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On Some Assignment Problems

Abstract

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The personnel assignment problem, a particularly simple example of an extremization problem involving permutations, is as follows: Given n persons and n jobs, and a set of n^2 scores measuring the value of each person in each job; to find the assignment of persons to jobs, that will maximize the sum of the scores. Methods of solution for this problem are known and consist in the reduction either to a linear programming problem (Votaw and Orden) or to a two-person zero-sum game (von Neumann). The present paper endeavors to bring out the price or value imputations implicit in these solutions, to formulate a similar linear assignment problem in the economic theory of location, and to propose a generalization of the latter to a quadratic problem which, although seemingly simple, offers mathematical difficulties that have so far not been overcome.

The locational version of the linear assignment problem is as follows: Given are a number of plants and a greater or equal number of locations, and the possible profit from production by each plant at each location. This profit is a "gross" profit, before rents on the location or plant. It is further assumed that the profit of a given plant-location combination is independent of the locations of all other plants. What is the assignment of plants to locations which maximizes total profits?

The value imputations we are interested in turn up under the guise of the probabilities that define the strategy for one of the players in the game solution, and as the variable of the dual problem in the linear programming

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solution; they can also be visualized as rents on the use of locations and plants. The following relations among these "efficiency" rents are reminiscent of a competitive market equilibrium, but in fact merely express conditions for allocative efficiency. The efficiency rents themselves are then looked upon as parameters used in characterizing an optimal assignment: That plant is placed at a given location for which the excess of profit (at that location) over rent on plant is maximal, this excess then being equal to the rent on the location in question. Conversely, that location is assigned to a given plant for which the excess of profit (of that plant) over rent on the location is maximal, and this excess equals the rent on that plant.

Any imputation of rents to plants and locations for which one of these two sets of conditions is satisfied (the other set then follows automatically) produces an optimal assignment. Conversely with each optimal assignment is associated at least one set of efficiency rents. Unlike many other linear programming problems, the efficiency prices associated with an optimal assignment are not unique. This is attributable to the presence of indivisible resources in the assignment problem.

The non-trivial nature of these efficiency price conditions is best seen when contrasted with the pricing situation in the generalized (quadratic) locational assignment problem. Again let there be given a number of plants and a greater or equal number of locations, and the profit c_{ki} that the k-th plant would make at the i-th location. Furthermore let the amount a_{km} of commodities (in weight units) that needs to be transported from the k-th plant to the m-th plant be given, and let the cost of transportation (per weight unit) from the i-th to the j-th locations be b_{ij}. What is the assignment of plants to locations that maximizes total profit after costs of transportation among plants are met?

If we include among the plants a number of inactive dummy plants sufficient to match the number of locations, an assignment can be characterized by a permutation matrix $P = (p_{ki})$, i.e. a matrix such that $p_{ki} = 0$ except that for each given k there exists exactly one i with $p_{ki} = 1$ and that for each i there exists exactly one k with $p_{ki} = 1$. We thus obtain the following problem formulation: to maximize trace $(C'P - A'PBP')$ with respect to P over the set of permutation matrices. The original, linear, assignment problem is obtained by suppressing the second, quadratic, term.

The device that leads to solution of the linear assignment problem is to regard the permutation matrices as the vertices of a polyhedron made up by the so-called doubly-stochastic matrices (all elements non-negative, and all column- and row-sums = 1). The difficulty in using the same continuization with a quadratic maximand is that, depending on the convexity or concavity properties possessed by or given to the maximand, one can have the property that each maximum is a vertex (as desired), or that each local maximum is an absolute maximum, but not both of these properties (which together led to solution of the linear case). Likewise, we have not succeeded in finding a value imputation for plants and locations, associated with an optimal assignment in the quadratic case, which can be used to decentralize locational choices.

The difficult nature of the quadratic assignment problem is further illustrated by the fact that the as yet unsolved travelling salesman problem represents a special case of it, namely the case in which A also is a (given) permutation matrix. The travelling salesman problem is as follows: Suppose that each of a given set of points must be visited exactly once in a round trip involving return to the point of departure, and that the costs of transportation are known between each pair of points, what is the least-cost route of travel?

It should be added that in all the assignment problems discussed, there is, of course, the obvious brute force method of enumerating all assignments, evaluating the maximand at each of these, and selecting the assignment giving the highest value. This is too costly in most cases of practical importance, and by a method of solution we have meant a procedure that reduces the computational work to manageable proportions in a wider class of cases.

References

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