A COMBINATORIAL ALGORITHM*

THOMAS E. EASTERFIELD†.

1. Let us suppose that we are given a square $(n \times n)$ array of real numbers $\{a_{ij}\}$ $(i, j = 1, \ldots, n)$, and let it be required to find, among all sums $\sum_{i=1}^{n} a_{ir_i}$, in which $(r_1, \ldots, r_n)$ is a permutation of the numbers $1, \ldots, n$, that subset whose members have the least value. (We shall for brevity call this subset the set of minimum sums.)

(*Note.* The a's need not necessarily be all positive. It has seemed clearer in the examples shown to make them so; and if it is desired to deal only with positive a's in a given practical example, the algorithm set out below may be applied to a new set derived from the old simply by addition of a sufficiently large constant quantity to every a.)

2. It is obvious that *if* the pattern formed by those elements which are the least in their respective rows is such that it is possible to choose from among these elements (hereinafter to be called *row-minima*) a set containing one from each column and one from each row, then such a set yields a minimum sum; moreover, that only sums consisting of row-minima will belong to the set. (Thus for example in the pattern

1	1	3
4	1	1
1	5	1

the sums $a_{11}+a_{22}+a_{33}$, $a_{12}+a_{23}+a_{31}$, consist entirely of 1s. Any other sum of elements no two of which lie in the same row or column will contain elements greater than 1 but none less, and therefore will not be in our required set.)

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† In the course of a piece of organisational research into the problems of demobilisation in the R.A.F., it seemed that it might be possible to arrange the posting of men from disbanded units into other units in such a way that they would not need to be posted again before they were demobilised; and that a study of the numbers of men in the various release groups in each unit might enable this process to be carried out with a minimum number of postings. Unfortunately the unexpected ending of the Japanese war prevented the implications of this approach from being worked out in time for effective use. The algorithm of this paper arose directly in the course of the investigation.

3. In general, however, this condition will not be satisfied. We shall show that from our original array $\{a_{ij}\}$ we can produce another, $\{\hat{a}_{ij}\}$ with the properties:

(i) $\Sigma\, a_{ir_i}$ belongs to the required set of minimum sums if and only if $\Sigma\, \hat{a}_{ir_i}$ belongs to the set of minimum sums of the array $\{\hat{a}_{ij}\}$,

(ii) it is possible to choose from among the row-minima of $\{\hat{a}_{ij}\}$ a set of n of which no two lie in the same row or column,

(iii) any row-mimimum of $\{\hat{a}_{ij}\}$ lies in at least one minimum sum.

In fact, our derived array $\{\hat{a}_{ij}\}$ will enable us to find all permutations r_i such that $\Sigma\, a_{ir_i}$ is a minimum sum of the original array.

4. We shall proceed from the original array $\{a_{ij}\}$ to the final array $\{\hat{a}_{ij}\}$ by a series of steps, in each of which a constant quantity is subtracted from all elements of one or more columns. This process will alter the pattern of row-minima, but, since each column contributes just one term to any sum $\Sigma\, a_{ir_i}$, the difference between the sums for two different permutations will remain unaltered, and in particular the set of permutations giving rise to minimum sums will also remain unaltered.

5. We wish to be able to select a set of row-minima one from each row and no two in any column; but it will in fact prove easier to work in terms of another condition on the pattern of row-minima, which we shall show to be equivalent to the other.

Let us define a set of r ($\leqslant n$) columns as being *adjusted* if each subset of s ($\leqslant r$) columns from it contains row-minima from *at least s* rows.

An adjusted set of r columns will be called *exactly adjusted* if the whole set contains row-minima from exactly r rows.

In fact, as we shall show below (§ 9), a necessary and sufficient condition for it to be possible to choose one row-minimum from each of a set of *any number* of columns, each from a different row, is that the set should be adjusted. Before we prove this, however, we require the lemma of § 7.

6. Let us suppose that we have a set C of columns containing row-minima which all lie in a set R of rows, and that, further, some or all of the rows in R contain row-minima in columns not belonging to C. Then if we subtract from every element of every column in C a constant quantity x, the row-minima lying in the rows of R will now lie in columns belonging to C only; the other row-minima, previously in the set of rows R, have

been *suppressed.* Moreover, by choosing x sufficiently small we may ensure that no new row-minima appear in the set of columns C.

7. LEMMA. *If a set of r columns is adjusted, and a subset S of s $(<r)$ columns is exactly adjusted, then if we suppress those row-minima in the remaining $r-s$ columns which lie in the same rows as those in the subset S, the whole set of r columns remains adjusted.*

We must prove that, after the suppression of the said row-minima, each set of t columns contains row-minima from at least t rows.

If the t columns chosen lie in the subset S, they contain row-minima from exactly t rows by hypothesis.

If the t columns lie wholly outside the subset S, consider the subset of $s+t$ columns formed by adjoining to them the columns of the subset S. Before removal of certain row-minima, these columns contained row-minima from at least $s+t$ rows of which exactly s were supplied from the columns of the subset S. Hence at least t were supplied from the remaining t columns. Even, therefore, if we suppress those row-minima in the same rows as those of the subset S, the subset of t columns contains row-minima from at least t rows.

Finally, if the t columns consist of t_1 from the subset S, and t_2 which do not belong to the subset S, the first contain row-minima from at least t_1 rows, and the second from at least t_2 rows, the two sets of rows having no row in common; hence the set of t columns contains row-minima from at least t_1+t_2 rows.

This covers all cases.

8. It will be observed that the above process can be repeated as long as we can find fresh exactly adjusted subsets of columns. Thus any adjusted set can eventually be converted into a number of sets that are each exactly adjusted but contain no exactly adjusted proper subset, with possibly one set left over that is adjusted but not exactly adjusted, and with no exactly adjusted subset. The latter possibility cannot arise for the whole array, in which the number of rows is equal to the number of columns. It will be shown below that such an ultimate decomposition is unique (§ 13).

9. We shall now prove

THEOREM 1. (A) *A necessary and sufficient condition for it to be possible to choose one row-minimum from each column of a set S in such a way that* *no two of these minima lie in the same row is that the set be adjusted.* (B) *Further, if the set contain no exactly adjusted subset, we may include any one of its row-minima, arbitrarily chosen, in such a choice.*

The necessity of the condition of the first part is obvious. Further, the sufficiency is obvious for a single column. Let us assume as an inductive hypothesis that we have proved the first part for all numbers of columns up to r.

Then if we have an adjusted set of $r+1$ columns, there are two possibilities: that the set contains an exactly adjusted proper subset, or that it does not.

In the first case, by the method of § 6 above we can break it down into two mutually exclusive exactly adjusted subsets, each of r columns or less, and with the row-minima of one lying in wholly different rows from those of the other. Application of the inductive hypothesis to the two parts at once proves the desired result.

In the second case, in which there is no exactly adjusted proper subset of columns, let us consider any row-minimum arbitrarily chosen. Let us consider the r columns in which it does not lie, and in which we shall agree to ignore any row-minima in the same row as that in which it lies. Then, as every subset of s of these columns contains originally row-minima in at least $s+1$ rows, and as we have decided to ignore row-minima in one row only, any subset still contains non-ignored row-minima in at least s rows. Hence, by the inductive hypothesis, we can choose from each of the r columns a row-minimum of a different row, and none of these lies in the same row as that first chosen arbitrarily.

This completes the proof of (A); and (B) has been proved incidentally.

10. It remains now to prove

THEOREM 2. *Any array can be reduced to one which is adjusted.*

We shall show that we can arrange all sets of columns in an order which allows us to adjust each in turn without spoiling the adjustment of those that have been adjusted already. If we represent the i-th column by (i) and the set of columns (i), (j), (k), by (ijk) (where $i<j<k$), a suitable order is (1), (2), (12), (3), (13), (23), (123), (4), (14), (24), (124), (34), (134), (234), (1234); or, formally, $(a_1 a_2 \ldots a_r)$ precedes $(b_1 b_2 \ldots b_s)$ if $a_r < b_s$ *or* if $a_r = b_s$, and $(a_1 \ldots a_{r-1})$ precedes $(b_1 \ldots b_{s-1})$. (The null set is taken as preceding every other). This is not the only suitable order; it will be clear from the proof that any order will do in which a set is preceded by all its proper subsets.

11. We proceed as follows:

If column (1) contains no row-minimum, subtract from each of its elements the same quantity, the least required to introduce one.

Treat column (2) in the same way.

If columns (12) contain row-minima in one row only, reduce all elements of *both* by the same quantity, the least required to introduce another row-minimum into at least one column. This process cannot remove the row-minima already obtained in columns (1) or (2).

We go on thus : Let us suppose that we have adjusted all sets of columns up to that immediately preceding $(a_1 a_2 \dots a_{r-1} a_r)$. If the set $(a_1 a_2 \dots a_{r-1} a_r)$ is not adjusted, we subtract from all its elements the same quantity, the least required to introduce a row-minimum from one more row. This procedure cannot spoil the adjustment of any set of columns already adjusted, whether this contains a_r or not. For if $a_r = s+1$, then the complete set of columns $(1 \dots s)$ is adjusted before reduction of the elements of the set $(a_1 a_2 \dots a_{r-1} s+1)$; but if the latter set is not adjusted, $(a_1 a_2 \dots a_{r-1})$ is exactly adjusted, and hence, by § 7, the adjustment of $(1 \dots s)$ (which includes that of all its proper subsets) is not spoilt. Thus the adjustment of any set preceding $(a_1 a_2 \dots a_{r-1} s+1)$ and *not* containing $(s+1)$, is not spoilt.

Moreover, the set of m columns* $(a_i a_j \dots b_1 \dots b_t s+1)$, where $(b_1 \dots b_t)$ has no column in common with $(a_1 \dots a_{r-1})$, may be looked on as compounded of $(b_1 \dots b_t)$ and $(a_i \dots a_j s+1)$. The first of these contains row-minima from t rows at least, none of which contains row-minima lying in the columns $(a_1 \dots a_{r-1})$, (since this is an exactly adjusted set). When we reduce the elements of $(a_1 \dots a_{r-1}\ s+1)$ by a constant quantity, therefore, we do not suppress any row-minima in these t rows; for the quantity is just sufficient to introduce one fresh row-minimum into $(a_1 \dots a_{r-1} s+1)$, and thus can suppress row-minima only in rows which contain row-minima of $(a_1 \dots a_{r-1} s+1)$ before reduction. The set $(a_1 \dots a_{r-1} s+1)$ contains row-minima from $m-t$ rows, all containing row-minima lying in the columns $(a_1 \dots a_{r-1})$ (for if $(s+1)$ contained a row-minimum from another row, no subtraction would be necessary). After the reduction none of these latter row-minima will have been removed. Since the t rows corresponding to the set $(b_1 \dots b_t)$ and the $m-t$ rows

* For the purpose of this paragraph only, the designation $(a_i a_j \dots b_1 \dots b_t s+1)$ is intended to cover every set that precedes $(a_1 \dots a_r)$, including those in which the b's appear mixed up with the a's. The b's have been displaced from their correct order for ease of exposition. It will be observed that this does not affect the proofs.

corresponding to $(a_1 \ldots a_{r-1} s+1)$ have no elements in common, the whole set contains row-minima from m rows. It follows that

$$(a_i a_j \ldots b_1 \ldots b_t \ldots s+1)$$

remains adjusted.

This completes the proof that the adjustment of all preceding columns remains unspoilt.

12. Since at no stage do we spoil the adjustment of a set of columns already examined, we shall reach after a finite number of steps the set $(1 \ldots n)$, which will therefore be adjusted.

13. We may now reduce the whole array to one in which the columns fall into sets each of which is exactly adjusted but contains no exactly adjusted subset, as described in §§ 7 and 8.

Then, by §§ 2 and 9 (A), any set of elements of the final array which yields a minimum sum consists entirely of row-minima, and by § 9 (B) any row-minimum of the final array occurs in at least one minimum sum. Since the set of all elements occurring in at least one minimum sum is unique, the pattern that the row-minima form relative to the whole array is also unique.

This completes the proof of the statements of § 3.

14. For a large array the number of steps described above is very large; but in fact a little practice will enable the worker to take short cuts in the way of reducing the particular columns or sets of columns most likely to reach the final stage.

15. As an example we give the working on a 6×6 array:

$\underline{9}$	22	58	11	19	27
$\underline{43}$	78	72	50	63	48
41	$\underline{28}$	91	37	45	33
74	42	$\underline{27}$	49	39	32
36	$\underline{11}$	57	22	25	18
$\underline{3}$	56	53	31	17	28

(*Note.* We have underlined all row-minima in this and the successive derived arrays.)

Here all sets of columns up to (123) are adjusted. Subtract 2 from each element of (4) and obtain

$$\begin{array}{cccccc} \underline{9} & 22 & 58 & \underline{9} & 19 & 27 \\ \underline{43} & 78 & 72 & 48 & 63 & 48 \\ 41 & \underline{28} & 91 & 35 & 45 & 33 \\ 74 & 42 & \underline{27} & 47 & 39 & 32 \\ 36 & \underline{11} & 57 & 20 & 25 & 18 \\ \underline{3} & 56 & 53 & 29 & 17 & 28. \end{array}$$

All sets of columns up to (1234) are adjusted. Subtract 10 from each element of (5) and obtain

$$\begin{array}{cccccc} \underline{9} & 22 & 58 & \underline{9} & \underline{9} & 27 \\ \underline{43} & 78 & 72 & 48 & 53 & 48 \\ 41 & \underline{28} & 91 & 35 & 35 & 33 \\ 74 & 42 & \underline{27} & 47 & 29 & 32 \\ 36 & \underline{11} & 57 & 20 & 15 & 18 \\ \underline{3} & 56 & 53 & 29 & 7 & 28 \end{array}$$

(15), (25), (125), (35), (135), (235), and (1235) are adjusted. (45) is not. Subtract 2 from each element of each of these columns and obtain

$$\begin{array}{cccccc} 9 & 22 & 58 & \underline{7} & \underline{7} & 27 \\ \underline{43} & 78 & 72 & 46 & 51 & 48 \\ 41 & \underline{28} & 91 & 33 & 33 & 33 \\ 74 & 42 & \underline{27} & 45 & \underline{27} & 32 \\ 36 & \underline{11} & 57 & 18 & 13 & 18 \\ \underline{3} & 56 & 53 & 27 & 5 & 28. \end{array}$$

(145), (245), (1245) are adjusted. (345) is not. Subtract 2 from each element of all three columns and obtain

$$\begin{array}{cccccc} 9 & 22 & 56 & \underline{5} & \underline{5} & 27 \\ \underline{43} & 78 & 70 & 44 & 49 & 48 \\ 41 & \underline{28} & 89 & 31 & 31 & 33 \\ 74 & 42 & \underline{25} & 43 & \underline{25} & 32 \\ 36 & \underline{11} & 55 & 16 & \underline{11} & 18 \\ \underline{3} & 56 & 51 & 25 & \underline{3} & 28. \end{array}$$

Every set up to (12345) is now adjusted. (6) is not. Subtract 5 from each element of (6) and obtain

$$\begin{array}{cccccc} 9 & 22 & 56 & \underline{5} & \underline{5} & 22 \\ \underline{43} & 78 & 70 & 44 & 49 & \underline{43} \\ 41 & \underline{28} & 89 & 31 & 31 & \underline{28} \\ 74 & 42 & \underline{25} & 43 & \underline{25} & 27 \\ 36 & \underline{11} & 55 & 16 & \underline{11} & 13 \\ \underline{3} & 56 & 51 & 25 & \underline{3} & 23. \end{array}$$

The whole set is now adjusted. We may apply the process of §§ 6 and 7, which we may exhibit by subtracting 1 from each element of columns 3 and 4, and obtain

9	22	55	$\underline{4}$	5	22
$\underline{43}$	78	69	43	49	$\underline{43}$
41	$\underline{28}$	88	30	31	$\underline{28}$
74	42	$\underline{24}$	42	25	27
36	$\underline{11}$	54	15	$\underline{11}$	13
$\underline{3}$	56	50	24	$\underline{3}$	23.

The sets of columns (3), (4), and (1256) are all exactly adjusted but contain no exactly adjusted subset.

It will be found that there are just two minimum sums,

$$a_{12}+a_{25}+a_{34}+a_{41}+a_{56}+a_{63} \quad \text{and} \quad a_{16}+a_{23}+a_{34}+a_{41}+a_{55}+a_{62},$$

which are, in the original,

$$43+11+27+11+17+33 = 142 \quad \text{and} \quad 3+28+27+11+25+48 = 142.$$

27 Middleton Road,
London, N.W.11.

IRREDUCIBLE MATRIX REPRESENTATIONS OF CERTAIN FINITE GROUPS

H. O. FOULKES*.

1. *Introduction.* Matrix representations of abstract finite groups can be set up in various ways. Burnside† and others have given methods whereby the actual matrices can be constructed when the group characters are known, and a method involving Young tableaux has been given‡ for the symmetric group. Some groups, however, have representations which can be expressed in monomial form§, and it is with certain groups of this kind that this note is concerned.

It is known|| that groups of orders pq, pqr, and p^3, where p, q, r are primes, have all their irreducible representations transformable to monomial form.

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† W. Burnside, *Theory of groups of finite order*, 2nd edition (Cambridge, 1911), Chap. 15.

‡ D. E. Littlewood, *Group characters and matrix representations of groups* (Oxford, 1940), Chap. 5.

§ Burnside, *op. cit.*, § 242; A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung*, 1st edition (Berlin, 1923), § 37.

|| Speiser, *op. cit.*, § 51; Burnside, *op. cit.*, § 258.