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Establishing the Matching Polytope

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This paper gives an elementary, inductive proof-"graphical" in spirit-of a theorem of Edmonds' which specifies the convex hull of the matchings of an arbitrary, finite, undirected graph in terms of a finite system of linear inequalities.

INTRODUCTION

Given an undirected finite graph $G = \{N, E\}$ with node-set N and edge-set E, each edge of E consisting of an unordered pair of distinct nodes, a *matching* (or *simple-matching*) M of G is a subset of E with the property that no two edges of M meet the same node. In other terms, a matching is a feasible solution to the constraints

$$\sum_{e \in i} x(e) \leqslant 1 \quad \text{for each} \quad i \in N, \quad \text{and} \quad x(e) = 0$$
$$\text{or} \quad 1 \quad \text{for each} \quad e \in E, \tag{1}$$

where $e \in i$ means the summation extends over those edges e which are incident to the node i. In words: weights $x(e)$ are to be assigned to edges e in such a way that the sum of the weights incident to any node i does not exceed 1, and each weight must be 0 or 1. The *maximum-cardinality matching problem* is to find a matching having a maximum number of edges. This is a much studied problem for which several good algorithms have been devised [2, 4, 6].

One view of the maximum matching problem is as an integer program ("on a graph"), that is, a linear program in which variables must take on integer values. As is well known (see, e.g., [1]) a principal approach to the

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solution of integer programs consists in generating supplementary linear constraints, in our case implied by the constraints of (1), until a derived linear program is found whose solution is integer valued and hence a matching. Thus, both for its inherent interest and to gain greater understanding of integer programming by knowing "best" supplementary constraints or "cuts," it is of particular interest to be able to specify the convex hull of the matchings or the feasible integer solutions to (1), which we now call the *matching polytope*. In [3] Edmonds determined this polytope. To explain this theorem we introduce the notation: $|S|$ for $S \subset N$ denotes the cardinality of S; $x(S, T)$, $S, T \subset N$ denotes the sum of the weights on distinct edges e having one end in S and the other end in T.

THEOREM [3]. *Given a graph $G = \{N, E\}$, the extreme points of*

$$x(i, N) = \sum_{e \in i} x(e) \leqslant 1 \quad \text{each} \quad i \in N, \qquad x(e) \geqslant 0 \quad \text{each} \quad e \in E, \tag{2}$$

$x(S, S) \leqslant s$ *each odd set of nodes S, $|S| = 2s + 1$, are integer valued.*

It is obvious that any matching must satisfy the constraints of (2), for the new constraints simply say (for $x(e) = 0$ or 1) that $2s + 1$ nodes can contain at most s matching edges. The fact that (2) *is the matching polytope* is also immediate, given the truth of the theorem, for any extreme point of (2), being integer valued, is clearly a matching. On the other hand, any matching x is an extreme point of (2) for, otherwise, x could be expressed $x = \frac{1}{2}x' + \frac{1}{2}x''$, x', x'' feasible for (2) and different from x, which is a contradiction since each component of x' and x'' must lie between 0 and 1 inclusive.

The proof of Edmonds is difficult and rather indirect. This paper provides an elementary, inductive proof which is "graphic" in spirit and directly displays any non-integer feasible solution x of (2) as a convex combination of other feasible solutions. On the other hand, an important aspect of Edmond's method [3] is that his proof of the theorem is given in terms of a good algorithm for solving the *maximum-weight matching problem*: that is, given arbitrary real "profits" $c(e)$, $e \in E$, find -weights $x(e)$ satisfying (1) and maximizing $\sum_E c(e)\, x(e)$. Edmonds, Johnson, and Lockhart [5] have further developed the method to solve a more general class of integer programs "on graphs," sometimes also referred to as generalized matching problems.

In Section 1, "Preliminaries," various combinatorial lemmas, needed in the subsequent arguments, are stated and proved together with a number of facts regarding the structure of feasible (non-integer) solutions to the constraints (2) which, while needed for the proof of the theorem, are of some interest in themselves. Section 2 contains the proof.

1. Preliminaries[1]

An *arc* is taken to be a connected graph which has two nodes of degree 1 and whose other nodes all have degree 2. A *circuit* is a finite connected graph all of whose nodes have degree 2. The set of nodes (edges) of a graph G will be denoted $N(G)$ (respectively, $E(G)$). A circuit C is odd or even, respectively, if $|N(C)|$ is odd or even. A graph is *separable* if it is either disconnected or the union of two subgraphs H and K such that $E(H) \neq \varnothing$, $E(K) \neq \varnothing$, $E(H) \cap E(K) = \varnothing$, $|N(H) \cap N(K)| = 1$. A *block* of a graph G is a maximal non-separable subgraph of G. A *cut* in a graph $G = \{N, E\}$ is a partition of the nodes into two non-void disjoint sets N' and $N \sim N'$, the *cut-edges* of this cut are the edges joining nodes in N' to nodes in $N \sim N'$, and the *value* of the cut is the number of such edges.

Lemma 1. *Let $G = \{N, E\}$ be a finite connected graph such that $|N| > 1$ and G contains no even circuit. Then, every block of G is either an edge or an odd circuit.*

Let B be a block of G. If B contains no circuit, it must be an edge.
If B contains a circuit it necessarily contains an odd circuit C. Suppose that $B \neq C$. Let H be the subgraph of B consisting of the edges in $E(B) \sim E(C)$ and their incident nodes. Since B is non-separable each component of H has two or more nodes in common with C and therefore H contains at least one arc joining two distinct nodes of C. Let A be a minimal arc of this kind, and let A_1 and A_2 be the two arcs contained in C which join the end-nodes of A. Since A is minimal and contained in H, $E(A) \cap E(C)$ is empty and $N(A) \cap N(C)$ includes only the two end-nodes of A. Therefore, $A \cup A_1$, $A \cup A_2$, and $A_1 \cup A_2 (= C)$ are all circuits. Since these circuits cannot all be odd, the hypothesis that G contains no even circuit is contradicted. Hence $B = C$, and so B is an odd circuit.

Lemma 2. *Let $G = \{N, E\}$ be a finite graph such that $|N| > 1$, G contains no even circuit, and the values of all cuts of G are at least 2. Then every block of G is an odd circuit.*

The proof is immediate from the above and the observation that, if a block were a single edge, then that edge would be the sole cut-edge of a cut of G with value 1.

We turn now to the "structure" of feasible solutions x to (2).

[1] The author is indebted to a referee, who identified some faulty argument and suggested the definitions, reformulations, and proofs through Lemma 2.

LEMMA 3. *If x is feasible and $T \subset N$ then*

$$x(T, T) \leqslant \tfrac{1}{2}\{|T| - x(T, N \sim T)\}.$$

From $x(i, N) \leqslant 1$ we have $|T| \geqslant \sum_{i \in T} x(i, N) = 2x(T, T) + x(T, N \sim T)$.

Given a feasible x we say that an odd set of nodes S, $|S| = 2s + 1$, is *minimal* if $x(S, S) = s$ and $T \subset S$, $|T| = 2t + 1 (>1)$ implies $x(T, T) < t$.

LEMMA 4. *Suppose $S, T \subset N$, $|S| = 2s + 1$, $|T| = 2t + 1$, and S is minimal. If $|S \cap T| = 2i + 1 > 1$ and $S \not\subset T$, then $x(T, T) < t$.*

If $T \subset S$, the statement is true by definition. Otherwise, suppose $x(T, T) = t$. From $S \cap T \subset S$ and $x(S \cap T, S \cap T) < i$ we have

$$\begin{aligned} &x(S \cup T, S \cup T) \\ &\quad = x(S, S) + x(T, T) + x(S \sim T, T \sim S) - x(S \cap T, S \cap T) \\ &\quad \geqslant x(S, S) + x(T, T) - x(S \cap T, S \cap T) > s + t - i, \end{aligned}$$

a contradiction, since $|S \cup T| = 2(s + t - i) + 1$ and x is feasible.

LEMMA 5. *Suppose $S, T \subset N$, $|S| = 2s + 1$, $|T| = 2t + 1$, and S is minimal. If $|S \cap T| = 2i$ and $|S \sim T| > 1$, then $x(T, T) < t$.*

Again, if $T \subset S$, the statement is true by definition. Otherwise, since $|S \sim T| = 2(s - i) + 1 > 1$, and using Lemma 3 for $x(S \cap T, S \cap T)$, we find

$$\begin{aligned} s &= x(S, S) \\ &= x(S \sim T, S \sim T) + x(S \cap T, S \cap T) + x(S \sim T, S \cap T) \\ &< s - i + \tfrac{1}{2}\{|S \cap T| - x(S \cap T, S \sim T) - x(S \cap T, T \sim S)\} \\ &\quad + x(S \sim T, S \cap T) \\ &= s - i + i + \tfrac{1}{2}\{x(S \cap T, S \sim T) - x(S \cap T, T \sim S)\}, \end{aligned}$$

implying $x(S \cap T, S \sim T) > x(S \cap T, T \sim S)$. Using the same formula for $x(T, T)$, recalling $|T \sim S| = 2(t - i) + 1$, we have, by symmetry,

$$x(T, T) \leqslant t + \tfrac{1}{2}\{x(S \cap T, T \sim S) - x(S \cap T, S \sim T)\} < t.$$

LEMMA 6. *Suppose $|S| = 2s + 1 > 3$, S is minimal, and H_S is the subgraph of nodes S and edges joining nodes of S for which $x(e) > 0$. If H_S contains no even circuit then it is an odd circuit of $2s + 1$ edges.*

We show, first, that the value of all cuts of H_S is at least two. For suppose there is a cut of value zero, i.e., H_S is disconnected. Then S, the nodes of H_S, are partitioned into two sets S_1, S_2 with $x(S_1, S_2) = 0$, $|S_1| = 2i + 1$,

$| S_2 | = 2(s - i)$. If $| S_1 | > 1$, then, since $x(S_1, S_1) < i$, $x(S_2, S_2) \leqslant s - i$, we find $x(S, S) = x(S_1, S_1) + x(S_2, S_2) < s$, a contradiction. If $| S_1 | = 1$, let k be any node of S_2. Then

$$x(S, S) = x(k, N) + x(S_2 \sim \{k\}, S_2 \sim \{k\}) < 1 + s - 1 = s,$$

a contradiction. Now suppose H_S has a cut of value 1. Let $e = (k_1, k_2)$ be the cut-edge with S partitioned into S_1, S_2, $k_i \in S_i$, and $| S_1 | = 2i + 1$, $| S_2 | = 2(s - i)$. If $| S_1 | > 1$,

$$x(S, S) = x(S_1, S_1) + x(S_2 \cup \{k_1\}, S_2 \cup \{k_1\}) < i + s - i = s,$$

a contradiction. If $| S_1 | = 1$,

$$x(S, S) = x(k_2, N) + x(S_2 \sim \{k_2\}, S_2 \sim \{k_2\}) < 1 + s - 1 = s,$$

a contradiction.

So the values of all cuts of H_S are at least two, H_S contains no even circuit, and $| N(H_S)| = | S | = 2s + 1 \geqslant 5$. Hence, by Lemma 2, each block of H_S is an odd circuit. If H_S had more than one block, it would have a block B with just one node in common with the union, call it U, of the other blocks of H_S. Then, taking $| N(B)| = 2i + 1$, we would have $| N(U)| = 2(s - i) + 1$ and therefore

$$x(S, S) = x(N(B), N(B)) + x(N(U), N(U)) < i + s - i = s,$$

a contradiction. Hence, H_S can have only one block and is an odd circuit.

This simple sturcture of H_S allows $x(e)$, $e \in (S, S)$, to be represented easily in terms of (0, 1)-matchings of the subgraph H_S. To explain this let $x(H_S)$ be the vector of values $x(e)$, $e \in H_S$, and let $y^i(H_S)$ be the vector representing the *unique* (0, 1)-matching of H_S in which node i

FIGURE 1. Matching $y^i(H_S)$ of H_S

is exposed (i.e., $y^i(i, S) = 0$, see Figure 1).

LEMMA 7. *Suppose* $| S | = 2s + 1$, S *is minimal and* H_S, *the subgraph defined in Lemma* 6, *an odd circuit of* $2s + 1$ *edges or, if* $| S | = 3$, *an arc or circuit. If* $q_i = 1 - x(i, S)$, $i \in S$ *then*

$$x(H_S) = \sum_{i \in S} q_i y^i(H_S), \qquad \sum_{i \in S} q_i = 1, \quad q_i \geqslant 0. \tag{3}$$

We have

$$\sum q_i = 2s + 1 - \sum x(i, S) = 2s + 1 - 2x(S, S) = 1. \qquad (4)$$

In fact, for $|S| > 3$, $q_i > 0$, for, if $q_i = 0$, S would not be minimal. The "excesses" q_i, $i \in S$, uniquely determine the weights $x(e)$, $e \in H_S$. Let the edges of H_S be $(1, 2), \ldots, (2s, 2s + 1), (2s + 1, 1)$. Then

$$x(j, j + 1) = (q_{j+2} + q_{j+4} + \cdot\cdot) + (q_{j-1} + q_{j-3} + \cdot\cdot),$$

summation extending to where it makes sense, since then

$$x(j - 1, j) + x(j, j + 1) = x(j, S) \quad \text{and} \quad x(j, S) + q_j = 1, \quad j \in S.$$

But, with this notation, $y^i(j, j + 1) = 1$ if $i = j + 2, j + 4, \ldots$ or $i = j - 1, j - 3, \ldots$, establishing the left equation of (3).

In order to prove the theorem we need one construction. Given a subgraph H_S as in Lemma 7 *shrink* H_S to obtain the *reduced graph* G/H_S consisting of all nodes of G not in H_S and a new node i_S (replacing all nodes S), and consisting of all edges of G not incident to a node of S and edges (j, i_S) for $j \notin S$ if $(j, i) \in E$, an edge of G, for some $i \in S$. Given a feasible x for G, define weights $\bar{x}$ for G/H_S by letting $\bar{x}(e) = x(e)$ for edges e not incident to S, and by letting $\bar{x}(j, i_S) = \sum_{i \in S} x(j, i)$ for $j \notin S$ (see Figure 2).

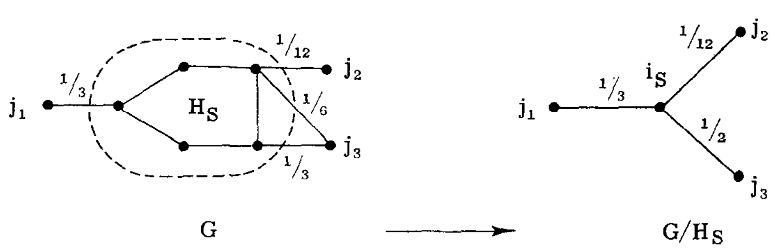


FIGURE 2

LEMMA 8. *If x is feasible for constraints* (2) *corresponding to G, then $\bar{x}$ is feasible for G/H_S, where H_S is as in Lemma* 7.

Since $x(S, S) = s$, we have $x(S, N \sim S) \leqslant 1$. Thus $\bar{x}(i_S, N \sim S) \leqslant 1$. If T is an odd set in G/H_S, $|T| = 2t + 1$, including i_S and $\bar{x}(T, T) > t$, then since $x(S \cup T) = 2(s + t) + 1$ in G we would have

$$x(S \cup T, S \cup T) = x(S, S) + \bar{x}(T, T) > s + t,$$

a contradiction. The remaining constraints are automatically satisfied.

2. The Proof

We will show that, if x is feasible for (2) but not all integer valued, then x is not an extreme point of (2). If (Case 1) no constraint of type $x(S, \bar{S}) \leqslant s$, $|S| = 2s + 1$, holds as an equation, we directly display x as a convex combination of two "close" feasible points. If, on the other hand, some one or more constraints of that type hold strictly, then we choose a minimal S. If (Case 2) the subgraph $H_S = \{e \in (S, \bar{S}); x(e) > 0\}$ contains an even circuit, then we again display $\bar{x}$ as a convex combination of two "close" feasible points. But, if (Case 3) H_S contains no even circuit, then we use induction on the number of nodes of the graph to display $\bar{x}$ as a convex combination of integer extreme points for the graph G/H_S and show how this convex combination can be extended to display x as a convex combination of feasible integer points.

Case 1. Suppose no constraint of type $x(S, \bar{S}) \leqslant s$ holds as an equation for the given x, i.e., all hold as strict inequalities. Consider a connected subgraph consisting of edges e for which $x(e) \neq 0, 1$ and nodes i for which $x(i, N) = 1$. This means we are considering a connected "subgraph" in which there may be some "dangling" edges having only one end in the graph; the other end, say node k, is "dangling," $0 < x(k, N) < 1$.

(a) Suppose the subgraph contains an even circuit of edges, say $e_1, e_2, \ldots, e_{2p}$. Define x^ϵ to be the same as x except that

$$x^\epsilon(e_j) = x(e_j) + (-1)^j \epsilon.$$

Then x^ϵ and $x^{-\epsilon}$ are feasible for small enough $\epsilon > 0$ while $x = \frac{1}{2}x^\epsilon + \frac{1}{2}x^{-\epsilon}$.

(b) Suppose (a) does not hold but that the subgraph contains at least one dangling edge e, with dangling node k. Then, following a path of incident edges in the subgraph beginning with e, either another dangling edge is reached (see Figure 3a) or a node is reached that is already incident to two edges of the traced path (see Figure 3b). In the first case there is an arc $e_1, \ldots, e_p$ joining two dangling nodes. Define x^ϵ to be the same as x

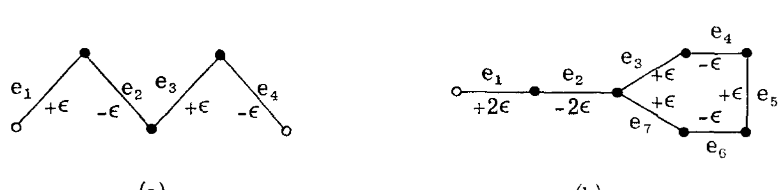


Figure 3

except that $x^\epsilon(e_j) = x(e_j) + (-1)^j \epsilon$. Then x^ϵ, $x^{-\epsilon}$ are feasible for small $\epsilon > 0$ and $x = \frac{1}{2}x^\epsilon + \frac{1}{2}x^{-\epsilon}$. In the second case there is an arc $e_1, \ldots, e_p$ together with an odd circuit of edges $e_{p+1}, \ldots, e_{p+2q+1}$ having e_p, e_{p+1}, and e_{p+2q+1} as the only three edges incident at one node. Define x^ϵ to be the same as x except that $x^\epsilon(e_j) = x(e_j) + 2(-1)^j \epsilon$ for $j = 1, \ldots, p$, and $x^\epsilon(e_j) = x(e_j) + (-1)^j \epsilon$ for $j = p + 1, \ldots, p + 2q + 1$. Then x^ϵ, $x^{-\epsilon}$ are feasible for small $\epsilon > 0$ and $x = \frac{1}{2}x^\epsilon + \frac{1}{2}x^{-\epsilon}$.

(c) Suppose neither (a) nor (b) holds. Then, by Lemma 1, every block of the subgraph is either an edge or an odd circuit. Suppose there exists a block which is an edge. Assign it a right and a left. March through the sequence of adjacent blocks going to the right until an odd circuit block is encountered. Such a block must be encountered or the sequence would terminate in a dangling edge. Considering the same sequence going to the left we find that there is a simple path $e_1, \ldots, e_p$ joining two odd circuits, $e_{p+1}, \ldots, e_{p+2q+1}$ and $e_2', \ldots, e_{2r}'$ with edges e_1, e_2', e_{2r}' incident at a node and e_p, e_{p+1}, e_{p+2q+1} incident at a node. Define x^ϵ to be the same as x except that $x^\epsilon(e_j) = x(e_j) + 2(-1)^j\epsilon$ for $j = 1, \ldots, p$, $x^\epsilon(e_j) = x(e_j) + (-1)^j\epsilon$ for $j = p + 1, \ldots, p + 2q + 1$, and $x^\epsilon(e_j') = x(e_j') + (-1)^j \epsilon$ for $j = 2, \ldots, 2r$. Again, x^ϵ and $x^{-\epsilon}$ are feasible for small $\epsilon > 0$ and $x = \frac{1}{2}x^\epsilon + \frac{1}{2}x^{-\epsilon}$.

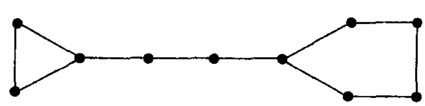


FIGURE 4

Therefore in all cases (a), (b), (c), unless in (c) the graph consists of blocks which are all odd circuits, x is not extreme. But in the latter case the graph contains an odd number of nodes S, $S = 2s + 1$, since the nodes may be counted by taking all nodes of one block, say B_1 (odd), adjoining a block B_2 having one node in common with B_1, thereby adding an even number of nodes, etc. Since the total weight on each node in the subgraph is exactly 1, we have, summing the weights of each node on S,

$$2x(S, S) = \sum_{i \in S} x(i, N) = 2s + 1, \qquad \text{or} \quad x(S, S) = s + \tfrac{1}{2},$$

contradicting the constraint $x(S, S) \leqslant s$. This cannot occur since x was assumed to be feasible.

Summarizing, if no odd-set constraint holds as an equation, then the theorem is easily proved directly by a simple adjustment argument.

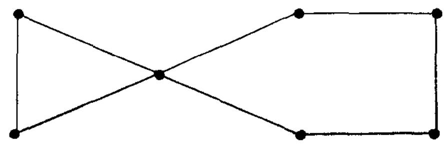


FIGURE 5

CASE 2. S is minimal and H_S contains an even circuit. Notice that this implies $|S| > 3$ and $x(e) \neq 1$ for $e \in (S, S)$. For, otherwise, let T be all of S save the two end points of e; $|T| = 2s - 1$. Then, since $x(T, S \sim T) = 0$, we must have $x(T, T) = s - 1$, contradicting minimality.

The adjustment of Case 1(a) can be made to display x as $x = \frac{1}{2}x^{\epsilon} + \frac{1}{2}x^{-\epsilon}$ for small $\epsilon > 0$, with x^{ϵ} and $x^{-\epsilon}$ feasible for (2) since no inequality of type $x(T, T) \leqslant t$, $|T| = 2t + 1$, can possibly be violated.

If $T \cap S = \varnothing$, the "adjustment" from x to $x^{\pm\epsilon}$ cannot effect the T-constraint.

If $T \subset S$, the minimality of S assures $x(T, T) < t$, which allows adjustment for small enough $\epsilon > 0$.

If $S \subset T$, then the even cycle of adjustment belongs to T and so the value of $x(T, T) = x^{\pm\epsilon}(T, T)$.

If $|T \cap S| = 1$, edges of (S, S) do not belong to (T, T) and the T-constraint cannot be effected. Otherwise, $|T \cap S|$ is odd or even.

If $|T \cap S| = 2i + 1 > 1$ then, by Lemma 4, $x(T, T) < t$, allowing adjustment for small enough $\epsilon > 0$.

If $|T \cap S| = 2i$ and $|S \sim T| > 1$ then , by Lemma 5, $x(T, T) < t$, again allowing adjustment.

Finally, if $|T \cap S| = 2i$ and $|S \sim T| = 1$, then any even simple cycle in H_S either belongs entirely to T or belongs to T save for two edges which are incident to the one node of $S \sim T$. In either case $x(T, T) = x^{\pm\epsilon}(T, T)$.

CASE 3. S is minimal and contains no even circuit. Therefore, H_S is an odd circuit of $2s + 1$ edges (or, if $|S| = 3$, perhaps an arc) by Lemma 6, and, by Lemma 7,

$$x(H_S) = \sum_S q_i y^i(H_S).$$

Shrink H_S to obtain a feasible $\bar{x}$ for the reduced graph G/H_S, having strictly fewer nodes than G. By induction, $\bar{x}$ is a convex combination of (0, 1)-matchings $\bar{z}^{j(t)}$ of G/H_S, say

$$\bar{x} = \sum_{t,j} \lambda_{j(t)} \bar{z}^{j(t)}, \qquad \sum_{t,j} \lambda_{j(t)} = 1, \qquad \lambda_{j(t)} > 0,$$

where each $\bar{z}^{j(t)}(t = 1,..., j_s)$ is a matching of G/H_S for which $\bar{z}^{j(t)}(i_S, j) = 1$, (i_S, j) an edge, and each $\bar{z}^{0(t)}$ is a matching for which $\bar{z}^{0(t)}(i_S, j) = 0$ for all j.

We show how the $\bar{z}^{j(t)}$ and the $y^i(H)$ can be "patched" together to form matchings of G, a convex combination of which is the feasible not all integer x. To begin, note that

$$\sum_t \lambda_{j(t)} = \sum_{i\in S} x(i,j) \qquad \text{and} \qquad \sum_t \lambda_{0(t)} = \sum_{i\in S} q_i - x(S, N \sim S). \tag{5}$$

The first equations hold since the right is the weight on edge (i_S, j) of G/H_S and the left the sum of λ's corresponding to z's having $\bar{z}(i_S, j) = 1$. Summing them over $j \neq 0$,

$$\sum_{j\neq 0, t} \lambda_{j(t)} = x(S, N \sim S),$$

which verifies the second equation of (5) since $\sum \lambda = 1 = \sum q$.

For each edge $(i,j) \in G$, $i \in S, j \notin S$, and each t of $j(t)$ above define a (0, 1)-matching $z^{ij(t)}$ of G as follows:

$$\begin{aligned} z^{ij(t)}(k, l) &= \bar{z}^{j(t)}(k, l), \qquad \text{for} \quad k, l \notin S \text{ (or } k, l \neq i_S); \\ z^{ij(t)}(i, j) &= 1, \qquad \text{for} \quad i \in S,\ j \notin S; \\ z^{ij(t)}(H) &= y^i(H) \qquad \text{(for edges of } H). \end{aligned} \tag{6}$$

Similarly, for each $i \in S$ and each t of $0(t)$, define a (0, 1)-matching $z^{i0(t)}$ of G:

$$\begin{aligned} z^{i0(t)}(k, l) &= \bar{z}^{0(t)}(k, l), \qquad \text{for} \quad k, l \notin S \text{ (or } k, l \neq i_S); \\ z^{i0(t)}(i, j) &= 0, \qquad \text{for} \quad i \in S,\ j \notin S; \\ z^{i0(t)}(H) &= y^i(H) \qquad \text{(for edges of } H). \end{aligned} \tag{7}$$

Then

$$x = \sum \delta_{ij(t)} z^{ij(t)}, \qquad \text{with} \quad \delta_{ij(t)} \geqslant 0 \qquad \text{and} \qquad \sum_{i,j,t} \delta_{ij(t)} = 1, \tag{8}$$

where

$$\delta_{ij(t)} = \frac{\lambda_{j(t)} x(i,j)}{\sum_t \lambda_{j(t)}} \qquad \text{and} \qquad \delta_{i0(t)} = \frac{\lambda_{0(t)}(q_i - x(i, N \sim S))}{\sum_t \lambda_{0(t)}}. \tag{9}$$

Clearly $\delta \geqslant 0$. That $\sum \delta = 1$ is immediate from (9) and (5). To show that

x is indeed the stated convex combination in (8) we verify over each of the three types of edges. If $(k, l) \in G$, $k, l \notin S$, then

$$x(k, l) = \sum_{i,j,t} \delta_{ij(t)} z^{ij(t)}(k, l) = \sum_{j,t} \lambda_{j(t)} \bar{z}^{j(t)}(k, l) = \bar{x}(k, l)$$

as required. The second equation follows from (5) through summation over i. If $(i, j) \in G$, $i \in S$, $j \notin S$ then

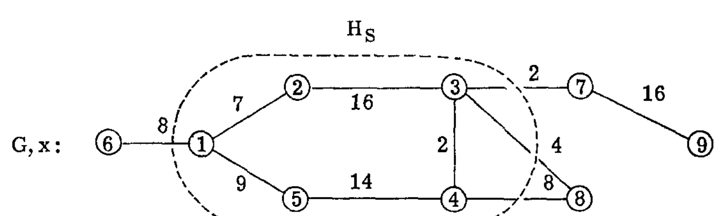
$$x(i, j) = \sum_{k,l,t} \delta_{kl(t)} z^{kl(t)}(i, j) = x(i, j)$$

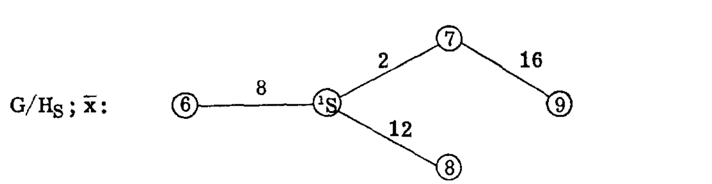
as required. Again, the second equation follows from (5) through summation over t. Finally, if $e \in H_S$,

$$\begin{aligned} x(H_S) &= \sum_{i,j,t} \delta_{ij(t)} z^{ij(t)}(H_S) \\ &= \sum_{i,j \neq 0} x(i, j) \cdot y^i(H_S) + \sum_i (q_i - x(i, N \sim S))\, y^i(H_S) \\ &= \sum_i [x(i, N \sim S) + q_i - x(i, N \sim S)]\, y^i(H_S) = \sum q_i y^i(H_S) \end{aligned}$$

as required. This completes the proof.

Example of Case 3. Let the graph G be as given below with weights $24x$ (to avoid denominators of 24) on edges and names of nodes encircled.



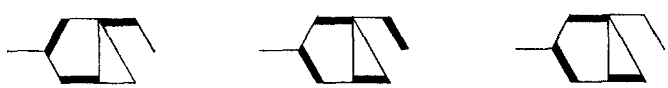


$$8\bar{z}^{6(1)} + 8\bar{z}^{8(1)} + 4\bar{z}^{8(2)} + 2\bar{z}^{7(1)} + 2\bar{z}^{0(1)}$$

$= \bar{x}$ and each t of j(t)

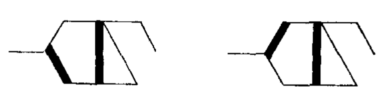
Therefore, taking each edge (i, j), $i \in S, j \notin S$, and each t of $j(t)$ we find:

$z^{16(1)}, \delta_{16(1)} = 8$ $\quad z^{38(1)}, \delta_{38(1)} = \frac{8\cdot 4}{12}$ $\quad z^{38(2)}, \delta_{38(2)} = \frac{4\cdot 4}{12}$



$z^{37(1)}, \delta_{37(1)} = 2$ $\quad z^{48(1)}, \delta_{48(1)} = \frac{8\cdot 8}{12}$ $\quad z^{48(2)}, \delta_{48(2)} = \frac{4\cdot 8}{12}$

Taking each $i \in S$ and each t of $0(t)$ (of which there is exactly 1) and, since $q_i - x(i, N \sim S) = 0$ for $i = 1, 3$, and 4, we find only:



$z^{20(1)}, \delta_{20(1)} = 1$ $\quad z^{50(1)}, \delta_{50(1)} = 1$

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