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# A MINIMAL TOTALLY DUAL INTEGRAL DEFINING SYSTEM FOR THE $b$-MATCHING POLYHEDRON*

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**Abstract.** Totally dual integral linear systems are intimately related to polyhedra that have the property that every nonempty face contains an integer point. A minimal totally dual integral defining system for a certain polyhedron related to $b$-matchings is given.

**1. Introduction.** The study of dual integrality is the study of integral optimal solutions to dual linear programs.

Dual integrality is studied in complexity combinatorics for several reasons. One is that often a combinatorial problem is better described as the dual of another problem. Another is to obtain combinatorial min-max theorems via the duality theorem of linear programming.

Alan Hoffman [5] introduced the concept of total dual integrality, which was latter studied and used by Edmonds–Giles [3].

A finite linear system $Ax \leqq b$, with $A$ and $b$ rational, is called totally dual integral (TDI) when the dual linear program of the linear program

$$\max\{cx : Ax \leqq b\}$$

has an integral optimal solution for integral $c$ such that it has an optimal solution.

TDI linear systems are intimately related to integer polyhedra (those polyhedra that have the property that every nonempty face contains an integer point).

This paper investigates the relation of TDI linear systems to a combinatorial problem known as the $b$-matching problem. A minimal TDI defining system for a certain integer polyhedron related to $b$-matchings is given. Pulleyblank [9] independently obtained this result in a different way, using the results contained in [8].

**2. TDI linear systems and integer polyhedra.** The relation of TDI linear systems to integer polyhedra is made clear by the following two theorems.

THEOREM 1 (Edmonds–Giles [3]). *If $Ax \leqq b$ is a* TDI *linear system with $b$ integral, then*

$$P = \{x \in \mathbb{R}^n : Ax \leqq b\}$$

*is an integer polyhedron.*

THEOREM 2 (Giles and Pulleyblank [4]). *Let*

$$P = \{x \in \mathbb{R}^n : Ax \leqq b\},$$

*where $A$ and $b$ are rational. If $P$ is an integer polyhedron, then there exists a* TDI *linear system $A'x \leqq b'$ with $b'$ integral such that*

$$P = \{x \in \mathbb{R}^n : A'x \leqq b'\}.$$

Theorem 1 is a nice generalization of a theorem of Hoffman [5].

The above theorems can be combined to produce an interesting and useful technique for proving that a particular linear system is a defining system for an integer

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polyhedron. Suppose the goal is to prove that

$$P=\{x \in \mathbb{R}^n : Ax \leqq b\}$$

is an integer polyhedron (often the goal is to prove that $P$ is the convex hull of a certain set of integral points). By multiplying the inequalities by a positive constant if necessary, it can be assumed that $b$ is integral. By Theorem 2, a set of inequalities that are implied by $Ax \leqq b$ can be added to $Ax \leqq b$ to form a TDI linear system with integral right-hand side. Theorem 1 implies that the polyhedron defined by the new system is an integer polyhedron. But the polyhedron defined by the new system is $P$.

The importance of the condition that $b$ be integral in the above development is shown by the following theorem.

THEOREM 3 (Giles and Pulleyblank [4]). *For any finite, rational, linear system $Ax \leqq b$, there is a positive rational number $d$ such that $dAx \leqq db$ is a* TDI *linear system.*

**3. $b$-matchings.** Before $b$-matchings are described, some notation will be given. For a real vector $x=(x_i : i \in I)$ and $S \subseteq I$, where $I$ is a finite set, let

$$x(S)=\sum \{x_i : i \in S\}.$$

Let $G$ be a graph with node set $VG$ and edge set $EG$. For $i$ in $VG$, let $N(i)$ denote the set of nodes adjacent to $i$ ($i$ is not adjacent to itself). For $S \subseteq VG$, let $\delta(S)$ denote the subset of edges of $G$ that are incident to exactly one node of $S$ (for $i$ in $VG$, $\delta(i)$ will denote $\delta(\{i\})$) and let $\gamma(S)$ denote the subset of edges having both ends in $S$. Let $G[S]$ denote the graph with node set $S$ and edge set $\gamma(S)$. For $j$ in $EG$, let $\psi(j)$ denote the subset of $VG$ that makes up the two ends of $j$ (each edge is assumed to have two distinct ends). If $\mathscr{S}$ is a collection of subsets of $VG$ and $j$ is an edge of $G$, let

$$\mathscr{S}(j)=\{R \in \mathscr{S} : j \in \gamma(R)\}.$$

Let $b=(b_i : i \in VG)$, where $b_i$ is a positive integer for each $i$ in $VG$. A $b$-matching in $G$ is an integral solution to the linear system

(3.1) $\qquad x_j \geqq 0 \quad$ for every $j$ in $EG$,

(3.2) $\qquad x(\delta(i)) \leqq b_i \quad$ for every $i$ in $VG$.

Let $P(G, b)$ denote the convex hull of the $b$-matchings of $G$.

Edmonds has proved the following theorem by means of a good algorithm known as the blossom algorithm.

THEOREM 4 (Edmonds [2]). *A defining system for $P(G, b)$ is* (3.1) *and* (3.2) *together with*

(3.3) $\qquad x(\gamma(S)) \leqq \lfloor b(S)/2 \rfloor$ *for all $S \subseteq VG$ such that $|S| \geqq 3$ and $b(S) \geqq 3$ is an odd integer.*

A perfect $b$-matching of a graph $G$ is a $b$-matching of $G$ such that $x(\delta(i))=b_i$ for all $i$ in $VG$. A near perfect $b$-matching of $G$ deficient at node $i$ is a $b$-matching of $G$ such that

$$x(\delta(i))=b_i-1$$

and

$$x(\delta(v))=b_v \quad \text{for all } v \text{ in } VG-\{i\}.$$

A graph $G$ is called $b$-critical if for every $i$ in $VG$ there exists a near perfect $b$-matching of $G$ deficient at node $i$ and $|VG| \geqq 3$.

A balanced edge of $G$ is a pair of nodes $i, j$ that are joined by one or more edges and satisfy $b_i = b_j$.

For a graph $G$ and positive, integral $b$, let

$$\mathcal{F} = \{S \subseteq VG: G[S] \text{ is } b\text{-critical and } G[S] \text{ contains no cutnode } i \text{ for which } b_i = 1\}$$

and

$$\mathcal{V} = \{i \in VG: i \text{ belongs to a component of } G \text{ that is a balanced edge; or } b(N(i)) > b_i \text{ and if } b(N(i)) = b_i + 1, \text{ then } \gamma(N(i)) = \varnothing\}.$$

A theorem of Pulleyblank can now be stated.

THEOREM 5 (Pulleyblank [7]). *A minimal set of inequalities that define $P(G, b)$ is* (3.1) *together with*

(3.4) $$x(\delta(i)) \leqq b_i \quad \textit{for every } i \textit{ in } \mathcal{V}$$

*and*

(3.5) $$x(\gamma(S)) \leqq \lfloor b(S)/2 \rfloor \quad \textit{for every } S \textit{ in } \mathcal{F}.$$

That (3.1), (3.4), and (3.5) is a defining system for $P(G, b)$ follows from a result of the next section, but the minimality seems more difficult to demonstrate.

**4. TDI linear systems and $b$-matchings.** The defining system for $P(G, b)$ given in Theorem 4 is not in general a TDI linear system. This can be seen by considering a triangle with $b_i = 2$ for each node $i$ and $c_j = 1$ for each edge $j$ in the objective function

$$\max \sum \{c_j x_j : j \in EG\}.$$

By Theorem 2, there does exist a TDI defining system for $P(G, b)$ which has integral right-hand side. Such a TDI defining system is given in the following theorem, which can be proven easily using Edmonds' blossom algorithm (see Pulleyblank [8]).

THEOREM 6. *A* TDI *defining system for $P(G, b)$ is* (3.1), (3.2), *and*

(4.1) $$x(\gamma(S)) \leqq \lfloor b(S)/2 \rfloor \quad \textit{for every } S \subseteq VG.$$

Theorem 6 has been proven without making use of the blossom algorithm by Hoffman and Oppenheim [6] and Schrijver and Seymour [11] (it should be noted that although [11] deals with the special case of 1-matchings, its proof generalizes easily to $b$-matchings).

The system given by Theorem 6 is much larger than necessary. The result will now be improved to get a smaller TDI defining system for $P(G, b)$.

Pulleyblank has introduced the idea of $b$-bicritical graphs in his study of dual integrality in $b$-matching problems. A graph $G$ is $b$-bicritical if $G$ is connected, $|VG| \geqq 3$, $b_i \geqq 2$ for all $i$ in $VG$, and for every $i$ in $VG$ there exists a $b$-matching of $G$ such that

$$x(\delta(i)) = b_i - 2$$

and

$$x(\delta(v)) = b_v \quad \text{for all } v \text{ in } VG - \{i\}.$$

Some results on the structure of $b$-critical and $b$-bicritical graphs are needed to proceed further.

For a graph $G$, positive integral $b$, and $S \subseteq VG$, let

$$\mathscr{C}^0(S) = \{i \in V - S \colon G[\{i\}] \text{ is a component of } G[V-S]\},$$

$$\mathscr{C}^1(S) = \{R \subseteq V - S \colon |R| \geqq 2,\ b(R) \text{ is odd, and } G[R] \text{ is a component of } G[V-S]\},$$

$$\mathscr{C}^2(S) = \{R \subseteq V - S \colon |R| \geqq 2,\ b(R) \text{ is even, and } G[R] \text{ is a component of } G[V-S]\}.$$

The following theorem of Tutte characterizes those graphs which have a perfect $b$-matching.

THEOREM 7 (Tutte [12]). *A graph, $G$, has a perfect $b$-matching if and only if for every $S \subseteq VG$*

$$b(S) \geqq b(U\mathscr{C}^0(S)) + |\mathscr{C}^1(S)|.$$

Using Theorem 7, the following two lemmas of Pulleyblank can be proven.

LEMMA 1 (Pulleyblank [7]). *A connected graph, $G$, is $b$-critical if and only if $b(VG)$ is odd, $|VG| \neq 1$, and for every nonempty $S \subseteq VG$*

$$b(S) \geqq b(U\mathscr{C}^0(S)) + |\mathscr{C}^1(S)| + 1.$$

LEMMA 2 (Pulleyblank [8]). *A connected graph, $G$, is $b$-bicritical if and only if $b(VG)$ is even, $|VG| \neq 1$, and for every nonempty $S \subseteq VG$*

$$b(S) \geqq b(U\mathscr{C}^0(S)) + |\mathscr{C}^1(S)| + 2.$$

It is useful to combine the above lemmas to get the following lemma, which can be proved by noting that if $G$ is a $b$-bicritical graph and $S$ is a subset of $VG$, then

$$b(S) + b(U\mathscr{C}^0(S)) + |\mathscr{C}^1(S)|$$

is an even number.

LEMMA 3. *A connected graph, $G$, is one of $b$-critical or $b$-bicritical if and only if $|VG| \neq 1$, and for every nonempty $S \subseteq VG$*

$$b(S) \geqq b(U\mathscr{C}^0(S)) + |\mathscr{C}^1(S)| + 1.$$

Using Lemma 3 and the TDI-ness of the system given by Theorem 6, a theorem which gives a smaller TDI defining system for $P(G, b)$ can be obtained. The result can also be obtained by using the results of Pulleyblank [8], but it is simpler to prove it directly. The proof uses an idea of Paul Seymour for proving the same result in the special case of 1-matchings.

For a graph $G$ and positive integral $b$, let

$$\mathscr{D} = \{S \subseteq VG \colon G[S] \text{ is } b\text{-critical or } G[S] \text{ is } b\text{-bicritical}\}.$$

THEOREM 8. *A* TDI *defining sytem for $P(G, b)$ is* (3.1), (3.2), *and*

$$x(\gamma(S)) \leqq \lfloor b(S)/2 \rfloor \quad \textit{for every } S \textit{ in } \mathscr{D}. \tag{4.2}$$

*Proof.* It will be shown that (3.1), (3.2), and (4.2) is a TDI linear system. That it is a defining system for $P(G, b)$ will then follow from the Edmonds–Giles theorem (Theorem 1) by noting that every $b$-matching of $G$ satisfies (3.1), (3.2), and (4.2).

Let $c$ be an integral vector and consider the linear program

$$\max \{\textstyle\sum (c_j x_j \colon j \in EG) \colon (3.1), (3.2), (4.2)\}. \tag{4.3}$$

The dual linear program of (4.3) is

$$\min \sum \{b_i y_i : i \in VG\} + \sum \{\lfloor b(S)/2 \rfloor Y_S : S \in \mathscr{D}\}$$
subject to

$$(4.4) \qquad \begin{aligned} &y(\psi(j)) + Y(\mathscr{D}(j)) \geqq c_j \quad \text{for every } j \text{ in } EG,\\ &y_i \geqq 0 \quad \text{for every } i \text{ in } VG,\\ &Y_S \geqq 0 \quad \text{for every } S \text{ in } \mathscr{D}. \end{aligned}$$

By Theorem 6, there exists an integral optimal solution, $(y, Y)$, to the dual linear program of

$$(4.5) \qquad \max \{\sum (c_j x_j : j \in EG) : (3.1), (3.2), (4.1)\}.$$

Suppose there exists $S \subseteq VG$ such that $Y_S > 0$ and $G[S]$ is not connected. Let $S_1, \cdots, S_k$ be the subsets of $S$ such that $G[S_1], \cdots, G[S_k]$ are the components of $G[S]$. Let

$$\begin{aligned} &Y'_S = 0,\\ &Y'_{S_i} = Y_{S_i} + Y_S \quad \text{for } i = 1, \cdots, k,\\ &Y'_R = Y_R \quad \text{for all other } R \subseteq VG. \end{aligned}$$

Now $(y, Y')$ is an integral optimal solution to the dual linear program of (4.5).

This procedure allows the assumption to be made that $(y, Y)$ is such that if $Y_S > 0$, then $G[S]$ is connected.

Suppose there exists $S \subseteq VG$ such that $Y_S > 0$ and $S$ is not in $\mathscr{D}$. Since $Y_S > 0$, $G[S]$ is connected. It can be assumed that $|S|$ is not equal to 1. By Lemma 3, there exists a nonempty $X \subseteq S$ such that in $G[S]$ (notation is relative to $G[S]$)

$$(4.6) \qquad b(X) < b(U\mathscr{C}^0(X)) + |\mathscr{C}^1(X)| + 1.$$

Let

$$\begin{aligned} &y'_v = y_v + Y_S \quad \text{for every } v \text{ in } X,\\ &y'_v = y_v \quad \text{for all other } v \text{ in } VG,\\ &Y'_S = 0,\\ &Y'_R = Y_R + Y_S \quad \text{for every } R \text{ in } \mathscr{C}^1(X) \cup \mathscr{C}^2(X),\\ &Y'_R = Y_R \quad \text{for all other } R \subseteq VG. \end{aligned}$$

It is easy to check that $(y', Y')$ is a feasible solution to the dual linear program of (4.5). To show that $(y', Y')$ is an optimal solution, it must be shown that

$$(4.7) \qquad \lfloor b(S)/2 \rfloor \geqq b(X) + b(U\mathscr{C}^1(X))/2 + b(U\mathscr{C}^2(X))/2 - |\mathscr{C}^1(X)|/2.$$

Since the right-hand side of (4.7) is integral, it suffices to show that

$$(4.8) \qquad \frac{b(S)}{2} \geqq b(X) + \frac{b(U\mathscr{C}^1(X))}{2} + \frac{b(U\mathscr{C}^2(X))}{2} - \frac{|\mathscr{C}^1(X)|}{2}.$$

Since

$$b(X) = b(S) - b(U\mathscr{C}^0(X)) - b(U\mathscr{C}^1(X)) - b(U\mathscr{C}^2(X)),$$

(4.8) is equivalent to

$$b(U\mathscr{C}^0(X)) + |\mathscr{C}^1(X)| \geqq b(X). \tag{4.9}$$

Now (4.9) follows from (4.6). So $(y', Y')$ is an integral optimal solution to the dual linear program of (4.5).

The above procedure makes it possible to assume that $(y, Y)$ is such that if $Y_S > 0$, then $S$ is in $\mathscr{D}$.

Let

$$\bar{Y} = (Y_S \colon S \in \mathscr{D}).$$

Now $(y, \bar{Y})$ is an integral optimal solution to (4.4). □

Using the techniques of Pulleyblank [8, see § 7], Theorem 8 can be sharpened.

For a graph $G$ and positive integral $b$, let

$$\mathscr{D}' = \mathscr{F} \cup \{S \subseteq VG \colon G[S] \text{ is } b\text{-bicritical and there does not exist a node } u \in S \text{ that is adjacent to } v \in VG - S \text{ with } b_v = 1\}.$$

where $\mathscr{F}$ is as defined in § 3. Let $\mathscr{V}$ be defined as in § 3.

THEOREM 9. *A* TDI *defining system for* $P(G, b)$ *is* (3.1),

$$x(\delta(i)) \leqq b_i \quad \textit{for every } i \textit{ in } \mathscr{V}, \tag{4.10}$$

and

$$x(\gamma(S)) \leqq \lfloor b(S)/2 \rfloor \quad \textit{for every } S \textit{ in } \mathscr{D}'. \tag{4.11}$$

*Proof.* Again, it suffices to show that (3.1), (4.10), and (4.11) is a TDI linear system.

Let $c$ be an integral vector. It will be shown that there exists an integral optimal solution, $(y, Y)$, to (4.4) such that if $Y_S > 0$ for some $S$ in $\mathscr{D}$, then $S$ is in $\mathscr{D}'$ and if $y_i > 0$ for some $i$ in $VG$, then $i$ is in $\mathscr{V}$. This will prove the theorem.

By Theorem 8, there exists an integral solution, $(y, Y)$, to (4.4). Suppose there is an $S$ in $\mathscr{D}$ such that $Y_S > 0$ and $S$ is not in $\mathscr{D}'$. If $G[S]$ is not $b$-critical, then letting

$$Y'_S = 0,$$
$$Y'_{S \cup \{v\}} = Y_{S \cup \{v\}} + Y_S,$$
$$Y'_R = Y_R \quad \text{for all other } R \text{ in } \mathscr{D},$$

where $v \in VG - S$ is adjacent to a node in $S$ and $b_v = 1$, gives an optimal solution $(y, Y')$ to (4.4) ($G[S \cup \{v\}]$ is $b$-critical). So it can be assumed that $G[S]$ is $b$-critical. Since $S$ is not in $\mathscr{D}'$, $G[S]$ is a $b$-critical subgraph with a cutnode $v$ such that $b_v = 1$. Let $S_1, \cdots, S_k$ be the subsets of $S - \{v\}$ such that $G[S_1], \cdots, G[S_k]$ are the components of $G[S - \{v\}]$. Let

$$S'_i = S_i \cup \{v\} \quad \text{for } i = 1, 2, \cdots, k.$$

Using the definition of a $b$-critical graph, it is easy to check that $G[S'_i]$ is a $b$-critical graph for $i = 1, 2, \cdots, k$. Let

$$Y'_S = 0,$$
$$Y'_{S'_i} = Y_{S'_i} + Y_S \quad \text{for } i = 1, 2, \cdots, k,$$
$$Y'_R = Y_R \quad \text{for all other } R \text{ in } \mathscr{D}.$$

The solution $(y, Y')$ is an optimal solution to (4.4). This procedure allows the assumption to made that $(y, Y)$ has the property that if $Y_S > 0$, then $S$ is in $\mathscr{D}'$.

Suppose there exists an $i$ in $VG$ such that $y_i > 0$ and $i$ is not in $\mathscr{V}$. If $b(N(i)) \leqq b_i$, let

$$y'_i = 0,$$
$$y'_v = y_v + y_i \quad \text{for } v \text{ in } N(i),$$
$$y'_v = y_v \quad \text{for all other } v \text{ in } VG.$$

The solution $(y', Y)$ is an optimal solution to (4.4). If $b(N(i)) = b_i + 1$, let

$$S = N(i) \cup \{i\}.$$

Since $\gamma(N(i)) \neq \varnothing$, $G[S]$ is a $b$-critical subgraph with no cutnode $v$ such that $b_v = 1$. Let

$$y'_i = 0,$$
$$y'_v = y_v \quad \text{for all other } v \text{ in } VG,$$
$$Y'_S = Y_S + y_i,$$
$$Y'_R = Y_R \quad \text{for all other } R \text{ in } \mathscr{D}.$$

Again, $(y', Y')$ is an optimal solution to (4.4). These two operations allow the assumption to be made that $(y, Y)$ is such that if $y_i > 0$, then $i$ is in $\mathscr{V}$. □

As was mentioned earlier, a result of this theorem is that (3.1), (3.4), and (3.5) is a defining system for $P(G, b)$. The stronger result of Pulleyblank (Theorem 5) will be needed to prove the minimality of the TDI defining system given in Theorem 9.

THEOREM 10. *A minimal TDI defining system for* $P(G, b)$ *is* (3.1), (4.10), *and* (4.11).

*Proof.* By Theorem 9, (3.1), (4.10), and (4.11) is a TDI defining system for $P(G, b)$. Write the system as (3.1), (4.10),

(4.12) $$x(\gamma(S)) \leqq \lfloor b(S)/2 \rfloor \quad \text{for every } S \text{ in } \mathscr{F},$$

and

(4.13) $$x(\gamma(S)) \leqq \lfloor b(S)/2 \rfloor \quad \text{for every } S \text{ in } \mathscr{D}' - \mathscr{F}.$$

By Pulleyblank's theorem (Theorem 5), (3.1), (4.10), and (4.12) is a minimal defining system for $P(G, b)$. Since $P(G, b)$ is a full dimensional polytope, any defining system for $P(G, b)$ must include some multiple of each inequality in (3.1), (4.10), and (4.12). So each inequality in (3.1), (4.10), and (4.12) is necessary for (3.1), (4.10), and (4.11) to be a defining system for $P(G, b)$.

To prove the theorem, all that remains to be shown is that if any inequality in (4.13) is removed, the resulting linear system is not TDI.

Let $G$ be a graph and let $S \subseteq VG$ be such that $S$ is in $\mathscr{D}'$ but not in $\mathscr{F}$. Now let (4.13′) be the set of inequalities (4.13) with

$$x(\gamma(S)) \leqq \lfloor b(S)/2 \rfloor$$

removed.

It must be shown that for some integral $c$, the dual linear program of

(4.14) $$\max \{\textstyle\sum (c_j x_j : j \in EG): (3.1), (4.10), (4.12), (4.13')\}$$

has no integral optimal solution. This is equivalent to showing that the dual linear program of

(4.15) $$\max \{\textstyle\sum (c_j x_j : j \in EG): (3.1), (4.10), (4.11)\}$$

has no integral optimal solution, $(y, Y)$, such that $Y_S = 0$.

Let

$$c_j = \begin{cases} 1 & \text{for every } j \text{ in } \gamma(S), \\ 0 & \text{for every other } j \text{ in } EG. \end{cases}$$

An optimal solution to (4.15) has objective value $b(S)/2$, since any $b$-bicritical graph contains a perfect $b$-matching. It will be shown that the dual linear program of (4.15) has no optimal solution, $(y, Y)$, such that $Y_S = 0$.

Since $c$ is 0, 1-valued, only 0, 1-valued solutions to the dual linear program of (4.15) need be considered. A 0, 1-valued solution to the dual linear program of (4.15) corresponds to a subset $Q$ of $\mathscr{V}$ and a subset $T$ of $\mathscr{D}'$ such that for every edge $j$ in $\gamma(S)$, either $j$ has an end in $Q$ or $j$ is contained in $\gamma(R)$ for some $R$ in $T$. Such a pair $(Q, T)$ is called a cover of $\gamma(S)$. The weight of a cover $(Q, T)$ of $\gamma(S)$ is

$$w(Q, T) = \sum (b_i : i \in Q) + \sum (\lfloor b(R)/2 \rfloor : R \in T).$$

It must be shown that there does not exist a cover $(Q, T)$ of $\gamma(S)$ such that $S$ is not in $T$ and $w(Q, T) \leqq b(S)/2$.

It is straightforward to check that

(4.16) if $S' \subseteq S$ is such that $G[S']$ is connected, then there does not exist a covering $(Q, T)$ of $\gamma(S')$ such that $Q = \varnothing$ and $w(Q, T) < \lfloor (S')/2 \rfloor$

and

(4.17) there does not exist a covering $(Q, T)$ of $\gamma(S)$ such that $w(Q, T) \leqq b(S)/2$ with $Q = \varnothing$ and $S$ not in $T$.

Now (4.16) and (4.17) will be used to finish the proof. Let $(Q, T)$ be a cover of $\gamma(S)$ such that $S$ is not in $T$. If $Q = \varnothing$, then $w(Q, T) > b(S)/2$. Suppose that $Q \neq \varnothing$. It can be assumed that $Q \subseteq S$. Since $G[S]$ is $b$-bicritical, by Lemma 3

$$b(Q) \geqq b(U\mathscr{C}^0(Q)) + |\mathscr{C}^1(Q)| + 1, \tag{4.18}$$

where all notation is with respect to $G[S]$.

Now (4.16) implies that

$$w(Q, T) \geqq b(Q) + \frac{b(U\mathscr{C}^1(Q))}{2} + \frac{b(U\mathscr{C}^2(Q))}{2} - \frac{|\mathscr{C}^1(Q)|}{2}. \tag{4.19}$$

By (4.18) and (4.19),

$$w(Q, T) > b(S)/2. \tag{4.20}$$

□

In the special case of 1-matchings, Theorem 9 implies that

(4.21) the minimal defining system for $P(G, 1)$ given by Theorem 5 is TDI.

This result on 1-matchings has been proven by Cunningham and Marsh [1].

Schrijver [10] has shown that every full dimensional, rational polyhedron is defined by a unique minimal TDI system with integral left-hand sides. This implies that the system given in Theorem 10 is the unique minimal integral TDI defining system for $P(G, b)$.

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