5. Electromagnetic Waves IV

5.1. Symmetry in EM

In applications, we often have symmetry in the structures we are interested in. For example, the slab waveguide we discussed above has reflection symmetry with respect to the center axis of the waveguide. It also has continuous translational symmetry along the z direction. In periodic structures, we have "discrete" translational symmetry. In photonic crystals, symmetry plays an important role of determining the dispersion relation. In this section, we briefly discuss some of the consequences due to symmetry. We will show that structures that exhibit the same symmetry have common properties in their solutions.

From Maxwell's equations, we have:

$$\nabla\times\left(\frac{1}{\varepsilon(\vec{r})}\nabla\times\vec{H}\right)=\omega^2\mu_0\vec{H} \tag{5.1}$$

Previously, we have studied the solution of this wave equation using its electric field counterpart. The reason of not using the electric field equation will be explained later. But now we would like to study the relationship between the symmetry of the structure (described by a real function $\varepsilon(\vec{r})$) and the solution $\vec{H}$. We define the operator $\hat{\Theta}$ as follows.

$$\hat{\Theta}=\nabla\times\left(\frac{1}{\varepsilon(\vec{r})}\nabla\times\right) \tag{5.2}$$

We can easily verify that $\hat{\Theta}$ is both a linear and a Hermitian operator. That means if $\vec{H}_1$ and $\vec{H}_2$ are both solutions of (5.1) and α and β are constants, $\hat{\Theta}$ satisfies:

$$\begin{aligned}&\hat{\Theta}(\alpha\vec{H}_1+\beta\vec{H}_2)=\alpha\hat{\Theta}\vec{H}_1+\beta\hat{\Theta}\vec{H}_2\\&\int d\vec{r}\vec{H}_1^*\cdot\hat{\Theta}\vec{H}_2=\int d\vec{r}\left(\hat{\Theta}\vec{H}_2\right)^*\cdot\vec{H}_1\end{aligned} \tag{5.3}$$

To verify the hermicity of $\hat{\Theta}$, we carry out the integral by parts in (5.3) as follows[3].

$$\begin{aligned}&\int d\vec{r}\vec{H}_1^*\cdot\nabla\times\left(\frac{1}{\varepsilon(\vec{r})}\nabla\times\vec{H}_2\right)\\&=\int d\vec{r}\left(\nabla\times\vec{H}_1\right)^*\cdot\left(\frac{1}{\varepsilon(\vec{r})}\nabla\times\vec{H}_2\right)=\int d\vec{r}\left(\frac{1}{\varepsilon(\vec{r})}\nabla\times\vec{H}_1\right)^*\cdot\left(\nabla\times\vec{H}_2\right)\\&=\int d\vec{r}\left(\nabla\times\frac{1}{\varepsilon(\vec{r})}\nabla\times\vec{H}_1\right)^*\cdot\vec{H}_2\end{aligned} \tag{5.4}$$

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[3] We use the following identity (integration by parts): $\int_\Omega \vec{F}\cdot\nabla\times\vec{G}d^3r=-\int_\Gamma \vec{F}\cdot\vec{G}\times\hat{n}ds+\int_\Omega \vec{G}\cdot\nabla\times\vec{F}d^3r$ where Γ is the surface of Ω.

where surface terms vanish because it is reasonable to assume the field vanishes at infinity.

Because $\hat{\Theta}$ is a linear Hermitian operator, it has several properties:

1. $\hat{\Theta}$ has real eigenvalues --> ω is real.
2. Solutions to (5.1) that correspond to different eigenvalues are orthogonal to each other. That is $\int d\vec{r}\vec{H}_1^* \cdot \vec{H}_2 = 0$.

You can verify that if instead of magnetic fields, you use the electric field counterpart of (5.1), you get:

$$\nabla\times\left(\nabla\times\frac{1}{\varepsilon(\vec{r})}\vec{D}\right)=\omega^2\mu_0\vec{D} \tag{5.5}$$

But it can be shown that the operator $\nabla\times\left(\nabla\times\frac{1}{\varepsilon(\vec{r})}\right)$ is not Hermitian. Hence we will use (5.1) instead of (5.5) in the following discussions.

We discuss two types of symmetry operations that are often encountered in the nanophotonic problems. One is the translational symmetry and the other is the symmetry with respect to a fixed point, i.e. rotation, inversion, and reflection symmetry. We will find the similarities between the EM waves and the electrons in a solid state lattice.

5.1.1. Translational symmetry (Bloch's theorem)

If the periodicity of the structure can be described by a set of unit vectors $\{\vec{a}_i, i=1,2,3\}$, we can expand the inverse of the dielectric function $\varepsilon^{-1}(\vec{r})$ into a Fourier's series as follows.

$$\varepsilon^{-1}(\vec{r})=\sum_{\vec{G}}\varepsilon^{-1}(\vec{G})e^{i\vec{G}\cdot\vec{r}} \tag{5.6}$$

where $\vec{G}$ represents a discrete set of vectors comprising of $\vec{G}=l\vec{b}_1+m\vec{b}_2+n\vec{b}_3$ (l,m,n are integers and $\vec{a}_i\cdot\vec{b}_j=2\pi\delta_{ij}$.) We say that $\vec{b}_i$'s are the unit vectors for the reciprocal lattice. If $\varepsilon(\vec{r})$ is real everywhere, $\varepsilon^{-1}(\vec{G})=\left[\varepsilon^{-1}(-\vec{G})\right]^*$. But the following discussions hold true even when $\varepsilon(\vec{r})$ is complex. Now we want to solve for $\vec{H}$ in (5.1) with $\varepsilon(\vec{r})$ given by (5.6). We first expand $\vec{H}$ in the plane-wave basis as in (2.25):

$$\vec{H}(\vec{r})=\int d\vec{k}\vec{H}_k(\vec{k})e^{i\vec{k}\cdot\vec{r}} \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.1), we get:

$$\begin{aligned}&\int d\vec{k}\sum_{\vec{G}}\varepsilon^{-1}(\vec{G})\vec{k}\times\vec{k}\times\vec{H}_k(\vec{k})e^{i(\vec{k}+\vec{G})\cdot\vec{r}}+\omega^2\mu_0\int d\vec{k}\vec{H}_k(\vec{k})e^{i\vec{k}\cdot\vec{r}}=0\\ &\Rightarrow\sum_{\vec{G}}\varepsilon^{-1}(\vec{G})\left(\vec{k}-\vec{G}\right)\times\left(\vec{k}-\vec{G}\right)\times\vec{H}_k(\vec{k}-\vec{G})+\omega^2\mu_0\vec{H}_k(\vec{k})=0\end{aligned} \tag{5.8}$$

(5.8) is a set of linear eigenvalue equations with $\vec{H}_k(\vec{k})$ linked by the reciprocal lattice $\vec{G}$. Therefore the solution must only be a superposition of these linked components, that is:

$$\vec{H}_{\vec{k}}(\vec{r}) = \sum_{\vec{G}} \vec{H}_k(\vec{k}-\vec{G}) e^{i(\vec{k}-\vec{G})\cdot\vec{r}} \equiv \vec{v}_{\vec{k}}(\vec{r}) e^{i\vec{k}\cdot\vec{r}} \tag{5.9}$$

where the Bloch's function $\vec{v}_{\vec{k}}(\vec{r}) = \sum_{\vec{G}} \vec{H}_k(\vec{k}-\vec{G}) e^{-i\vec{G}\cdot\vec{r}}$ has the same periodicity as $\varepsilon(\vec{r})$ as we can easily verify as follows.

$$\begin{aligned}\vec{v}_{\vec{k}}(\vec{r}+l\vec{a}_1+m\vec{a}_2+n\vec{a}_3) &= \sum_{\vec{G}} \vec{H}_k(\vec{k}-\vec{G}) e^{-i\vec{G}\cdot(\vec{r}+l\vec{a}_1+m\vec{a}_2+n\vec{a}_3)} \\ &= \sum_{\vec{G}} \vec{H}_k(\vec{k}-\vec{G}) e^{-i\vec{G}\cdot\vec{r}} e^{-2\pi i} \\ &= \sum_{\vec{G}} \vec{H}_k(\vec{k}-\vec{G}) e^{-i\vec{G}\cdot\vec{r}} = \vec{v}_{\vec{k}}(\vec{r})\end{aligned} \tag{5.10}$$

On the other hand because of the periodicity in the reciprocal lattice $\vec{G}$, we only need to consider $\vec{k}$ in the first Brillouin zone which is the unit lattice of the reciprocal lattice.

As an example in a 1D periodic structure with a period Λ along the z direction, the unit vector $\vec{a} = \Lambda\hat{z}$ and the unit vector in the reciprocal space is $\vec{b} = 2\pi/\Lambda\hat{z}$. The dielectric function can be expanded as:

$$\varepsilon^{-1}(z) = \varepsilon_0^{-1}\left(\kappa_0 + \kappa_1 e^{i(2\pi/\Lambda)z} + \kappa_{-1} e^{-i(2\pi/\Lambda)z} + \cdots\right) \tag{5.11}$$

The TM solution ($\vec{H} = \hat{y}H$) can be expanded as follows.

$$H_{k_z}(z) = \sum_{m=-\infty}^{\infty} H_m\left(k_z - \frac{2\pi m}{\Lambda}\right) e^{i(k_z - 2\pi m/\Lambda)z} \tag{5.12}$$

Substituting (5.12) and (5.11) into (5.1) and neglecting all the higher order terms in (5.11), we get[4]:

$$\omega^2\mu_0\varepsilon_0 H_m = \kappa_0\left(k_z - \frac{2\pi m}{\Lambda}\right)^2 H_m + \kappa_1\left(k_z - \frac{2\pi(m+1)}{\Lambda}\right)^2 H_{m+1} + \kappa_{-1}\left(k_z - \frac{2\pi(m-1)}{\Lambda}\right)^2 H_{m-1} \tag{5.13}$$

We have for m=0:

$$\left(\omega^2\mu_0\varepsilon_0 - \kappa_0 k_z^2\right) H_0 = \kappa_1\left(k_z - \frac{2\pi}{\Lambda}\right)^2 H_1 + \kappa_{-1}\left(k_z + \frac{2\pi}{\Lambda}\right)^2 H_{-1} \tag{5.14}$$

For m=1:

$$\left[\omega^2\mu_0\varepsilon_0 - \kappa_0\left(k_z - \frac{2\pi}{\Lambda}\right)^2\right] H_1 = \kappa_1\left(k_z - \frac{4\pi}{\Lambda}\right)^2 H_2 + \kappa_{-1} k_z^2 H_0 \tag{5.15}$$

When $k_z \approx \pi/\Lambda$, H_1 represents a counterpropagating wave along the –z direction. Let $k_z = (\pi/\Lambda)(1+\delta)$ with $\delta \approx 0$. In this case the dominating terms in (5.14) and (5.15) are H_0 and H_1. We have:

$$\left[\omega^2\mu_0\varepsilon_0 - \kappa_0\left(\frac{\pi}{\Lambda}\right)^2(1+\delta)^2\right]\left[\omega^2\mu_0\varepsilon_0 - \kappa_0\left(\frac{\pi}{\Lambda}\right)^2(1-\delta)^2\right] = \kappa_1\kappa_{-1}\left(\frac{\pi}{\Lambda}\right)^4(1+\delta)^2(1-\delta)^2 \tag{5.16}$$

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4 $\nabla\times\nabla\times H_{k_z}(z)\hat{y} = -\left[i(k_z - 2\pi m/\Lambda)\right]^2 H_{k_z}(z)\hat{y}$

If $\varepsilon(\bar{r})$ is real, $\kappa_1 = \kappa_{-1}^*$. If we only keep terms up to δ^2, we can derive the dispersion relation from (5.16) as:

$$\omega = c\left(\frac{\pi}{\Lambda}\right)\sqrt{\kappa_0 \pm \kappa_1}\left(1 - \frac{\kappa_0 \pm \kappa_1 \pm 2\kappa_0^2/\kappa_1}{2(\kappa_0 \pm \kappa_1)}\delta^2\right) \tag{5.17}$$

5.1.2. Symmetry with respect to a fixed point

Let $\hat{T}$ be a symmetry operator on $\varepsilon(\bar{r})$ with respect to a fixed point, that is $\hat{T}\varepsilon(\bar{r}) = \varepsilon(\bar{r})$. We know that if $\left[\hat{T},\hat{\Theta}\right] = \hat{T}\hat{\Theta} - \hat{\Theta}\hat{T} = 0$, $\hat{\Theta}$ and $\hat{T}$ have a simultaneous set of eigenfunctions. This is indeed the case for any $\hat{T}$ that is a combination of rotation, inversion, and reflection symmetry on $\varepsilon(\bar{r})$. $\hat{T}$ can be written in a matrix form. For example, a rotation operation along the z-axis by an angle φ can be written as follows.

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.18}$$

When $\hat{T}$ is acting on a vector field $\bar{F}(\bar{r})$:

$$\hat{T}\bar{F}(\bar{r}) = R\bar{F}(R^{-1}\bar{r}) \tag{5.19}$$

We will discuss more details later when we discuss photonic crystals.

5.2. Scaling in EM

We know that EM waves can be classified by their wavelengths. In this section, we study the scaling behavior of EM waves. We ask ourselves the following question. If we design a device structure that works for infrared waves, what do we do to make it to work for visible lights? In general, if $\vec{H}(\bar{r})$ is a solution to:

$$\nabla_{\bar{r}} \times \left(\frac{1}{\varepsilon(\bar{r})}\nabla_{\bar{r}} \times \vec{H}(\bar{r})\right) = \omega^2\mu_0\vec{H}(\bar{r}) \tag{5.20}$$

If we scale the structure such that the new dielectric function is identical to the original dielectric function in (5.20) but with its magnitude and linear dimension scaled as follows.

$$\varepsilon'(\bar{r}) = m\varepsilon(s\bar{r}) \tag{5.21}$$

where *m* and *s* are two scaling factors. For example, if $\varepsilon(\bar{r})$ is a period function with a period of *a*. The new dielectric function will have the dielectric constant *m* times bigger and with a period of *a/s*. If we make the coordinate transformation in (5.20) as $\bar{r}' = \bar{r}/s$ and use $\nabla_{\bar{r}'} = s\nabla_{\bar{r}}$, (5.20) becomes:

$$\frac{1}{s}\nabla_{\bar{r}'} \times \left(\frac{1}{\varepsilon(s\bar{r}')}\frac{1}{s}\nabla_{\bar{r}'} \times \vec{H}(s\bar{r}')\right) = \omega^2\mu_0\vec{H}(s\bar{r}') \tag{5.22}$$

Using (5.21) in (5.22), we get:

$$\frac{1}{s}\nabla_{\vec{r}\,'} \times \left(\frac{m}{\varepsilon'(\vec{r}\,')} \frac{1}{s} \nabla_{\vec{r}\,'} \times \vec{H}(s\vec{r}\,') \right) = \omega^2 \mu_0 \vec{H}(s\vec{r}\,')$$
$$\Rightarrow \nabla_{\vec{r}\,'} \times \left(\frac{1}{\varepsilon'(\vec{r}\,')} \nabla_{\vec{r}\,'} \times \vec{H}(s\vec{r}\,') \right) = \left(\frac{\omega s}{\sqrt{m}} \right)^2 \mu_0 \vec{H}(s\vec{r}\,') \quad (5.23)$$

We see that with the new dielectric function defined by (5.21), $\vec{H}(s\vec{r}\,')$ is a solution. That is if $\vec{H}(\vec{r})$ is a solution for a dielectric function $\varepsilon(\vec{r})$, $\vec{H}(s\vec{r})$ is a solution to a "scaled" problem with a dielectric function given by $m\varepsilon(s\vec{r})$ but with a frequency at $\omega s/\sqrt{m}$. For example, if we design an antireflection coating using a periodic structure at a certain frequency ω, we can reduce the period of the structure by two times and the same device will work at a frequency 2ω.