution to the minimum fixed-charges transportation problem.

For a complete discussion of this result, cf. Warren N. Hirsch and George Dantzig, The Fixed Charge Problem, RAND Corporation Paper P-648, December 1954.

up to the sum of some subset of activity excesses.[2] There are two reasons, however, why looking for problems which happen to be degenerate is not a satisfactory expedient:

1. There is no guarantee that a significant proportion of the problems which are encountered in practice will turn out to be degenerate.[3]

2. Even when a problem is degenerate, identification and consideration of all of its degeneracies is likely to be a long computational process. For, essentially, this requires the computation of all partial sums of excesses and of all partial sums of requirements and their comparison in order to see which, if any, of these partial sums happen to be equal. The number of combinations involved mounts very rapidly with the scale of the problem. (See "Comments on the Fixed-Charge Problem.")

Therefore, it was decided to undertake a systematic extension of a current Navy clerical procedure. Often clerks will simply decide that very small requirements or excesses are not worth the trouble of a special shipment. In effect, this decision amounts to the elimination of a fixed charge by forcing a degeneracy onto the problem. A small clerical change in requirements or excess figures has been used to eliminate one or more shipments.

In generalizing this procedure, it will usually be possible to <u>impose</u> degeneracy on such problems by making insignificant changes in surplus and deficit figures which make some of their partial sums equal. This is clearly desirable, so long as the resulting savings in fixed charges are greater than any costs which are produced by changing the surplus or shortage figures. That is the approach which was taken by the algorithm which was developed in this study and which is described in detail later under the unhappy title, "forced degeneracy."

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[2] Actually this is a special case of the general linear-programming definition of degeneracy. The dependence of the column vector of requirements and excesses upon fewer than m + n - 1 columns of constraint coefficients easily implies the stated problem. To illustrate this, consider the two-shipper-two-destination transportation problem which has the following constraints:

$$\begin{array}{ccccccccl} x_{11} & + & x_{12} & & & & & = & E_1 \\ & & & & x_{21} & + & x_{22} & = & E_2 \\ x_{11} & & & + & x_{21} & & & = & R_1 \\ & & x_{12} & & & + & x_{22} & = & R_2 , \end{array}$$

where E_1 and E_2 are the excesses of surplus installations 1 and 2 and R_1 and R_2 are the requirements of the two deficit installations. The matrix of coefficients is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & E_1 \\ 0 & 0 & 1 & 1 & E_2 \\ 1 & 0 & 1 & 0 & R_1 \\ 0 & 1 & 0 & 1 & R_2 \end{bmatrix} .$$

Note that only three of the first four columns are linearly independent. Degeneracy means that the last column is linearly dependent on a subset of fewer than $m + n - 1 = 3$ of the others. Suppose, e.g., we have $\underline{a}$ × col 1 + $\underline{b}$ × col 2 = col 5, where $\underline{a}$ and $\underline{b}$ are any constants. Then, by substitution, $\underline{a} + \underline{b} = E_1$, $\underline{a} = R_1$, and $\underline{b} = R_2$ so that we have $E_1 = R_1 + R_2$ – a partial sum of requirements equals a partial sum of surpluses, as was asserted. This argument is easily extended into a formal proof.

[3] In a sample of the 34 problems in our test sample which on prior inspection appeared most likely to be degenerate, only 5 turned out to exhibit useful degeneracy.

THE APPROXIMATION METHODS TESTED

It was necessary for our assignment to test out a variety of approximative computing methods for the ordinary transportation problem and to compare them with the results of a precise optimality calculation.

For this purpose, a sample of 100 actual Navy transportation problems was collected and used in making a number of tests. The nature of the sample and the results of the calculations are described in detail later in this report.

In addition to the optimal solutions and the modified minimum distance (so-called proximity table) solutions actually arrived at by the installation with which we worked, both as determined by the electronic computer and as adjusted subsequently by clerks, other types of solutions were investigated.

Ship Most at Least Cost (SMALC)

The first approximation method tested was labelled ship most at least cost and was given the unfortunate mnemonic name SMALC. The basic idea is to find which route involves costs lower than any other and to ship as much as possible along this route. Then we ship as much as possible along the second-lowest cost route and so on, until all excesses have been eliminated and all requirements have been filled.

Specifically, let x_{ij} represent the amount shipped from activity[4] i to activity j, let c_{ij} be the unit cost of that shipment, let E_i represent the excess at installation i, and let R_j represent the requirement at installation j. Then the method proceeds as follows:

In the cost matrix choose the minimum cost figure c_{ij}. Set x_{ij} = smaller of E_i and R_j and remove the corresponding row i or column j from consideration.

Replace

$$E_i \text{ by } E_i - x_{ij}$$

$$R_j \text{ by } R_j - x_{ij}$$

(hence at least one goes to zero) and repeat on the new smaller matrix; continue until the problem is solved.

The basic idea is illustrated by the following tables. In the left-hand table the entries on the outside represent installations, excess quantities, and requirement quantities. The spaces in the table represent shipping routes, and the numbers which are entered in these spaces represent the unit cost (distance) of a shipment along that route. For example, the 832 entered in the upper left-hand corner represents the cost of shipping one unit from installation A to installation D.

		D	E	F	G	
		18	301	52	4	Installation Requirement
A	267	832	771	940	2488	
B	104	531	30	663	3247	
C	4	3336	3380	3444	73	

Installation Excess

Cost Table

	D	E	F	G
A	18	197	52	
B		104		
C				4

Solution Table

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[4] In Navy supply system parlance, an installation is called "an activity."

The minimum c_{ij} is clearly the 30 shipping cost from installation B (whose excess is 104 units) to installation E (whose requirement is 301 units). Hence, the maximum amount which can be shipped along this route is 104 units, and this is the amount entered in the corresponding position in the right-hand solution table.

Now installation B's excess figure is reduced to zero while E's requirement is reduced to 197. With this change in the original table, proceed to assign as large a shipment as possible (4 units) along the second lowest cost route (the 73 figure in the lower right-hand corner). This procedure is followed until all requirements are met and all excesses are eliminated in the manner shown in the solution table.

Vogel's Approximation Method (VAM)

A second method of approximation tested was a modification of Vogel's Approximation Method (VAM). This method is a bit more difficult to explain and involves more computer time. For each possible excess installation, i, one finds the lowest and the second-lowest cost shipping routes which begin at installation i. Similarly one determines the lowest and second-lowest cost routes which terminate at each requirement installation. The difference between these lowest and second-lowest cost figures may be referred to as the error penalty which would result if the second-lowest cost route were inadvertently chosen. The basic idea of VAM is to seek to avoid these error penalties. Thus, now having an error penalty figure for each installation, we pick that installation which has the largest error penalty figure. On the argument that here is where the most expensive mistake can be made, we make as much of that installation's shipment as possible along its least cost route to avoid such a costly error. We now reexamine (and, if necessary, recompute) the error penalty figures, and next take care of the installation with the largest remaining error penalty, and so on.

More explicitly, the following procedure was employed:

Locate the minimum element in the cost matrix in each row and column. Call these

$$\begin{array}{lll} & \text{Rows} & \text{Columns} \\ (1) & c_1, \ldots, c_m & d_1, \ldots, d_n . \end{array}$$

Locate the next largest element in each row and column. Call these

$$(2) \quad c'_1, \ldots, c'_m \qquad d'_1, \ldots, d'_n .$$

Then the "unit penalties" incurred by not shipping on the routes located at the entries (1) are

$$c'_1 - c_1, \ldots, c'_m - c_m, \qquad d'_1 - d_1, \ldots, d'_n - d_n .$$

Instead of these unit penalties, the computation employed "absolute penalties." For example, if $c_1 = c_{1j}$ and is in row 1 and column j, we can ship only $x_{1j} = \min(E_1, D_j)$ on this route. Hence the "absolute penalty" for that route is $x_{1j}(c'_1 - c_1)$.

The largest possible shipments are then made along routes where this absolute penalty is greatest.

EXAMPLE: By use of the same problem as before, the cost table may be rewritten as

$$\begin{array}{llr|cccc} & & & D & E & F & G \\ \hline & & & 18 & 301 & 52 & 4 \\ & A & 267 & 832 & \underline{771} & 940 & 2488 \\ \text{Cost matrix} & B & 104 & \overline{531} & \overline{\underline{30}} & \overline{633} & 3247 \\ & C & 4 & 3336 & 3380 & 3444 & \overline{\underline{73}} \end{array}$$

where:

row minima are underlined thus $\underline{\quad}$

and

column minima overlined thus $\overline{\quad}$.

The row unit penalties = (61; 501; 3263)

The column unit penalties = (301; 741; 307; 2415)

The row absolute penalties = (16,287; 52,104; 13,052)

The column absolute penalties = (5,418; 223,041; 15,964; 9,660).

The 223,041 figure is clearly the greatest absolute penalty. Hence, the first shipment must be assigned in accord with the second column minimum entry, i.e., the first shipment must go from installation B to installation E, which, by coincidence, is the same as in the SMALC method.

The Degeneracy Forcing Algorithm

The last method tested is the one which was especially designed for the fixed-charges problem. It has these central features:

1. A device for adjusting excess and requirements figures which is designed to produce a reasonable amount of degeneracy by equating excess and requirements figures which differ by only a small amount or by eliminating very small excess or requirements figures.

2. An optimality computation device which is an extension of the SMALC method described earlier.

Specifically, the method is the following:

SMALC—DEGENERACY FORCING ALGORITHM: Scan the matrix for $\min_{i,j} c_{ij}$. Suppose this is achieved at $i = k$ and $j = m$. If $|E_k - R_m| \leqq \Delta$, set $x_{km} = \max$[5] $[E_k, R_m]$ and delete both row k and column m. If $|E_k - R_m| > \Delta$, as in the usual SMALC algorithm, set $x_{km} = \min[E_k, R_m]$ and then set

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[5] Note that this always results in the "rounding up" of excess or requirement figures as needed. This decision was reached on the basis of Navy opinions that the cutting down of any such figures would only postpone them until the next period and would not eliminate the requirement or excess quantity in question. However, the algorithm can easily be changed to work the other way by substituting "min" for "max" at this point. Indeed, on the basis of later tests conducted by the Navy, a modification of this sort was recommended. See C. M. Allender and James Encimer, Redistribution Decisions: The Transportation Problem, The Advanced Logistic Research and Development Branch, Ships Parts Control Center, Mechanicsburg, Pa., Sept. 1960.

$$\left.\begin{array}{l} E'_k = E_k - x_{km} \\ \\ R'_m = 0 \end{array}\right\} \quad \text{if } \min\left[E_k, R_m\right] = R_m$$

and

$$\left.\begin{array}{l} E'_k = 0 \\ \\ R'_m = R_m - x_{km} \end{array}\right\} \quad \text{if } \min\left[E_k, R_m\right] = E_k \ .$$

EXAMPLE: Using Δ = 1 consider the following transportation cost matrix:

		4	4	6	2	4	2	Requirements
Excesses	5	9	12	9	6	9	10	
	6	7	3	7	7	5	5	
	2	6	5	9	11	3	11	
	9	6	8	11	2	2	10	

The preceding algorithm may readily be checked to yield the solution

		4	4	7	2	4	2	Requirements
Excesses	5			5				7 shipments
	6		4				2	Amount shipped = 23
	2			2				Total cost = 121
	10	4			2	4		Average cost = 5.3 .

This may be compared with the following optimal solution obtained by the simplex method:

		4	4	6	2	4	2	Requirements
Excesses	5			5				9 shipments
	6		3	1			2	Amount shipped = 22
	2	1	1					Total cost = 112
	9	3			2	4		Average cost = 5.1 .

In this algorithm, Δ, the degeneracy limiting constant, is the maximum amount by which excess or requirement figures are permitted to be revised in order to produce degeneracy. A recommendation was made that a different Δ be employed for each item and that the value of each Δ be revised at each review. The computation involved can be made very simple, and the factors affecting the appropriate value of Δ can vary sharply over time and from item to item, so that a more inflexible Δ figure appears to be undesirable.

The tested value of Δ was based on the values of the following two variables:

1. The average level of inventory on hand at the review date in those installations which carry the item. For, the higher the level of inventory on hand, the less significant, relatively,

will be a given readjustment in requirement or excess figures. Hence Δ should vary directly with the average stock level.

2. The price of the item (as a rough index of military essentiality). Clearly, the more essential the item the less the adjustment in excess and requirement figures which can be permitted. Hence Δ should vary inversely with the price of the item.

In our trial calculation Δ was arrived at as follows. Define D by the expression

$$D = \frac{\text{Average weekly system demand for the item}}{\text{Number of installations showing demand}} \times \text{a price adjustment factor}.$$

Then Δ is equal to D rounded up to the nearest integer.

The price adjustment factor was developed on the argument that less essential items can be assigned larger Δ's, i.e., that it is appropriate to go further in forcing degeneracy on such items. The relatively conservative but rather arbitrary rule which was developed for our trial run is summarized as follows:

Price Adjustment Factor	Unit Price of Item[6]
1.5	$100.01 and over
2.0	$ 50.01 to $100.00
2.5	Up to $50.00

In effect, this means that for expensive items Δ is kept down to 1-1/2 weeks of average installation demand; similarly, for medium priced items Δ is set at 2 weeks demand.

RESULTS OF THE COMPUTATIONS

Table 1 summarizes the results of the computations, and shows average figures for the 100-problem sample. For example, the first figure in the first column indicates that the average problem incurred 1,127,000 item-miles of transportation when redistributed in accord with the current Navy decision process. This was an average of some 12,600 item-miles more than would have been involved in an optimal solution. For the average problem[7] this represents a 2.60 percent increase in transport costs over the optimal solution.

The last three entries in this column are meant to indicate the representativeness of these results and the largest deviations from them which have been encountered. As an index of the variability of the percentage increase in the cost figures we see that the standard deviation of that percentage figure is 5.50. Moreover, the largest absolute excess in cost for any of the sample problems of the Navy calculation over the optimal solution is 215,000 item-miles. The largest percentage differences for any problem is 29 percent. These last two figures are meant to be indicative of the maximum risk incurred in using the approximation methods to solve a particular problem.

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[6] The break points were developed partly on the basis of the information that more than 50 percent of the type of shipments investigated involve items worth less than $50 while some 75 percent of the shipments involve items worth less than $100.

[7] Notice this figure is not the percentage over-all saving. Rather, it is obtained by getting the percentage saving for each of the 100 problems and averaging them. This is clearly the arithmetic mean which must be used in computing the standard deviation.

TABLE 1

Transportation "Costs": Results for an Average Trial Problem*

Type of Computation	Actual Shipments Ordered	Computed by Simplex (optimum) Method	Computed by SMALC Method	Computed by VAM	Shipments Recommended by the Computer	Computed by Forced Degeneracy Method
Cost (item-miles)†	1,127,000	1,115,000	1,119,000	1,132,000	1,123,000	1,119,000
Excess over optimum	12,600	0	4,100	17,100	8,700	4,700
Percentage excess over optimum (per problem)....	2.60	0	0.45	1.60	1.95	0.20
Overall percentage excess over optimum	1.13	0	0.37	1.5	0.8	0.4
Standard deviation of percentage excess over optimum	5.50	0	1.30	3.30	4.50	4.60
Maximum absolute excess over optimum	215,000	0	111,000	628,000	164,000	88,000
Maximum percentage excess over optimum	29.20	0	7.20	21.30	28.90	19.70

*Some inconsistencies have crept in as a result of rounding errors.

†Number of miles moved times the number of items in each shipment (added over all shipments).

It is to be noted that (ignoring for the moment the forced degeneracy method) the SMALC method comes out best after the simplex method on any one of the relevant criteria. The SMALC method has the smallest excess in transportation cost over the simplex result, taken either absolutely or percentagewise. Moreover, the excesses have a smaller maximum variation and standard deviation than any other method. Roughly, we may conclude that the SMALC method will involve more than a 2 percent saving in transportation costs on the average redistribution as against present methods. (This conclusion, of course, applies only to larger problems.)

It is to be noted that the forced degeneracy calculation, despite the fact that it involves about a 9 percent reduction in number of shipments for the average problem (Table 2), compares very favorably with SMALC in the variable transportation costs it involves. This result is

TABLE 2

Comparison of Forced Degeneracy Calculation with SMÁLC

Number of Shipments (Line Items)		Absolute Difference	Percent Difference
In the Simplex Solution	In the Forced Degeneracy Solutions		
707	642	-65	-9.2

explained by the fact that the forced degeneracy method automatically eliminates some trivially small shipments and hence can result in an over-all decrease in the total amount shipped in some problems. However, in general this method does involve considerable variability in the extent to which it approximates the transportation costs of the optimal solution. (In some problems its transportation costs will even be substantially below the simplex cost figure.) Hence, it seems advisable to maintain a conservative interpretation of the low average cost incurred by the forced degeneracy method. That is, it does not seem appropriate to consider this a reliable method for reducing transportation costs unless fixed costs also are substantial. It is remarkable that this method is able to achieve such a substantial reduction in number of shipments with Δ figures as moderate as those which were employed in the trial calculation.[8]

COMMENTS ON THE FIXED-CHARGE PROBLEM

Any theoretical attack on the fixed-charge problem must start from the two theoretical results of Hirsch and Dantzig [2]. An outline of a proof has been given earlier in footnote 1.

THEOREM 1: If the underlying transportation problem is nondegenerate, and the fixed charges are positive and equal on all routes, then any basic optimal distribution (i.e., with $m + n - 1$ routes in use) for the underlying transportation problem will solve the fixed-charge problem.

THEOREM 2: In the general case, where the fixed charges may vary from route to route, an optimal distribution may always be achieved as a basic feasible distribution (although it will not necessarily be optimal for the underlying transportation problem).

These results suggested an attempt to use an approach analogous to the simplex method, to examine neighboring basic feasible distributions in order to see if any of these involves a lower cost than the current basis, and to take the current basis to be optimal if none of its immediate neighbors is cheaper. The following example exhibits the difficulties inherent in such an approach:

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[8] It is noteworthy that these conclusions were confirmed and, in some cases, strengthened in independent tests subsequently conducted by the Navy. Cf. Allender and Encimer, op. cit.

$$\begin{array}{c|ccc|} & R_1 & R_2 & R_3 \\ \hline E_1 & c_{11} & c_{12} & c_{13} \\ E_2 & c_{21} & c_{22} & c_{23} \\ \hline \end{array} = \begin{array}{c|ccc|} & 8 & 5 & 3 \\ \hline 9 & 6 & 5 & 3 \\ 7 & 3 & 2 & 1 \\ \hline \end{array}$$

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix} .$$

To exhibit the basic feasible solutions graphically, let $x = x_{11}$ and $y = x_{12}$. Then the feasible region is the shaded area in Figure 2. The extreme feasible distributions are tabulated below:

$$A = \begin{bmatrix} 6 & 0 & 3 \\ 2 & 5 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 8 & 0 & 1 \\ 0 & 5 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 8 & 1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix} , \quad E = \begin{bmatrix} 1 & 5 & 3 \\ 7 & 0 & 0 \end{bmatrix} .$$

The total costs are as follows:

A: 67
B: 66
C: 65
D: 67
E: 66 .

Thus E is a local minimum (the neighboring basic feasible distributions are A and D) but is not a global minimum. That is, despite the fact that no neighboring basic solution is cheaper than E, it is C and not E which represents the over-all least cost solution.

It is inevitable that there can be no general theorem assuring an optimal solution covering the degenerate case, even when the fixed charges are constant, if the basic computational routine is approximate. However, somewhat trivially, we can be sure that we are moving in the right direction from a current approximation. This result will be dignified with the name of theorem although it is little more than common sense.

THEOREM 3: Given a fixed-charge problem with constant fixed charges on all routes, let x_{ij} be a distribution with average transport cost A involving N shipments, and let x^*_{ij} be a distribution with average transport cost A with $N' < N$ shipments. Then x^*_{ij} involves less total cost than x_{ij}.

PROOF: The total costs involved are

$$A \sum_i E_i + NK > A \sum_i E_i + N'K .$$

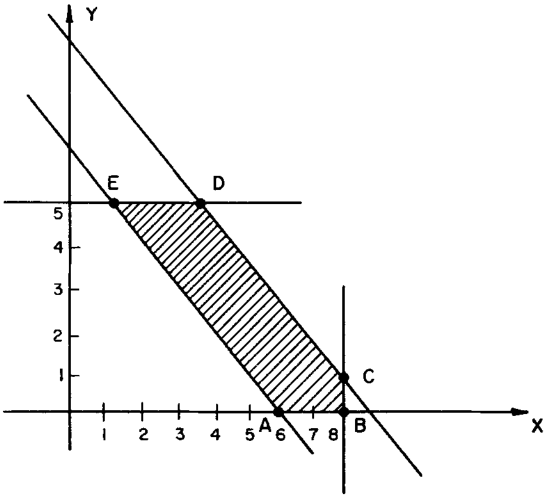

Figure 2

Recall now that a degenerate distribution is one in which fewer than $m + n - 1$ routes are used. Such distributions are possible if, and only if, there are two subsets S and T of the excesses and requirements, respectively, such that

$$\sum_{i \in S} E_i = \sum_{j \in T} R_j .$$

On the assumption that the average transportation cost resulting from the approximation techniques used is not changed significantly by slight alterations in the excesses and requirements, it is clear by Theorem 3 that forcing degeneracy decreases total costs. Two complementary remarks remain to be made.

The first deals with the difficulties involved in recognizing rather than forcing degeneracy. A rough estimate of the number of comparisons needed to check the equality above is provided by $\frac{1}{2}\left(2^m - 2\right)\left(2^n - 2\right)$. (This is merely the number of nontrivial subsets of excesses compared with the number of nontrivial subsets of requirements. In one case, only one set of each complementary pair need be used; hence the factor of 1/2.) This estimate can be reduced somewhat by a partial order of the subsets involved, a subset being built one element at a time until it exceeds or equals a given comparison subset from the other class. Thus, we need never go past the point where the comparison subset is exceeded or equaled, and a large number of comparisons is avoided. At best, however, the number of comparisons is usually prohibitive.

The second observation deals with the method adopted for forcing degeneracy. In any method of constructing a feasible solution which adds one shipment at each stage, in order to achieve the total of $m + n - 1$ shipments, it must exhaust exactly one current excess or fulfill one current requirement until the last stage. Then, because of the balance equation $\Sigma E_i = \Sigma R_j$, both an excess and a requirement are cancelled. The method for forcing degeneracy proposed in the main body of this report is based on the fact that it cancels both an excess and a requirement with one shipment. The alterations in the given excesses and requirements are bounded by the factor Δ. This method is only the first in a class of methods in which higher order comparisons are made. For example, we could ask:

$$\text{Is } E_{i_1} + E_{i_2} - R_j \leqq 2\Delta \text{ ?}$$

If this holds, we would increase both E_{i_1} and E_{i_2} by an amount less than or equal to Δ and force a degeneracy. This is illustrated in the following example:

$$\begin{array}{c|cc|} & R_1 & R_2 \\ \hline E_1 & x_{11} & x_{12} \\ E_2 & x_{21} & x_{22} \\ E_3 & x_{31} & x_{32} \\ E_4 & x_{41} & x_{42} \\ \hline \end{array} = \begin{array}{c|cc|} & 8 & 7 \\ \hline 3 & 4 & 0 \\ 3 & 4 & 0 \\ 3 & 0 & 0 \\ 6 & 0 & 7 \\ \hline \end{array}$$

$$\Delta = 1 .$$

Here E_1 and E_2 can be increased by $\Delta = 1$ to make them add up to R_1, thus producing a degeneracy.

These higher-order partial sums were not recommended for two reasons: (1) they require more complicated programming than seems feasible on the electronic computers available in most Navy installations, and (2) the order in which the partial sums are constructed is unlikely to coincide with the least cost entries which are at the heart of SMALC.

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